Three-Dimensional Elasticity Solution for Sandwich Plates With Orthotropic Phases: The Positive Discriminant Case

George A. Kardomateas
Professor
Fellow ASME
School of Aerospace Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0150

A three-dimensional elasticity solution for rectangular sandwich plates exists only under restrictive assumptions on the orthotropic material constants of the constitutive phases (i.e., face sheets and core). In particular, only for negative or zero discriminant of the cubic characteristic equation, which is formed from these constants (case of three real roots). The purpose of the present paper is to present the corresponding solution for the more challenging case of positive discriminant, in which two of the roots are complex conjugates. [DOI: 10.1115/1.2966174]

Introduction

Elasticity solutions are significant because they provide a benchmark for assessing the performance of the various plate or shell theories or the various numerical methods such as the finite element method. For monolithic anisotropic bodies, such solutions have been developed primarily by Lekhnitskii [1]. For laminated composite or sandwich structures a few closed form solutions exist, namely, for a plate configuration by Pagano [2] for the two-dimensional case and [3] for the three-dimensional case (both under restrictive assumptions) and for a sandwich shell configuration by Kardomateas [4]. The purpose of this work is to extend the paper for the three-dimensional elastic solution by Pagano [3]. Specifically, the material constants of each phase (layer in composites or face-sheet or core in sandwich) result in a cubic characteristic equation. In Ref. [3] only the case of negative discriminant of the cubic equation, which is the case of three unequal real roots, was treated. The isotropic case, in which there are three equal real roots, was also treated. In the present paper we present the solution for the case of positive discriminant, which results in two complex conjugate roots and one real root of the cubic equation. Although the case of negative discriminant is probably more frequent with composite layers, including the transversely isotropic layers [3], the positive discriminant seems to appear frequently in sandwich construction with orthotropic cores, in which the stiffness in the transverse direction is greater than that of the in-plane directions (e.g., realistic honeycomb cores as shown in the example in the Results and Discussion section). Therefore, the solution given in the present paper completes Pagano's original work [3] for all cases of material constants.

Elasticity Formulation

We consider a sandwich plate consisting of orthotropic face-sheets of thickness f_1 and f_2 and an orthotropic core of thickness $2c$, such that the various axes of elastic symmetry are parallel to the plate axes x, y, and z (Fig. 1). The body is simply supported. A normal traction $\sigma_z=q_0(x,y)$ is applied on the upper surface but the lower surface is traction-free.

Let us denote each phase by i, where $i=f_1$ for the upper face-sheet, $i=c$ for the core, and $i=f_2$ for the lower face-sheet. Then, for each phase, the orthotropic strain-stress relations are in the same form as in Eqs. (1) and (2) of Ref. [3], with c_{ij} denoting the stiffness constants. Using the strain-displacement relations and the equilibrium relations and the simply supported plate solution for the displacements as in Eqs. (6)–(8) of Ref. [3] results in the following characteristic equation for a solution to exist in each of the sandwich phases:

$$A_0 s^6 + A_1 s^4 + A_2 s^2 + A_3 = 0 \tag{1}$$

where

$$A_0 = -c_{33}c_{44}c_{55} \tag{2a}$$

$$A_1 = p^2[c_{44}(c_{11}c_{33}-c_{13}^2)+c_{55}(c_{33}c_{66}-2c_{13}c_{44})] + q^2[c_{55}(c_{22}c_{33}-c_{23}^2)+c_{44}(c_{33}c_{66}-2c_{23}c_{55})] \tag{2b}$$

$$\begin{aligned}A_2 = {} & -p^4[c_{66}(c_{11}c_{33}-c_{13}^2)+c_{55}(c_{11}c_{44}-2c_{13}c_{66})] \\ & + p^2q^2[-c_{11}(c_{22}c_{33}-c_{23}^2) \\ & -2(c_{12}+c_{66})(c_{13}+c_{55})(c_{23}+c_{44}) \\ & -2c_{44}c_{55}c_{66}+2c_{11}c_{23}c_{44}+c_{12}c_{33}(c_{12}+2c_{66}) \\ & +c_{13}c_{22}(c_{13}+2c_{55})] - q^4[c_{66}(c_{22}c_{33}-c_{23}^2) \\ & +c_{44}(c_{22}c_{55}-2c_{23}c_{66})]\end{aligned} \tag{2c}$$

$$\begin{aligned}A_3 = {} & p^6c_{11}c_{55}c_{66} + p^4q^2[c_{55}(c_{11}c_{22}-c_{12}^2)+c_{66}(c_{11}c_{44}-2c_{12}c_{55})] \\ & + p^2q^4[c_{44}(c_{11}c_{22}-c_{12}^2)+c_{66}(c_{22}c_{55}-2c_{12}c_{44})] + q^6c_{22}c_{44}c_{66}\end{aligned} \tag{2d}$$

With the substitution

$$\beta = s^2 \tag{3}$$

Eq. (1), which defines the parameter s, can be written in the form of a cubic equation as

$$\beta^3 + a_1\beta^2 + a_2\beta + a_3 = 0, \quad a_i = A_i/A_0 \quad (i=1,2,3) \tag{4}$$

This is what we would call the "characteristic equation" for the elasticity solution. Let

$$Q = \frac{3a_2 - a_1^2}{9}, \quad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}, \quad D = Q^3 + R^2 \tag{5}$$

The last quantity, D, is the discriminant and determines the nature of the solution. If $D<0$, then all roots are real and unequal. This case was treated by Pagano [3]. We consider next the case of positive discriminant, which has not yet been treated.

Solution for Positive Discriminant

If $D>0$, then the cubic equation (4) has one real root and two complex conjugates.

With R and D defined in Eq. (5), we further define

$$S = \sqrt[3]{R+\sqrt{D}}, \quad T = \sqrt[3]{R-\sqrt{D}} \tag{6a}$$

Then if

$$\mu_R = -\tfrac{1}{2}(S+T) - \frac{a_1}{3}, \quad \mu_I = \tfrac{1}{2}\sqrt{3}(S-T) \tag{6b}$$

the two complex conjugate roots are

$$\beta_1 = \mu_R + i\mu_I, \quad \beta_2 = \mu_R - i\mu_I \tag{6c}$$

The real root is

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received November 16, 2007; final manuscript received May 24, 2008; published online November 5, 2008. Review conducted by Anthony Wass.

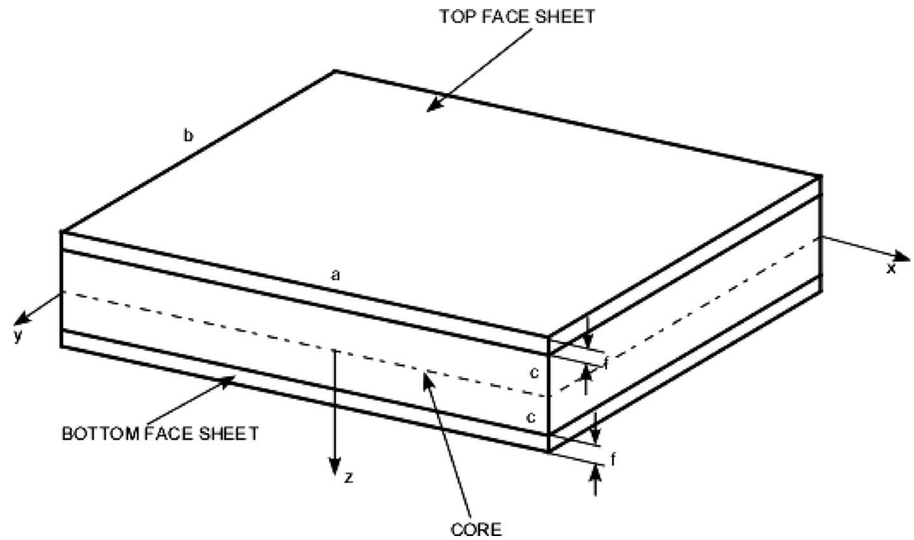

Fig. 1 Definition of the geometrical and loading configuration for the sandwich plate

$$\beta_3 = S + T - \frac{a_1}{3} \tag{6d}$$

We will consider how to deal with the complex conjugate roots first. In terms of the modulus r and amplitude θ of these complex numbers,

$$r = \sqrt{\mu_R^2 + \mu_I^2}, \quad \theta = \arctan\left(\frac{\mu_I}{\mu_R}\right) \tag{6e}$$

these roots can be set in the form

$$\beta_1 = r(\cos\theta + i\sin\theta), \quad \beta_2 = r(\cos\theta - i\sin\theta) \tag{6f}$$

From Eq. (3), we seek now the square roots of the β_is. Thus, in terms of

$$\gamma_1 = \sqrt{r}\cos\frac{\theta}{2}, \quad \gamma_2 = \sqrt{r}\sin\frac{\theta}{2} \tag{6g}$$

the corresponding roots of the sixth order equation (7), s_i, are

$$s_{1,2} = \pm(\gamma_1 + i\gamma_2), \quad s_{3,4} = \pm(\gamma_1 - i\gamma_2) \tag{6h}$$

Corresponding to these four roots, the displacement functions take the form

$$U_\eta(z) = a_{1\eta}e^{\gamma_1 z}\cos\gamma_2 z + a_{2\eta}e^{\gamma_1 z}\sin\gamma_2 z + a_{3\eta}e^{-\gamma_1 z}\cos\gamma_2 z + a_{4\eta}e^{-\gamma_1 z}\sin\gamma_2 z, \quad \eta = u,v,w \tag{7}$$

where $\eta = u,v,w$ corresponds to the U, V, W displacements and the $a_{1\eta}$ are constants. Of the 12 constants appearing in Eq. (7) only 4 are independent. The eight relations that exist among these constants are found by substituting the displacements along with Eqs. (6)–(8) of Ref. [3] into the equilibrium Eq. (3) of Ref. [3].

For convenience, let us set

$$r_1 = c_{44}(\gamma_1^2 + \gamma_2^2) + c_{66}p^2 + c_{22}q^2 \tag{8a}$$

$$r_2 = c_{44}(\gamma_1^2 + \gamma_2^2) - c_{66}p^2 - c_{22}q^2 \tag{8b}$$

$$r_3 = c_{55}(\gamma_1^2 + \gamma_2^2) + c_{11}p^2 + c_{66}q^2 \tag{8c}$$

$$r_4 = c_{55}(\gamma_1^2 + \gamma_2^2) - c_{11}p^2 - c_{66}q^2 \tag{8d}$$

and

$$e_1 = r_1(c_{13} + c_{55}) - q^2(c_{12} + c_{66})(c_{23} + c_{44}) \tag{8e}$$

$$e_2 = r_2(c_{13} + c_{55}) + q^2(c_{12} + c_{66})(c_{23} + c_{44}) \tag{8f}$$

$$e_3 = r_3(c_{23} + c_{44}) - p^2(c_{12} + c_{66})(c_{13} + c_{55}) \tag{8g}$$

$$e_4 = r_4(c_{23} + c_{44}) + p^2(c_{12} + c_{66})(c_{13} + c_{55}) \tag{8h}$$

In this way, we obtain the following relations for the coefficients in the displacement expression for $V(z)$, Eq. (7), in terms of the coefficients in the expression for $U(z)$:

$$a_{1v} = \xi_{11}a_{1u} + \xi_{12}a_{2u}, \quad a_{2v} = \xi_{21}a_{1u} + \xi_{22}a_{2u} \tag{9a}$$

$$a_{3v} = \xi_{33}a_{3u} + \xi_{34}a_{4u}, \quad a_{4v} = \xi_{43}a_{3u} + \xi_{44}a_{4u} \tag{9b}$$

where

$$\xi_{11} = \xi_{22} = \xi_{33} = \xi_{44} = \frac{q(e_1e_3\gamma_2^2 + e_2e_4\gamma_1^2)}{p(\gamma_2^2e_1^2 + \gamma_1^2e_2^2)} \tag{9c}$$

$$\xi_{12} = -\xi_{21} = -\xi_{34} = \xi_{43} = \frac{q\gamma_1\gamma_2(e_2e_3 - e_1e_4)}{p(\gamma_2^2e_1^2 + \gamma_1^2e_2^2)} \tag{9d}$$

Also, the following relations for the coefficients in the expression for $W(z)$, Eq. (7), in terms of the coefficients in the expression for $U(z)$:

$$a_{1w} = f_{11}a_{1u} + f_{12}a_{2u}, \quad a_{2w} = f_{21}a_{1u} + f_{22}a_{2u} \tag{10a}$$

$$a_{3w} = f_{33}a_{3u} + f_{34}a_{4u}, \quad a_{4w} = f_{43}a_{3u} + f_{44}a_{4u} \tag{10b}$$

where

$$f_{11} = f_{22} = -f_{33} = -f_{44} = \frac{(c_{12} + c_{66})pq\gamma_1 - r_2\gamma_1\xi_{11} - r_1\gamma_2\xi_{21}}{q(c_{23} + c_{44})(\gamma_1^2 + \gamma_2^2)} \tag{10c}$$

$$f_{12} = -f_{21} = f_{34} = -f_{43} = -\frac{(c_{12} + c_{66})pq\gamma_2 + r_2\gamma_1\xi_{12} + r_1\gamma_2\xi_{22}}{q(c_{23} + c_{44})(\gamma_1^2 + \gamma_2^2)} \tag{10d}$$

Now, coming to the real root β_3 of Eq. (4), this is treated in the same manner as in Ref. [3], i.e., if we set

$$m_3 = \sqrt{|\beta_3|} \tag{11}$$

then, if $\beta_3 < 0$, the corresponding two roots of Eq. (1) are $s_{5,6} = \pm im_3$ and the corresponding displacements can be set in the form

$$U_\eta(z) = a_{5\eta}\cos m_3 z + a_{6\eta}\sin m_3 z, \quad \eta = u,v,w \tag{12}$$

If $\beta_3 > 0$ then $s_{5,6} = \pm m_3$ and in an analogous fashion, we can set

$$U_\eta(z) = a_{5\eta} \cosh m_3 z + a_{6\eta} \sinh m_3 z, \quad \eta = u,v,w \tag{13}$$

Again, of the six unknown constants in Eq. (12) and (13) only two are independent and the four relations among them are found again by substituting these expressions into Eqs. (6)–(8) and (3) of Ref. [3].

Hence, if we consider as independent the constants a_{1u}, a_{2u}, a_{3u}, a_{4u}, a_{5u}, and a_{6u}, which we rename for convenience as g_1, g_2, g_3, g_4, g_5, and g_6, respectively, the displacement $U(z)$ is in the form

$$U(z) = d_{u1}g_1 + d_{u2}g_2 + d_{u3}g_3 + d_{u4}g_4 + d_{u5}g_5 + d_{u6}g_6 \tag{14}$$

with the z-dependent coefficients defined as

$$d_{u1} = e^{\gamma_1 z} \cos \gamma_2 z, \quad d_{u2} = e^{\gamma_1 z} \sin \gamma_2 z \tag{15a}$$

$$d_{u3} = e^{-\gamma_1 z} \cos \gamma_2 z, \quad d_{u4} = e^{-\gamma_1 z} \sin \gamma_2 z \tag{15b}$$

$$d_{u5} = \begin{cases} \cos m_3 z & \text{if } \beta_3 < 0 \\ \cosh m_3 z & \text{if } \beta_3 > 0 \end{cases} \tag{15c}$$

$$d_{u6} = \begin{cases} \sin m_3 z & \text{if } \beta_3 < 0 \\ \sinh m_3 z & \text{if } \beta_3 > 0 \end{cases} \tag{15d}$$

Similar expressions can be found for $V(z)$, $W(z)$, and the stresses.

From this analysis, we can see that within each phase (i), where $i = f_1, c, f_2$, there are six constants: $g_j^{(i)}$, $j = 1, \ldots, 6$. Therefore, for the three phases, this gives a total of 18 constants to be determined.

There are three traction conditions at each of the two core/face-sheet interfaces, giving a total of six conditions. In a similar fashion, there are three displacement continuity conditions at each of the two core/face-sheet interfaces, giving another six conditions. Finally, there are three traction boundary conditions on each of the two plate bounding surfaces, giving another six conditions, for a total of 18 equations.

Results and Discussion

As an illustration of the above, let us consider a sandwich plate with unidirectional graphite/epoxy faces and hexagonal glass/phenolic honeycomb core. Such sandwich construction is quite common in the aerospace/rotorcraft industry. The orthotropic graphite/epoxy facing moduli are (in gigapascals) as follows: $E_1^f = 181.0$, $E_2^f = E_3^f = 10.3$, $G_{23}^f = 5.96$, and $G_{12}^f = G_{31}^f = 7.17$ and facing Poisson's ratios are as follows: $\nu_{12}^f = \nu_{13}^f = 0.277$ and $\nu_{32}^f = 0.400$. The orthotropic honeycomb core moduli are (in gigapascals) as follows: $E_1^c = E_2^c = 0.032$, $E_3^c = 0.300$, $G_{23}^c = G_{31}^c = 0.048$, and $G_{12}^c = 0.013$ and core's Poisson's ratios are as follows: $\nu_{12}^c = \nu_{32}^c = \nu_{31}^c = 0.25$. The thickness of each face-sheet is $f_1 = f_2 = 2$ mm and the core $2c = 16$ mm. The plate is square with $a = b = 10h_{\text{tot}}$, where h_{tot} is the total thickness of the plate. We further assume that a transverse loading is applied at the top face-sheet of the form represented by Eq. (25) of Ref. [3], and in the definition of p and q in Eq. (7) of Ref. [3], we further assume $m = n = 1$, i.e., the applied loading is in the form $q_0(x,y) = \sigma \sin(\pi x/a)\sin(\pi y/b)$.

Substituting the corresponding constants leads to the following βs:

Face-sheets, $D > 0$, therefore two complex conjugate roots and one real root:

$$\beta_1^f = 342.5 + i316.3, \quad \beta_2^f = 342.5 - i316.3, \quad \beta_3^f = 6150.2$$

Core, $D > 0$, therefore again two complex conjugate roots and one real root:

$$\beta_1^c = 158.9 + i49.2, \quad \beta_2^c = 158.9 - i49.2, \quad \beta_3^c = 131.6$$

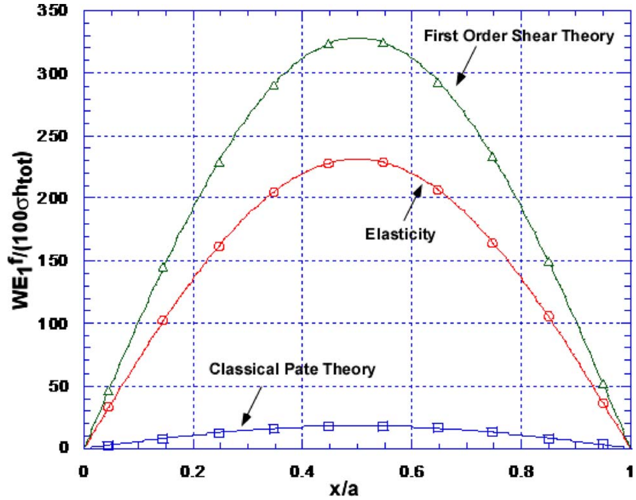

Fig. 2 Transverse displacement, *W*, at the top face-sheet and at $y = b/2$ as a function of x for $a = b = 10h_{\text{tot}}$

Since for both the face-sheet and the core we have positive discriminant, the formulas for the coefficients in the expressions of the displacements and stresses given in the present paper are applicable. Note that if one of the phases had a negative discriminant, then we would have to use the corresponding formulas in Ref. [3].

The solution is determined by imposing the following:

(a) three traction conditions at the lower face-sheet/core interface:

$$\sigma_{zz}^{(c)} = \sigma_{zz}^{(f2)}, \quad \tau_{yz}^{(c)} = \tau_{yz}^{(f2)} \quad \text{and } \tau_{xz}^{(c)} = \tau_{xz}^{(f2)} \quad \text{at } z = -c$$

(b) three displacement continuity conditions at the lower core/face-sheet interfaces:

$$U^{(c)} = U^{(f2)}, \quad V^{(c)} = V^{(f2)} \quad \text{and } W^{(c)} = W^{(f2)} \quad \text{at } z = -c$$

(c) three analogous traction conditions at the upper face-sheet/core interface, $z = +c$

(d) three analogous displacement continuity conditions at the upper face-sheet/core interface, $z = +c$

(e) three traction-free conditions at the lower bounding surface:

$$\sigma_{zz} = 0, \quad \tau_{yz} = 0 \quad \text{and } \tau_{xz} = 0 \quad \text{at } z = -(c + f_2)$$

and finally,

(f) three traction conditions at the upper bounding surface where the transverse load q_0 is applied:

$$\sigma_{zz} = q_0, \quad \tau_{yz} = 0 \quad \text{and } \tau_{xz} = 0 \quad \text{at } z = (c + f_1)$$

Therefore, we have a system of 18 linear algebraic equations in the 18 unknowns, $g_j^{(f2)}$, $g_j^{(c)}$, and $g_j^{(f1)}$, $j = 1, 6$.

The resulting transverse displacement w at the top, i.e., at $z = c + f_1$, and at $y = b/2$ is shown in Fig. 2. In this figure, we also show the predictions of the simple classical plate theory [5,6], which does not include transverse shear.

Furthermore, the resulting displacement profile from the first order core shear theory (based on the shear being carried exclusively by the core) [5,6] is also shown in Fig. 2. It can be seen that the classical plate is too nonconservative and very inaccurate. Furthermore, the first order shear is too conservative and also quite inaccurate (although considerably better than the classical plate). Figure 3 shows the corresponding displacement profiles for a plate five times longer, i.e., with $a = b = 50h_{\text{tot}}$. We can see that for this case of larger ratio of length over thickness, the classical and first order shear theories come closer to the elasticity, as expected; the

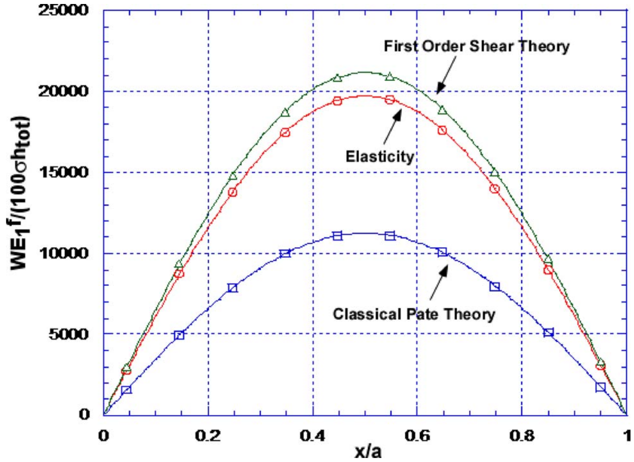

Fig. 3 Transverse displacement, W, at the top face-sheet and at $y=b/2$ as a function of x for $a=b=50h_{tot}$

classical plate is still quite inaccurate but much less so with the first order shear. These figures demonstrate clearly the large effect of transverse shear, which is an important feature of sandwich structures.

Summary

A three-dimensional elasticity solution for a rectangular sandwich plate with positive discriminant orthotropic phases is presented. This is a case frequently encountered in realistic sandwich construction. The solution is closed form. This work completes Pagano's original work [3], which was done for the negative discriminant orthotropic phases and for the isotropic phases.

Acknowledgment

The financial support of the Office of Naval Research, Grant No. N00014-07-10373, and the interest and encouragement of the Grant Monitor, Dr. Y.D.S. Rajapakse, are both gratefully acknowledged.

References

[1] Lekhnitskii, S. G., 1963, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, Inc., San Francisco.
[2] Pagano, N. J., 1969, "Exact Solutions for Composite Laminates in Cylindrical Bending," J. Compos. Mater., **3**, pp. 398–411.
[3] Pagano, N. J., 1970, "Exact Solutions for Rectangular Bidirectional Composites and Sandwich Plates," J. Compos. Mater., **4**, pp. 20–34.
[4] Kardomateas, G. A., 2001, "Elasticity Solutions for a Sandwich Orthotropic Cylindrical Shell Under External Pressure, Internal Pressure and Axial Force," AIAA J., **39**(4), pp. 713–719.
[5] Vinson, J. R., 1999, *The Behavior of Sandwich Structures of Isotropic and Composite Materials*, Technomic, Lancaster, PA.
[6] Allen, H. G., 1969, *Analysis and Design of Structural Sandwich Panels*, Pergamon, Oxford, Chap. 8.