An Elasticity Solution for the Global Buckling of Sandwich Beams/Wide Panels With Orthotropic Phases

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There exist several formulas for the global buckling of sandwich plates, each based on a specific set of assumptions and a specific plate or beam model. It is not easy to determine the accuracy and range of validity of these rather simple formulas unless an elasticity solution exists. In this paper, we present an elasticity solution to the problem of global buckling of wide sandwich panels (equivalent to sandwich columns) subjected to axially compressive loading (along the short side). The emphasis on this study is on the global (single-wave) rather than the wrinkling (multiwave) mode. The sandwich section is symmetric, and all constituent phases, i.e., the facings and the core, are assumed to be orthotropic. The buckling problem is formulated as an eigenboundary-value problem for differential equations, with the axial load being the eigenvalue. The complication in the sandwich construction arises due to the existence of additional "internal" conditions at the face-sheet/core interfaces. Results are produced for a range of geometric configurations, and these are compared with the different global buckling formulas in the literature. [DOI: 10.1115/1.3173758]

1 Introduction

The compressive strength of thin sheets can be realized only if they are stabilized against buckling. In sandwich construction, two such sheets (face sheets) are bonded to a core slab of different (light) material. Both the core and the face sheets can be isotropic or anisotropic.

Panels of this construction give rise to a set of problems of strength, stiffness, and stability analogous to, but by no means identical with, the well-known problems of ordinary homogeneous elastic beam/plates. One of these is "cylindrical buckling." Referring to Fig. 1, the panel is so wide that lines along the y axis can be taken as uncurved. Therefore, a unit width can be treated as an Euler column. Buckling is either like column buckling (Euler or global buckling) or a short wave "wrinkling" of the face sheets. In the former, the core may exhibit a substantial shearing deformation; in the latter, it acts like an elastic foundation and the buckling deformation is mainly confined to the layers adjacent to the face sheets.

A few global buckling formulas for sandwich construction can be found in the literature. In particular, a whole chapter is devoted to buckling in Allen's book [1]. Two formulas are presented, one for thin faces and one for thick faces. Another formula is in Bazant and Cedolin's book [2]. In fact, the global buckling formulas for sandwich structures in the literature are essentially ways of defining the equivalent rigidity and the shear correction factors for the sandwich construction in the old Engesser's shear correction formula for column buckling [3]. Recently, Huang and Kardomateas [4] derived a new shear correction formula for sandwich sections, which can be used with either the Engesser [3] or the Haringx [5,6] shear correction formula. This shear correction formula is not exclusively based on the shear modulus of the core, but instead includes the shear modulus of the faces and the extensional modulus of the core; therefore, it can account for sandwich constructions with stiffer cores and/or more compliant faces.

As far as wrinkling, several investigations have led to simple formulas, with the most known those by Hoff and Mautner [7] and the one in Allen's book [1].

Sandwich structures may also exhibit another form of local buckling, namely, edge buckling, which is a nonperiodic buckling deformation mode. Ji and Waas [8] showed, through a two-dimensional elasticity study, that a sandwich beam having a core with a negligible stiffness compared with the face sheets is prone to failure by edge buckling.

The existence of different buckling formulas based on various beam or plate models underscores the need for an elasticity solution, in order to compare the accuracy of the predictions from the simple beam/plate formulas. Elasticity solutions for buckling have become available mainly for the axisymmetric cylindrical shell geometry due to the availability of three-dimensional elasticity solutions for the prebuckling state and the ease of formulation afforded by the axisymmetry [9–11]. As far as sandwich structures, a three-dimensional elasticity solution for the buckling of a sandwich long shell under external pressure was recently done by Kardomateas and Simitses [12]. In all these studies, a prerequisite to obtaining elasticity solutions for shell buckling is the existence of three-dimensional elasticity solutions to the prebuckling problem. For the monolithic homogeneous cylindrical shells, the elasticity solutions for orthotropy provided by Lekhnitskii [13] were used, whereas for the sandwich shells, the elasticity solution of Kardomateas [14] was used.

In this paper we make again the simplifying assumption of a two-dimensional problem by considering a wide plate. Because the plate is wide, lines along the long dimension can be taken as uncurved during buckling, and the problem reduces to a two-dimensional (equivalent to a beam rather than a plate assumption) one. Under these assumptions, an elasticity solution for the wrinkling of a sandwich wide plate/beam with generally orthotropic phases under axial loading was presented by Kardomateas [15]. In this paper we focus on the global buckling behavior, and the governing buckling equations along with the corresponding boundary conditions are derived by including all terms. These reduce to an

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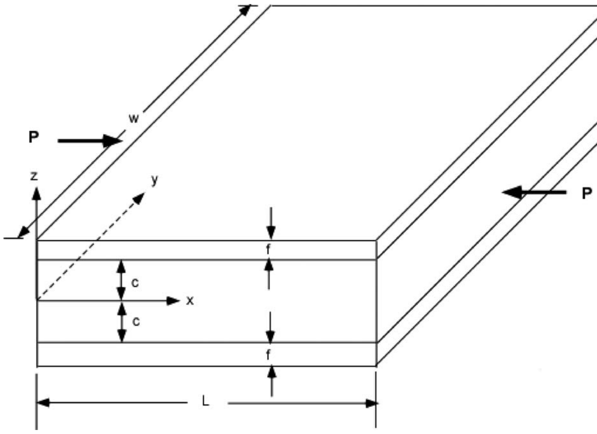

Fig. 1 Definition of the geometry for a sandwich wide panel/ beam under axial compression

eigenboundary-value problem for differential equations with the axial load being the eigenvalue. The complication in the sandwich construction arises due to the existence of additional "internal" conditions at the face sheet/core interfaces. The shooting method is used to solve the problem thus formulated.

2 Formulation

The significance of the proper conjugate incremental stress and incremental strain measure for buckling was introduced by Bazant [16] and in Refs. [17,18] it was shown that, for sandwich-type structures with a soft core, the Green–Lagrange strain measure must be used if the strains are small and the elastic moduli are kept constant throughout the analysis. The objective stress measure, which is energetically conjugate to the Green–Lagrange strain measure, is the second Piola–Kirchhoff stress, σ_{ij} and the corresponding incremental stress measure is the Trefftz stress. Based on this, the buckling equations that follow can be accordingly derived from this general formulation [16] or the fundamental approach presented in Ref. [19]. Since sandwich panels are characterized by large shear deformations in the soft core, we include all terms herein. Specifically, by considering the equations of equilibrium in terms of the second Piola–Kirchhoff stress tensor, subtracting these at the perturbed and initial conditions, the buckling equations for a Cartesian coordinate system can be obtained as follows:

$$\frac{\partial}{\partial x}\left[\sigma_{xx}+\epsilon_{xx}\sigma_{xx}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\tau_{xz}^{0}\right]+\frac{\partial}{\partial y}\left[\tau_{xy}+\epsilon_{xx}\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\sigma_{yy}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\tau_{yz}^{0}\right]+\frac{\partial}{\partial z}\left[\tau_{xz}+\epsilon_{xx}\tau_{xz}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\tau_{yz}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\sigma_{zz}^{0}\right]=0 \tag{1a}$$

$$\frac{\partial}{\partial x}\left[\tau_{xy}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\sigma_{xx}^{0}+\epsilon_{yy}\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\tau_{xz}^{0}\right]+\frac{\partial}{\partial y}\left[\sigma_{yy}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\tau_{xy}^{0}+\epsilon_{yy}\sigma_{yy}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\tau_{yz}^{0}\right]+\frac{\partial}{\partial z}\left[\tau_{yz}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\tau_{xz}^{0}+\epsilon_{yy}\tau_{yz}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\sigma_{zz}^{0}\right]=0 \tag{1b}$$

$$\frac{\partial}{\partial x}\left[\tau_{xz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\sigma_{xx}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\tau_{xy}^{0}+\epsilon_{zz}\tau_{xz}^{0}\right]+\frac{\partial}{\partial y}\left[\tau_{yz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\sigma_{yy}^{0}+\epsilon_{zz}\tau_{yz}^{0}\right]+\frac{\partial}{\partial z}\left[\sigma_{zz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\tau_{xz}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\tau_{yz}^{0}+\epsilon_{zz}\sigma_{zz}^{0}\right]=0 \tag{1c}$$

In the previous equations, σ_{ij}^0 are the values of stresses at the initial equilibrium position (prebuckling state), and σ_{ij} and ω_j are the values of stresses and rotations at the perturbed position (buckled state).

The foregoing equations include shear stresses and normal strains at the buckled state, therefore, they are complete. We have retained these terms because the core is weak in shear, and transverse shear stresses may be significant. This is unlike the simplified equations used in previous work on monolithic composites [9–11], in which it was assumed that rotations substantially exceed strains.

The boundary conditions associated with Eq. (1) can be obtained from the traction (stress resultant) relationships in terms of the second Piola–Kirchhoff stress tensor, and in the general case of an external hydrostatic pressure loading (in which case the magnitude of the surface load remains invariant under deformation, but its direction changes) [19]. By writing these equations for the initial and the perturbed equilibrium position and then subtracting them, the following boundary conditions on a surface, which has outward unit normal $(\hat{l},\hat{m},\hat{n})$ (no hydrostatic pressure acting), are obtained (with shear stresses and normal strains at the buckled state included):

$$\left[\sigma_{xx}+\epsilon_{xx}\sigma_{xx}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\tau_{xz}^{0}\right]\hat{l}+\left[\tau_{xy}+\epsilon_{xx}\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\sigma_{yy}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\tau_{yz}^{0}\right]\hat{m}+\left[\tau_{xz}+\epsilon_{xx}\tau_{xz}^{0}+\left(\frac{1}{2}\gamma_{xy}-\omega_{z}\right)\tau_{yz}^{0}+\left(\frac{1}{2}\gamma_{xz}+\omega_{y}\right)\sigma_{zz}^{0}\right]\hat{n}=0 \tag{2a}$$

$$\left[\tau_{xy}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\sigma_{xx}^{0}+\epsilon_{yy}\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\tau_{xz}^{0}\right]\hat{l}+\left[\sigma_{yy}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\tau_{xy}^{0}+\epsilon_{yy}\sigma_{yy}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\tau_{yz}^{0}\right]\hat{m}+\left[\tau_{yz}+\left(\frac{1}{2}\gamma_{xy}+\omega_{z}\right)\tau_{xz}^{0}+\epsilon_{yy}\tau_{yz}^{0}+\left(\frac{1}{2}\gamma_{yz}-\omega_{x}\right)\sigma_{zz}^{0}\right]\hat{n}=0 \tag{2b}$$

$$\left[\tau_{xz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\sigma_{xx}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\tau_{xy}^{0}+\epsilon_{zz}\tau_{xz}^{0}\right]\hat{l}+\left[\tau_{yz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\tau_{xy}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\sigma_{yy}^{0}+\epsilon_{zz}\tau_{yz}^{0}\right]\hat{m}+\left[\sigma_{zz}+\left(\frac{1}{2}\gamma_{xz}-\omega_{y}\right)\tau_{xz}^{0}+\left(\frac{1}{2}\gamma_{yz}+\omega_{x}\right)\tau_{yz}^{0}+\epsilon_{zz}\sigma_{zz}^{0}\right]\hat{n}=0 \tag{2c}$$

For the bounding surfaces, $\hat{l}=\hat{m}=0$ and $\hat{n}=\pm 1$. These conditions will also be used when we impose traction continuity at the core/face-sheet interfaces.

The face sheets and core are assumed to be homogeneous and linearly elastic orthotropic solids. Consequently, it is assumed that the perturbed stresses are related to the perturbed strains in the same manner as in the prebuckling state. Therefore, the stress-strain relationship for the face sheet, $i=f$, or the core, $i=c$ is assumed to be

$$\begin{bmatrix} \sigma_{xx}^{(i)} \\ \sigma_{yy}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{yz}^{(i)} \\ \tau_{xz}^{(i)} \\ \tau_{xy}^{(i)} \end{bmatrix} = \begin{bmatrix} c_{11}^i & c_{12}^i & c_{13}^i & 0 & 0 & 0 \\ c_{12}^i & c_{22}^i & c_{23}^i & 0 & 0 & 0 \\ c_{13}^i & c_{23}^i & c_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^i \end{bmatrix} \begin{bmatrix} \epsilon_{xx}^{(i)} \\ \epsilon_{yy}^{(i)} \\ \epsilon_{zz}^{(i)} \\ \gamma_{yz}^{(i)} \\ \gamma_{xz}^{(i)} \\ \gamma_{xy}^{(i)} \end{bmatrix}, \quad (i=f,c) \quad (3)$$

where c_{kl}^i are the stiffness constants (we have used the notation $1 \equiv x$, $2 \equiv y$, and $3 \equiv z$, see Fig. 1). In a similar fashion, we shall denote by a_{ij} the corresponding compliance constants.

Note that the specific elastic moduli corresponding to the incremental stresses are used [16], and the strains associated with the incremental displacements are the linearized strains, as outlined in Sec. 2.2.

2.1 Prebuckling State. The prebuckling state is that of normal prebuckling stresses in the form (for $i=f,c$)

$$\sigma_{xx}^{0(i)} = Pd_i, \quad \sigma_{yy}^{0(i)} = \sigma_{zz}^{0(i)} = 0 \quad (4a)$$

whereas the shear stresses are zero

$$\tau_{xy}^0 = \tau_{xz}^0 = \tau_{yz}^0 = 0 \quad (4b)$$

Imposing the condition of same axial strain $\epsilon_{xx}^{0(i)} = a_{11}^i \sigma_{xx}^{0(i)}$ and the condition of the resultant applied compressive load, $\int \sigma_{xx} w dz = -P$, leads to two equations for the constants, i.e.,

$$a_{11}^f P d_f = a_{11}^c P d_c, \quad Pd_f f + Pd_c c = -P/(2w) \quad (4c)$$

and subsequently to the prebuckling stresses

$$\sigma_{xx}^{0f} = -P \frac{a_{11}^c}{2w(a_{11}^c f + a_{11}^f c)}, \quad \sigma_{xx}^{0c} = -P \frac{a_{11}^f}{2w(a_{11}^c f + a_{11}^f c)} \quad (4d)$$

2.2 Perturbed State. The buckling equations (1) can be written in terms of the buckling displacements u, v, and w by using the strain versus displacement relations

$$\epsilon_{xx} = u_{,x}, \quad \epsilon_{yy} = v_{,y}, \quad \epsilon_{zz} = w_{,z} \quad (5a)$$

$$\gamma_{xy} = u_{,y} + v_{,x}, \quad \gamma_{xz} = u_{,z} + w_{,x}, \quad \gamma_{yz} = v_{,z} + w_{,y} \quad (5b)$$

and rotation versus displacement relations

$$2\omega_x = w_{,y} - v_{,z}, \quad 2\omega_y = u_{,z} - w_{,x}, \quad 2\omega_z = v_{,x} - u_{,y} \quad (5c)$$

and then using the stress-strain relations (3). The following three equations are obtained for the prebuckling stress field equations (4*a*) and (4*b*). These equations apply at every point through the thickness, but for convenience we have dropped the superscript i

$$(c_{11} + \sigma_{xx}^0) u_{,xx} + c_{66} u_{,yy} + c_{55} u_{,zz} + (c_{12} + c_{66}) v_{,xy} + (c_{13} + c_{55}) w_{,xz} = 0 \quad (6a)$$

$$c_{22} v_{,yy} + (c_{66} + \sigma_{xx}^0) v_{,xx} + c_{44} v_{,zz} + (c_{12} + c_{66}) u_{,xy} + (c_{44} + c_{23}) w_{,yz} = 0 \quad (6b)$$

$$c_{33} w_{,zz} + (c_{55} + \sigma_{xx}^0) w_{,xx} + c_{44} w_{,yy} + (c_{13} + c_{55}) u_{,xz} + (c_{44} + c_{23}) v_{,yz} = 0 \quad (6c)$$

The corresponding from Eq. (2) traction boundary conditions at the bounding surfaces for $\hat{l} = \hat{m} = 0$ and $\hat{n} = 1$ are

$$c_{55} u_{,z} + c_{55} w_{,x} = 0 \quad (7a)$$

$$c_{44} w_{,y} + c_{44} v_{,z} = 0 \quad (7b)$$

$$c_{13} u_{,x} + c_{23} v_{,y} + c_{33} w_{,z} = 0 \quad (7c)$$

In the petrurbed position we seek two-dimensional equilibrium modes as follows:

$$u_i = U_i(z) \cos \lambda x, \quad v_i = 0, \quad w_i = W_i(z) \sin \lambda x, \quad \lambda = \frac{\pi}{L} \quad i = f,c \quad (8)$$

It should be mentioned that according to the general bifurcation formulation, if we denote by u_0, v_0, and w_0 the x, y, and z components of the displacement field at the primary (prebuckling) state, the perturbed (buckled) position is denoted by $u_1 = u_0 + \alpha u$, $v_1 = v_0 + \alpha v$, and $w_1 = w_0 + \alpha w$, where α is an infinitesimally small quantity. Here $\alpha u(x,y,z)$, $\alpha v(x,y,z)$, and $\alpha w(x,y,z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. Thus, displacements (8) are simply on top of the prebuckling displacements and therefore, the fact that the prebuckling displacement field corresponding to Eq. (4) would imply $v_0 \neq 0$ does not present a contradiction.

Now, substituting into Eq. (7), results in the following two linear homogeneous ordinary differential equations of the second order for $U_i(z)$, $W_i(z)$, where $i=c$ for $0 \leq z \leq c$ and $i=f$ for $c \leq z \leq (c+f)$:

$$c_{55}^{(i)} U_i'' - (c_{11}^{(i)} + \sigma_{xx}^{0(i)}) \lambda^2 U_i + (c_{13}^{(i)} + c_{55}^{(i)}) \lambda W_i' = 0 \quad (9a)$$

and

$$c_{33}^{(i)} W_i'' - (c_{55}^{(i)} + \sigma_{xx}^{0(i)}) \lambda^2 W_i - (c_{13}^{(i)} + c_{55}^{(i)}) \lambda U_i' = 0 \quad (9b)$$

The associated boundary conditions are as follows:

(a) at the bounding surfaces, $z=c+f$, we have the following two traction-free conditions:

$$c_{55}^{(f)} U_f' + c_{55}^{(f)} \lambda W_f = 0 \quad (10a)$$

$$c_{33}^{(f)} W_f' - c_{13}^{(f)} \lambda U_f = 0 \quad (10b)$$

(b) at the face-sheet/core interface, $z=c$, we have the following four conditions at each of the interfaces.

For displacement continuity,

$$U_f = U_c, \quad W_f = W_c \quad (10c)$$

For traction continuity,

$$c_{55}^{(f)} U_f' + c_{55}^{(f)} \lambda W_f = c_{55}^{(c)} U_c' + c_{55}^{(c)} \lambda W_c \quad (10d)$$

$$c_{33}^{(f)} W_f' - c_{13}^{(f)} \lambda U_f = c_{33}^{(c)} W_c' - c_{13}^{(c)} \lambda U_c \quad (10e)$$

(c) at the axis of symmetry, $z=0$, we have the following antisymmetry conditions:

$$U_c = W_c' = 0 \quad (10f)$$

Notice that since the construction is assumed to be symmetric, only half of the sandwich needs to be considered.

2.3 Solution of the Eigenboundary-Value Problem for Differential Equations. Equations (9) and (10) constitute an eigenvalue problem for differential equations, with the axial load, P, the parameter (two point boundary-value problem). An important point is that the prebuckling stresses $\sigma_{jj}^{0(i)}(z)$, depend linearly on the applied axial load, P (the parameter) through expressions in the form of Eq. (4), and this makes possible the direct application of standard solution techniques.

With respect to the method used, there is a difference between the present problem and the homogeneous orthotropic body (apart from being shell geometry) solved by Kardomateas [9]. The complication in the present problem is due to the fact that the displacement field is continuous but has a slope discontinuity at the face-sheet/core interfaces. This is the reason that the displacement field was not defined as one function but as two distinct functions for $i=f$ and $i=c$, i.e., the face sheet and the core. Our formulation of the problem employs, hence, internal boundary conditions at the face-sheet/core interface, as outlined above. Due to this complication, the shooting method [20] was deemed to be the best way to solve this eigenboundary-value problem for differential equations. A special version of the shooting method was formulated and programmed for this problem. In fact, for each of the two constituent phases of the sandwich structure, we have five variables: $y_1=U_i$, $y_2=U_i'$, $y_3=W_i$, $y_4=W_i'$, and $y_5=P$. The five differential equations are $y_1'=y_2$, the first equilibrium Eq. (9*a*), $y_3'=y_4$, the second equilibrium Eq. (9*b*), and $y_5'=0$.

The method starts from the middle of the core, $z=0$ and integrates the five first order differential equations from $z=0$ to the face-sheet/core interface $z=c$ (i.e., through the core). At the start point, $z=0$, we have three conditions as follows: $U_c=y_1=0$, $W_c'=y_4=0$ and a third condition of (arbitrarily) setting $W_c=y_3=1.0$, therefore we have two freely specifiable variables, the $P=y_5$ and the $U_c'=y_2$.

The freely specifiable starting values at $z=0$ are taken as the values from the simple plate/beam theory solutions available in the literature (described later); in particular, we have used Allen's [1] solution; therefore, we input the Allen's [1] solution as a guess and then obtain the elasticity solution by the shooting method described. Further details of the solution method can be found in Ref. [15].

3 Global Buckling Formulas in the Literature, Results and Discussion

In the following, we list the formulas in the literature for global buckling of sandwich columns.

(a) A formula for thin faces, which accounts for transverse shear, is in Allen's book [1] as follows:

$$\frac{1}{P_{\mathrm{cr},A}^{\mathrm{thin}}}=\frac{1}{P_{E1}}+\frac{1}{P_c} \tag{11a}$$

where

$$P_{E1}=E_f w f(2c+f)^2\frac{\pi^2}{2L^2},\quad P_c=G_c\frac{w(2c+f)^2}{2c} \tag{11b}$$

i.e., P_{E1} represents the Euler load of the sandwich column in the absence of core shear strains and with the bending stiffness of the core ignored and with the local bending stiffness of the faces ignored as well (because they are assumed thin); P_c may be described as the shear buckling load.

Although not explicitly stated, the structure of the formula indicates that it is an adaptation for sandwich configurations of the Engesser's [3] column buckling formula.

(b) For thick faces, Allen gives another formula

$$P_{\mathrm{cr},A}^{\mathrm{thick}}=P_{E2}\left\{\frac{1+\dfrac{P_{Ef}}{P_c}-\dfrac{P_{Ef}}{P_c}\dfrac{P_{Ef}}{P_{E2}}}{1+\dfrac{P_{E2}}{P_c}-\dfrac{P_{Ef}}{P_c}}\right\} \tag{12a}$$

where

$$P_{E2}=E_f\frac{\pi^2}{L^2}\left[\frac{wf^3}{6}+\frac{wf(2c+f)^2}{2}\right] \tag{12b}$$

$$P_{Ef}=E_f\frac{\pi^2}{L^2}\frac{wf^3}{6},\quad P_c=G_c\frac{w(2c+f)^2}{2c} \tag{12c}$$

i.e., P_{E2} represents the Euler load of the sandwich column in the absence of core shear strains and with the bending stiffness of the core ignored, but with the local bending stiffness of the faces included; P_{Ef} represents the sum of the Euler loads of the two faces when they buckle as independent struts (i.e., when the core is absent) and P_c is the shear buckling load, same as in Eq. (11*b*).

(c) Another formula is in Bazant and Cedolin's book [2], which was derived from adapting Engesser's [3] formula for the sandwich configuration

$$P_{\mathrm{cr},BC}=\frac{P_{E2}}{1+\dfrac{P_{E2}}{(GA)_0}} \tag{13a}$$

where P_{E2} is given in Eq. (8) and

$$(GA)_0=G_c w(f+2c)\left[1+\frac{f^2}{3(f+2c)^2}\right] \tag{13b}$$

(d) A shear correction formula for sandwich sections was derived by Huang and Kardomateas [4] from a shear energy equivalency. This shear correction is usually expressed by the ratio $\beta/(AG_{\mathrm{eq}})$, where A is the total cross-sectional area of the column, G_{eq} is the "equivalent" or "effective" modulus in shear, and β is a numerical factor depending on the shape of the cross section, which accounts for the fact that shear is not uniformly distributed throughout the section. If the section is rectangular and the column homogeneous isotropic, then $G_{\mathrm{eq}}=G=$shear modulus of the homogeneous material and $\beta=1.2$. For sandwich construction, which is a nonhomogeneous section, the Huang and Kardomateas [4] formula is as follows:

$$s=\frac{\beta}{AG_{\mathrm{eq}}}=\frac{2w}{(EI)_{\mathrm{eq}}^2}\left(\frac{a_f}{G_f}+\frac{a_c}{G_c}\right) \tag{14a}$$

where

$$a_f=\frac{E_f^2}{4}\left[(f+c)^4 f-\frac{7}{15}(f+c)^5-\frac{c^5}{5}+\frac{2}{3}(f+c)^2c^3\right] \tag{14b}$$

$$a_c=E_f^2f^2c\left(\frac{f}{2}+c\right)^2+\frac{2}{15}E_c^2c^5+\frac{2}{3}E_fE_cf\left(\frac{f}{2}+c\right)c^3 \tag{14c}$$

and $(EI)_{\mathrm{eq}}$ is the equivalent rigidity of the section, which includes the local bending stiffness of the faces and the bending stiffness of the core

$$(EI)_{\mathrm{eq}}=2E_f\frac{wf^3}{12}+2E_fwf\left(\frac{f}{2}+c\right)^2+E_c\frac{w(2c)^3}{12} \tag{14d}$$

For a homogeneous section (this can be most easily seen by setting $c=0$, $A=2fw$), the calculations reduce to the simple value $\beta=6/5$, which is a well-known shear correction factor for a rectangular homogeneous section from classical bending theory [21,22].

Notice that this shear correction formula is not exclusively based on the shear modulus of the core, but instead includes the shear modulus of the faces and the extensional modulus of the core, therefore, it can account for sandwich constructions with stiffer cores and/or more compliant faces.

It should also be noted that a more general formula for the transverse shear correction coefficient β, which is applicable for a sandwich section with dissimilar faces can be found in Ref. [4].

The shear correction formula (14*a*) can be used by substituting in either the Engesser critical load formula [3]

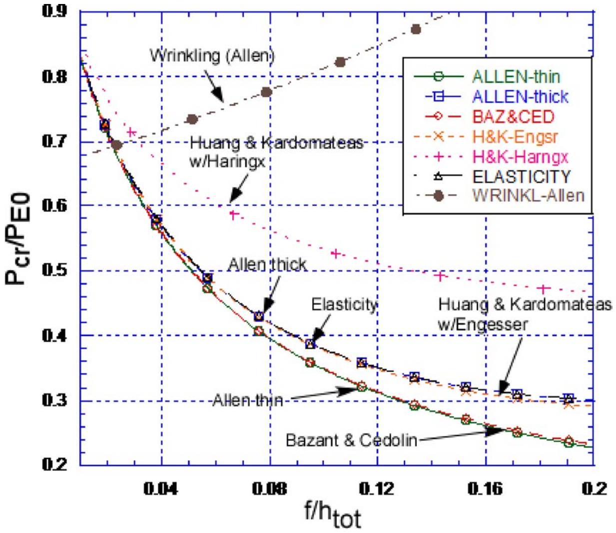

Fig. 2 Critical load for a wide range of face-sheet thicknesses for graphite/epoxy faces and glass/phenolic honeycomb core

$$P_{\mathrm{cr}}^{\mathrm{Engess}}=\frac{P_{E0}}{1+sP_{E0}} \tag{15}$$

or the Haringx one [5,6]

$$P_{\mathrm{cr}}^{\mathrm{Harngx}}=\frac{\sqrt{1+4sP_{E0}}-1}{2s} \tag{16}$$

where P_{E0} is the Euler load corresponding to Eq. (14*d*)

$$P_{E0}=\frac{\pi^2(EI)_{\mathrm{eq}}}{L^2} \tag{17}$$

In a recent paper, Bazant and Beghini [17] showed that the Engesser and Haringx-type theories are equivalent (i.e., one to follow from the other) provided that a proper transformation of the shear modulus of the core, G_c, is made. However, this transformation implies that G_c of the soft core is a function of the axial stress in the stiff skins. This paradox was clarified by showing that the energetic variational analysis merely requires that the shear stiffness of the cross section, characterized by G_c of the core, to be a function of the axial force in the skins. In other words, if the Haringx type theory was used with a constant shear modulus, the results, as in Eq. (16), would be obtained; however, if the shear modulus is updated as a function of the axial load, then the results are expected to agree with Engesser's formula.

One last point: For a wide panel, in the above formulas, E_f and E_c are substituted by the $E_f'=E_f/(1-\nu_{13}^f\nu_{31}^f)$ and the $E_c'=E_c/(1-\nu_{13}^c\nu_{31}^c)$.

3.1 Results and Discussion. Let us consider a sandwich construction with unidirectional graphite/epoxy faces and hexagonal glass/phenolic honeycomb core. Such sandwich construction is quite common in the aerospace/rotorcraft industry. The orthotropic graphite/epoxy facings moduli are (in GPa): $E_1^f=181.0$, $E_2^f=E_3^f=10.3$, $G_{23}^f=5.96$, and $G_{12}^f=G_{31}^f=7.17$; and the facings' Poisson's ratios: $\nu_{12}^f=\nu_{13}^f=0.277$ and $\nu_{32}^f=0.400$. The orthotropic honeycomb core moduli are (in GPa): $E_1^c=E_2^c=0.032$, $E_3^c=0.300$, $G_{23}^c=G_{31}^c=0.048$, and $G_{12}^c=0.013$; and the core's Poisson's ratios: $\nu_{12}^c=\nu_{32}^c=\nu_{31}^c=0.25$.

The total thickness is considered constant at $h_{\mathrm{tot}}=2f+2c=30$ mm, the length over total thickness, $L/h_{\mathrm{tot}}=30$, and we examine a range of face thicknesses defined by the ratio of face-sheet thickness over total thickness, f/h_{tot}, between 0.010 and 0.20. Figure 2 shows the critical load for a simply supported configuration, normalized with the Euler load without transverse shear, P_{E0}. The elasticity solution, along with the different formulas from the literature, is plotted. Detailed data for the critical loads are also given in Table 1, along with the percentage differences of the different formulas from the exact elasticity solution. Notice also that we use G_{31}^c in place of G_c in these formulas, which were originally derived for isotropy.

Table 1 Critical loads, P_{cr}/P_{E0} and in second line the % difference from elasticity

f/h_{tot}	Elasticity	Allen thin[a]	Baz and Ced[b]	Allen thick[c]	Engssr w/H and K [d]	Harngx w/H and K [e]	Wrinkl Allen[f]
0.02	0.7173	0.7106	0.7107	0.7149	0.7156	0.7665	0.6902
		(−0.94)	(−0.92)	(−0.33)	(−0.24)	(+6.86)	
0.04	0.5692	0.5568	0.5571	0.5678	0.5678	0.6642	0.7179
		(−2.18)	(−2.13)	(−0.25)	(−0.24)	(+16.69)	
0.06	0.4795	0.4609	0.4615	0.4786	0.4782	0.6031	0.7478
		(−3.89)	(−3.76)	(−0.19)	(−0.27)	(+25.78)	
0.08	0.4205	0.3955	0.3965	0.4199	0.4189	0.5620	0.7795
		(−5.95)	(−5.72)	(−0.15)	(−0.37)	(+33.65)	
0.10	0.3797	0.3481	0.3495	0.3793	0.3776	0.5325	0.8132
		(−8.34)	(−7.96)	(−0.12)	(−0.57)	(+40.24)	
0.12	0.3508	0.3122	0.3141	0.3505	0.3477	0.5107	0.8487
		(−11.03)	(−10.47)	(−0.09)	(−0.89)	(+45.57)	
0.14	0.3303	0.2840	0.2866	0.3301	0.3259	0.4944	0.8862
		(−14.01)	(−13.25)	(−0.06)	(−1.33)	(+49.67)	
0.16	0.3161	0.2614	0.2646	0.3160	0.3100	0.4823	0.9259
		(−17.29)	(−16.29)	(−0.04)	(−1.92)	(+52.58)	
0.18	0.3069	0.2429	0.2468	0.3068	0.2987	0.4735	0.9678
		(−20.84)	(−19.57)	(−0.01)	(−2.65)	(+54.33)	
0.20	0.3018	0.2274	0.2321	0.3018	0.2911	0.4676	1.012
		(−24.65)	(−23.09)	(0.0)	(−3.54)	(+54.94)	

[a]Eq. (11*a*).
[b]Eq. (13*a*).
[c]Eq. (12*a*).
[d]Eq. (15).
[e]Eq. (16).
[f]Ref. [1].

Since it is possible that wrinkling could dominate for the very thin face sheets, Fig. 2 shows also the critical wrinkling load from Allen's wrinkling formula [1], and we can see that wrinkling would dominate for f/h_{tot} below 0.02.

From these results we can make the following observations.

(a) Allen's thick formula [1] and the Engesser formula with the Huang and Kardomateas shear correction [4] give predictions, which are very close to the elasticity solution (the corresponding three curves can hardly be distinguished as different in Fig. 2). Between the two, Allen's thick formula [1] is more accurate in general, except for the very thin faces (f/h_{tot} between 0.01 and 0.02).

(b) Allen's thin formula [1] and the Bazant and Cedolin [2] formula are practically identical with the latter slightly more accurate, and both give predictions that are accurate within 5% of the elasticity solution for small f/h_{tot}, less than 0.07; however, both can be quite conservative for the thicker face sheets (of the order of 20% below the elasticity value for $f/h_{\text{tot}}=0.2$). In Fig. 2, the curves from these two formulas can hardly be distinguished as different.

(c) All the above mentioned formulas (Allen's thin and thick formulas [1], the Bazant and Cedolin [2] formula, and the Engesser formula with the Huang and Kardomateas shear correction [4]) are conservative.

(d) The Haringx formula with the Huang and Kardomateas shear correction [4] give predictions, which are nonconservative, and it is the most inaccurate, being of the order of 50% above the elasticity value for $f/h_{\text{tot}}=0.2$. Its accuracy improves, though, for the very thin face sheets. However, these results are based on a constant core shear modulus and the Haringx theory is more complicated and its proper application would imply a varying shear modulus, as discussed later.

(e) The transverse shear effect is very large and results in a critical load less than one-third of the Euler load for face-sheet ratios, f/h_{tot}, above 0.14.

To examine another case with a smaller anticipated transverse shear effect, we consider a sandwich construction with unidirectional E-glass/polyester faces and H100 cross-linked PVC foam core and with the same geometry as before. Such sandwich construction is more common in the marine industry. The orthotropic E-glass/polyester facings moduli are (in GPa): $E_1^f=40.0$, $E_2^f=E_3^f=10.0$, $G_{23}^f=3.5$, and $G_{12}^f=G_{31}^f=4.5$; and the facings Poisson's ratios: $\nu_{12}^f=0.065$, $\nu_{31}^f=0.26$, and $\nu_{32}^f=0.400$. The isotropic core modulus is $E^c=0.100$ GPa and the core's Poisson's ratio: $\nu^c=0.30$.

Figure 3 shows again the critical load and we can say that all of the conclusions reached earlier hold true except that the elasticity curve is now a bit more distinguishable from Allen's thick or Engesser with shear correction from Ref. [4] curves. Both of these curves are slightly below the elasticity curve with the latter being closer to the elasticity curve, especially for the thinner faces. These two curves are still the most accurate. In addition, the Haringx curve with shear correction from Ref. [4] curve is now closest to the elasticity for very small f/h_{tot}, less than 0.03.

Moreover, the effect of transverse shear is smaller as the critical load goes as low as about half the Euler load for $f/h_{\text{tot}}=0.2$.

To examine the effect of the core material, we present in Fig. 4 the critical loads for the case of graphite/epoxy faces and isotropic core material with varying modulus, E^c and Poisson's ratio $\nu_c=0.30$, as described by the ratio E_1^f/E^c between 100 and 5000. The results are produced for $f/h_{\text{tot}}=0.10$ and the same length and total thickness as before. We can see that all formulas converge for the lower ratios (stiffer cores) and basically the same general conclusions regarding the relative performance of each formula hold true.

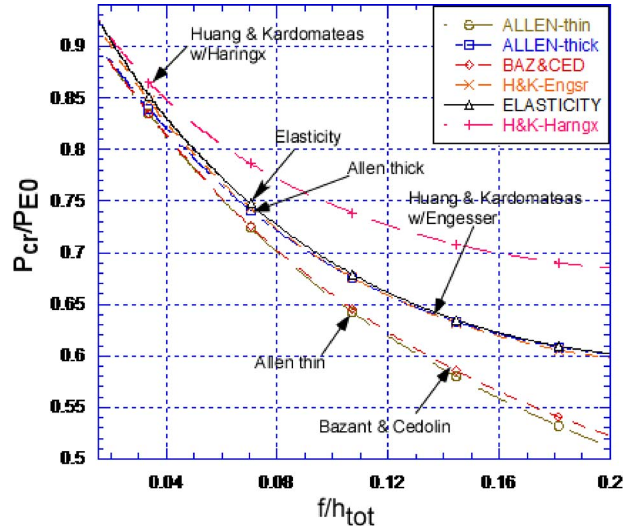

Fig. 3 Critical load for E-glass polyester faces and H100 cross-linked PVC foam core

One general observation is that the Haringx results stand out as being in much discrepancy with the elasticity results. This confirms the Bazant and Beghini's study [18] who showed that if a constant shear modulus G_c is used, then the correct theory is the Engesser-type theory and that the Haringx-type theory is usable only if the G_c of the core is considered to be a linear function of the axial stress in the skins.

There is another interesting difference between the Engesser and Haringx formulas, in that they have different limit for infinite slenderness, as discussed in Bazant and Cedolin [2]. Indeed, in the limit of zero rigidity (infinite slenderness), $P_{E0}\to 0$, both formulas give the same, zero critical load. However, in the limit of infinite rigidity (zero slenderness), that is $P_{E0}\to\infty$, the Haringx formula gives from Eq. (16), $P_{\text{cr}}^{\text{Harngx}}\to\infty$, while Engesser's formula with the shear correction from Ref. [4], gives from Eq. (15): $P_{\text{cr}}^{\text{Engess}}\to s^{-1}=AG_{\text{eq}}/\beta$, see Eq. (14*a*), which is a very different result.

Allen's thin formula gives $P_{\text{cr},A}^{\text{thin}}\to P_c$ with P_c given in Eq. (11*b*) and the Bazant and Cedolin formula $P_{\text{cr},BC}\to(GA)_0$ with $(GA)_0$

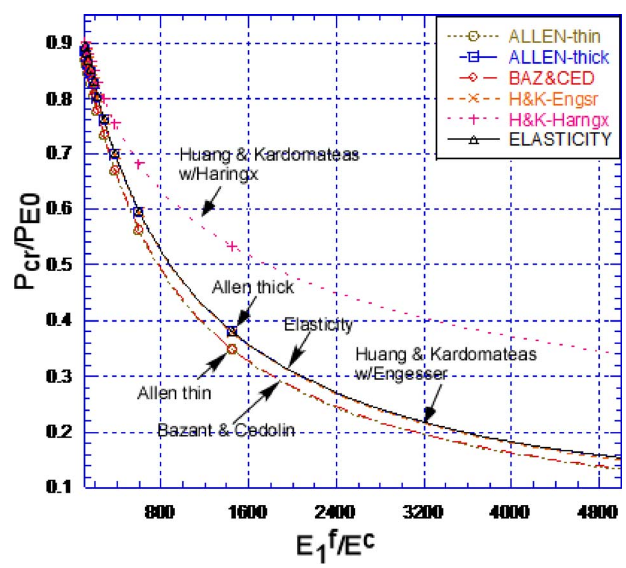

Fig. 4 The effect of relative core stiffness (case of $f/h_{\text{tot}}=0.10$)

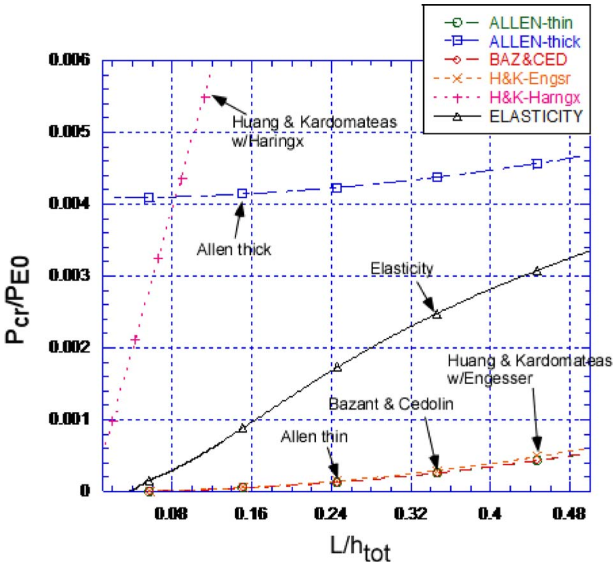

Fig. 5 The asymptotic behavior of the various sandwich buckling formulas (at the very small lengths)

given in Eq. (13*b*). However, Allen's thick formula (12) gives $P_{cr,A}^{thick} \to \infty$ when both $P_{E2} \to \infty$ and $P_{Ef} \to \infty$, and in that sense it has the same asymptotic behavior as the Haringx formula.

This very interesting behavior is illustrated in Fig. 5, which shows the case of E-glass/polyester faces and H100 cross-linked PVC foam core with $f/h_{tot}=0.10$, $h_{tot}=30$ mm, and a range of lengths L/h_{tot} between 0.01 and 0.50, i.e., for very small lengths. The elasticity solution is now between the Allen thick and the other Engesser-type formulas for this range of very small lengths, but tends rapidly to the latter ones as $L/h_{tot} \to 0$. This case of vanishing slenderness may not be of practical interest but it illustrates the complexities and differences among the different formulas.

4 Conclusions

An elasticity solution to the problem of global buckling of sandwich beams or wide sandwich panels subjected to axially compressive loading is presented. A symmetric section is considered with all constituent phases, i.e., the facings and the core, being orthotropic. The different global buckling formulas in the literature are compared with the elasticity predictions. The elasticity problem for buckling is formulated as an eigenboundary-value problem for differential equations, with the axial load being the eigenvalue. The complication in the sandwich construction arises due to the existence of additional "internal" conditions at the face-sheet/core interfaces. From the results it can be concluded that (1) Allen's thin and thick formulas, the Bazant and Cedolin formula and the Engesser formula with the Huang and Kardomateas shear correction are all conservative. (2) Allen's thick faces formula and the Engesser formula with the Huang and Kardomateas shear correction give predictions, which are very close to the elasticity solution and, in many cases, identical to the elasticity solution for the entire range of face thicknesses examined. (3) The Haringx formula, with the Huang and Kardomateas shear correction and the assumption of a constant core shear modulus, gives predictions that are nonconservative, and it is the most inaccurate, especially at the thicker faces. (4) Allen's thin formula and the Bazant and Cedolin formula give practically identical predictions (with the latter being slightly more accurate); they are both quite accurate for the thinner faces. (5) The transverse shear effect can be very large in these sandwich configurations with the critical load being just a fraction of the Euler load. The solution presented herein provides a means of accurately assessing the limitations of simplifying analyses and the accuracy of simple formulas in predicting global buckling in sandwich beams or wide panels.

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