AIAA JOURNAL
Vol. 50, No. 8, August 2012

Global Buckling of Sandwich Beams Based on the Extended High-Order Theory

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DOI: 10.2514/1.J051454

The focus of this paper is the application of the recently introduced extended high-order sandwich panel theory to the global buckling of a sandwich beam/wide panel. Three different solution approaches using the extended high-order sandwich panel theory are presented to investigate the effects of simplifying the loading case by applying loads just on the face sheets and by including or excluding nonlinear axial strains in the core. The results are also compared with results from a benchmark elasticity solution and, furthermore, from the simple sandwich buckling formula by the earlier extended high-order sandwich panel theory. It is found that all three theories are close to the elasticity solution for "soft" cores with $E_1^c/E_1^f < 0.001$. However, for "moderate" cores, i.e., with $E_1^c/E_1^f > 0.001$, the theories diverge and the extended high-order sandwich panel theory is the most accurate.

Nomenclature

a	=	length of the sandwich beam
$a_{11}^{t,b,c}$	=	axial compliance of the top face, bottom face, and core, respectively
$C_{ij}^{t,b,c}$	=	Stiffness constants of the top face, bottom face, and core, respectively
c	=	half-thickness of the core (total core thickness is $2c$)
$E_1^{t,b}$	=	axial extensional (Young's) modulus of the top and bottom face, respectively
E_3^c	=	transverse extensional modulus of the core
$f_{t,b}$	=	thickness of the top face and bottom face, respectively
G_{31}^c	=	shear modulus of the core
h_{tot}	=	total thickness of the sandwich beam
$M^{t,b,c}$	=	moment stress resultant about their own centroids per unit width of the top face, bottom face and core, respectively
$N^{t,b,c}$	=	axial stress resultant per unit width of the top face, bottom face and core, respectively
$\tilde{N}^{t,b}$	=	end axial force (at the ends $x = 0, a$) per unit width at the top and bottom faces, respectively
$\tilde{n}^c$	=	end axial force (at the ends $x = 0, a$) per unit width at the core
P_{E0}	=	Euler load
$u^{t,b,c}$	=	axial displacement (along x) of the top face, bottom face, and core, respectively
V^c	=	shear stress resultant per unit width of core
$w^{t,b,c}$	=	transverse displacement (along z) of the top face, bottom face, and core, respectively
ϕ_0^c	=	slope at the centroid of the core
"0"	=	refers to the middle surface (centroid), as a subscript

I. Introduction

IN THE recent literature, several high-order theories have been presented to model the behavior of sandwich composites. The higher-order sandwich panel theory (HSAPT) Frostig et al. [1] accounts for the transverse and shear stiffness of the core, but neglects the axial stiffness of the core and assumes a constant shear stress in the core, which is also one of the generalized coordinates of the theory. The extended high-order sandwich panel theory (EHSAPT) Phan et al. [2] accounts for the axial, transverse, and shear stiffness in the core and considers the rotation at the centroid of the core as one of the generalized coordinates. These theories were compared in [2] to the elasticity solution for the case of a simply supported beam subjected to a distributed sinusoidal transverse load; it was found that while the HSAPT captures the stresses and strains in the core very well for very soft cores, the EHSAPT is able to do the same but for a wider range of core materials, i.e., not just soft cores.

In this paper, we are interested in how the theories perform with respect to elasticity, in predicting the global buckling of a sandwich beam/wide plate under compressive axial loading. The buckling behavior of sandwich composite beams using the HSAPT was studied by Frostig and Baruch [3]. Furthermore, a few global buckling formulas for sandwich construction can be found in the literature. These formulas are reviewed in detail in a whole chapter devoted to buckling in the recent book by Carlsson and Kardomateas [4]. As benchmark, an elasticity solution for the global buckling of a sandwich beam/wide plate was presented by Kardomateas [5]. In this paper, two formulas were found to be the most accurate (by comparing to elasticity). These were 1) the formula derived by Allen [6] for thick faces (note that there also exist a corresponding formula by Allen for thin faces, but this was less accurate), and 2) the Engesser's [7] critical load formula where the shear correction factor used is the one derived for sandwich sections by Huang and Kardomateas [8]. The latter shear correction formula is not exclusively based on the shear modulus of the core, but instead includes the shear modulus of the faces and the extensional modulus of the core, therefore, it can account for sandwich constructions with stiffer cores and/or more compliant faces. In this paper we shall use the Allen's [6] thick faces formula as a representative of the simple formulas to compare with the high-order theory results. It will also be proven that this formula would be the direct result of the HSAPT for the case of an incompressible core.

Thus, in this present paper, the buckling behavior for a general asymmetric sandwich beam/wide plate with different face sheet materials and face sheet thicknesses is presented. We have used the

Received 24 June 2011; revision received 23 December 2011; accepted for publication 20 January 2012.

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EHSAPT to solve for three cases: (a) axial load applied exclusively the face sheets and the geometric nonlinearities in the core are neglected (linear core); (b) uniform axial strain applied through the entire thickness and, again, linear core; and (c) uniform axial strain applied through the entire thickness but now the geometric nonlinearities in the core are included (nonlinear core). It will be shown that the critical load is nearly identical for cases (a) and (c) and a range of core materials and geometry but case (a) loading involves a simpler solution process. Moreover, this critical load is very close to the elasticity prediction. Therefore, this paper will show in detail the solution procedure for finding the critical load of the case (a) loading.

The format of this paper is the following: in Sec. II the formulation of the EHSAPT for the case of buckling in terms of stress resultants and generalized coordinates is presented; in Sec. III three solution approaches are outlined for the critical load for a simply supported sandwich beam with general asymmetric geometry, namely linear core and applied load on faces, linear core and applied uniform strain, and nonlinear core and applied uniform strain; in Sec. IV results are shown for a soft and a moderate core sandwich configurations with symmetric geometry. Section V concludes the paper.

II. EHSAPT in Terms of Stress Resultants

In Phan et al. [2] the EHSAPT was formulated but the governing equations were given in terms of displacements. In this section, we formulate the corresponding equations in terms of the applied axial loading and the stress resultants, which is appropriate for solving the buckling problem.

Figure 1 shows a sandwich panel of length a with a core of thickness $2c$ and top and bottom face sheet thicknesses f_t and f_b, respectively. A Cartesian coordinate system (x, y, z) is defined at the left end of the beam and its origin is placed at the middle of the core. Only loading in the x-z plane is considered to act on the beam, which solely causes displacements in the x and z directions, designated by u and w, respectively. The superscripts t, b, and c shall refer to the top face sheet, bottom face sheet, and core, respectively. The subscript 0 refers to the middle surface of the corresponding layer (top face, bottom face, or core). We should also note that in our formulation the rigidities and all applied loadings are per unit width.

The displacement field of the top and bottom face sheets are assumed to satisfy the Euler–Bernoulli assumptions: Therefore, the displacement field for the top face sheet, $c \le z \le c + f_t$, is:

$$w^t(x,z) = w_0^t(x); \qquad u^t(x,z) = u_0^t(x) - \left(z - c - \frac{f_t}{2}\right) w_{0,x}^t(x) \tag{1a}$$

and for the bottom face sheet, $-(c + f_b) \le z \le -c$:

$$w^b(x,z) = w_0^b(x); \qquad u^b(x,z) = u_0^b(x) - \left(z + c + \frac{f_b}{2}\right) w_{0,x}^b(x) \tag{1b}$$

The only nonzero strain in the faces is the axial strain, which for buckling should be in its general nonlinear form:

$$\epsilon_{xx}^{t,b}(x,z) = u_{,x}^{t,b}(x,z) + \frac{1}{2}[w_{0,x}^{t,b}(x)]^2 \tag{1c}$$

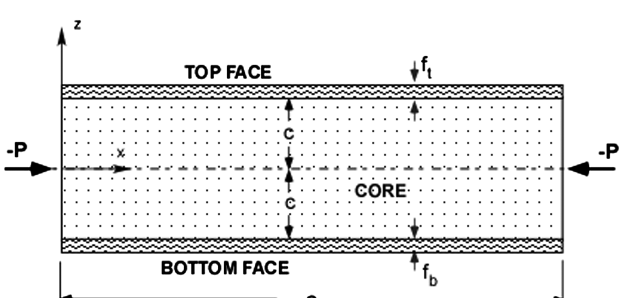

Fig. 1 Definition of the sandwich configuration.

While the face sheets can change their length only longitudinally, the core can change its height and length. For the EHSAPT, the transverse displacement in the core is (Phan et al. [2])

$$w^c(x,z) = \left(-\frac{z}{2c} + \frac{z^2}{2c^2}\right) w_0^b(x) + \left(1 - \frac{z^2}{c^2}\right) w_0^c(x) + \left(\frac{z}{2c} + \frac{z^2}{2c^2}\right) w_0^t(x) \tag{2a}$$

and the axial displacement in the core

$$\begin{aligned} u^c(x,z) = {} & z\left(1 - \frac{z^2}{c^2}\right)\phi_0^c(x) + \frac{z^2}{2c^2}\left(1 - \frac{z}{c}\right) u_0^b(x) \\ & + \left(1 - \frac{z^2}{c^2}\right) u_0^c(x) + \frac{z^2}{2c^2}\left(1 + \frac{z}{c}\right) u_0^t(x) \\ & + \frac{f_b z^2}{4c^2}\left(-1 + \frac{z}{c}\right) w_{0,x}^b(x) + \frac{f_t z^2}{4c^2}\left(1 + \frac{z}{c}\right) w_{0,x}^t(x) \end{aligned} \tag{2b}$$

where w_0^c and u_0^c are the transverse and in-plane displacements, respectively, and ϕ_0^c is the slope at the centroid of the core. This displacement field satisfies all displacement continuity conditions at the face/core interfaces (Phan et al. [2]).

Therefore, this theory is in terms of seven generalized coordinates (unknown functions of x): two for the top face sheet, w_0^t, u_0^t, two for the bottom face sheet, w_0^b, u_0^b, and three for the core, w_0^c, u_0^c, and ϕ_0^c.

The core strains can be obtained from the displacements using the strain-displacement relations.

In the following we use the notation $1 \equiv x$, $3 \equiv z$, and $55 \equiv xz$. We assume orthotropic face sheets, thus the nonzero stresses for the faces are

$$\sigma_{xx}^{t,b} = C_{11}^{t,b}\epsilon_{xx}^{t,b}; \qquad \sigma_{zz}^{t,b} = C_{13}^{t,b}\epsilon_{xx}^{t,b} \tag{3a}$$

Notice that the $\sigma_{zz}^{t,b}$ does not ultimately enter into the variational equation because the corresponding strain $\epsilon_{zz}^{t,b}$ is zero. We also assume an orthotropic core with stress-strain relations:

$$\begin{bmatrix} \sigma_{xx}^c \\ \sigma_{zz}^c \\ \tau_{xz}^c \end{bmatrix} = \begin{bmatrix} C_{11}^c & C_{13}^c & 0 \\ C_{13}^c & C_{33}^c & 0 \\ 0 & 0 & C_{55}^c \end{bmatrix} \begin{bmatrix} \epsilon_{xx}^c \\ \epsilon_{zz}^c \\ \gamma_{xz}^c \end{bmatrix} \tag{3b}$$

where $C_{ij}^{t,b,c}$, $ij = 11, 13, 33, 55$ are the corresponding stiffness constants.

The governing equations and boundary conditions are derived from the variational principle:

$$\delta(U + V) = 0 \tag{4}$$

where U is the strain energy of the sandwich beam, and V is the potential due to the applied loading. The first variation of the strain energy per unit width of the sandwich beam is

$$\begin{aligned} \delta U = \int_0^a \bigg[& \int_{-c+f_b}^{-c} \sigma_{xx}^b \delta\epsilon_{xx}^b \,\mathrm{d}z + \int_{-c}^{c} (\sigma_{xx}^c \delta\epsilon_{xx}^c + \sigma_{zz}^c \delta\epsilon_{zz}^c + \tau_{zx}^c \delta\gamma_{zx}^c) \,\mathrm{d}z \\ & + \int_c^{c+f_t} \sigma_{xx}^t \delta\epsilon_{xx}^t \,\mathrm{d}z \bigg] \mathrm{d}x \end{aligned} \tag{5a}$$

and the first variation of the external potential per unit width is

$$\delta V = -\tilde{N}^t \delta u_0^t]_{x=0}^a - \tilde{N}^b \delta u_0^b]_{x=0}^a - \left(\int_{-c}^{c} \tilde{n}^c(z) \delta u^c \,\mathrm{d}z\right)\bigg]_{x=0}^a \tag{5b}$$

where $\tilde{N}^t$ and $\tilde{N}^b$ are the concentrated axial forces (per unit width) on the top and bottom face sheets, respectively, and $\tilde{n}^c(z)$ is the distributed axial force (per unit width) applied to the core (at the ends $x = 0$ and $x = a$). In the following, we assume that $\tilde{n}^c$ is constant. Therefore, using Eq. (2b) gives

$$\int_{-c}^{c} \tilde{n}^c \delta u^c \,\mathrm{d}z = \tilde{n}^c c\left[\frac{1}{3}(\delta u_0^b + \delta u_0^t) + \frac{4}{3}(\delta u_0^c) - \frac{f_b}{6}\delta w_{0,x}^b + \frac{f_t}{6}\delta w_{0,x}^t\right] \tag{5c}$$

For the sandwich panels made out of orthotropic materials, we can substitute the stresses in terms of the strains from the constitutive relations, Eq. (3), and then the strains in terms of the displacements and the displacement profiles, Eqs. (1) and (2), and finally apply the variational principle, Eqs. (4) and (5) to obtain the governing equations and associated boundary conditions. These are listed in Appendix A.

We have also made use of the definitions of the axial stress resultants of the top face, bottom face, and core per unit width, respectively, $N^{t,b,c}$, and these are defined as

$$N^t(x) = \int_{c}^{c+f_t} \sigma_{xx}^t \,\mathrm{d}z = C_{11}^t f_t \epsilon_{xx}^t$$
$$N_x^b(x) = \int_{-c-f_b}^{-c} \sigma_{xx}^b \,\mathrm{d}z = C_{11}^b f_b \epsilon_{xx}^b \tag{6a}$$

$$N_x^c(x) = \int_{-c}^{c} \sigma_{xx}^c \,\mathrm{d}z = C_{13}^c(w_0^t - w_0^b) + \frac{cC_{11}^c}{3}\left(u_{0,x}^b + 4u_{0,x}^c + u_{0,x}^t + \frac{f_t}{2}w_{0,xx}^t - \frac{f_b}{2}w_{0,xx}^b\right) \tag{6b}$$

where the nonlinear axial strains $\epsilon_{xx}^{t,b}$ are given in Eq. (1c). The expression (6b), however, is based on the assumption of linear strains for the core. When nonlinear strains are included in the core, the following term is added to Eq. (6b):

$$N_{\mathrm{NL}}^c = \frac{c}{15}C_{11}^c[2w_{0,x}^b w_{0,x}^c - w_{0,x}^b w_{0,x}^t + 2(w_{0,x}^b)^2 + 2w_{0,x}^c w_{0,x}^t + 8(w_{0,x}^c)^2 + 2(w_{0,x}^t)^2] \tag{6c}$$

Also, $M^{t,b,c}$ are the moment stress resultants of the top face, bottom face, and core about their own centroids per unit width, respectively, and are defined as

$$M^t(x) = -\int_{c}^{c+f_t} \sigma_{xx}^t\left(z - c - \frac{f_t}{2}\right)\mathrm{d}z = C_{11}^t \frac{f_t^3}{12} w_{0,xx}^t \tag{6d}$$

$$M^b(x) = -\int_{-c-f_b}^{-c} \sigma_{xx}^b\left(z + c + \frac{f_b}{2}\right)\mathrm{d}z = C_{11}^b \frac{f_b^3}{12} w_{0,xx}^b \tag{6e}$$

$$M^c(x) = -\int_{-c}^{c} \sigma_{xx}^c z\,\mathrm{d}z = -\frac{2cC_{13}^c}{3}(w_0^b - 2w_0^c + w_0^t) - \frac{c^2 C_{11}^c}{30}[8c\phi_{0,x}^c + 6(u_{0,x}^t - u_{0,x}^b) + 3(f_b w_{0,xx}^b + f_t w_{0,xx}^t)] \tag{6f}$$

Again, the expression (6f) is based on the assumption of linear strains for the core. When nonlinear strains are included in the core, the following term is added to Eq. (6f):

$$M_{\mathrm{NL}}^c = \frac{c^2}{30}C_{11}^c(w_{0,x}^b - w_{0,x}^t)(3w_{0,x}^b + 4w_{0,x}^c + 3w_{0,x}^t) \tag{6g}$$

Finally, V^c is the shear stress resultant of the core and is defined as

$$V^c = \int_{-c}^{c} \tau_{zx}^c \,\mathrm{d}z = C_{55}^c\left[(u_0^t - u_0^b) + \frac{c}{3}(w_{0,x}^b + w_{0,x}^t) + \frac{1}{2}(f_b w_{0,x}^b + f_t w_{0,x}^t) + 8cw_{0,x}^c\right] \tag{6h}$$

III. Three Solution Approaches for the Global Buckling of a Simply Supported Sandwich Beam

An elasticity solution exists for the critical load of a sandwich beam undergoing compressive loading (uniform axial strain loading) Kardomateas [5]. The goal of this paper is to determine the critical load for a simply supported sandwich beam using the EHSAPT, and to compare its solution, as well as other sandwich panel theories, to the elasticity solution. Three solution approaches were pursued using the EHSAPT: 1) loading applied on the face sheets and the geometric nonlinearities for the core are neglected, i.e., linear strains assumed for the core, 2) uniform strain loading through the entire thickness and, again, linear strains assumed for the core, and 3) uniform strain loading through the entire thickness and the geometric nonlinearities for the core are now included, i.e., nonlinear strains assumed for the core. The following sections outline the solution procedures for determining the critical load using these approaches for a sandwich beam/wide plate of general asymmetric geometry and material configuration.

A. Case (a): Loading on the Face Sheets with Linear Axial Strains in the Core

In this case, $\tilde{N}^t$ and $\tilde{N}^b$ are applied on the top and bottom face sheets, respectively, such that the axial strains are equal on the top and bottom faces, and the net axial loading per unit width on each side of the beam is $-P$.

Imposing the condition of the same axial strain and the condition that the sum of the loads per unit width on the top and bottom face sheets equals $-P$ provides two equations for the unknown axial loads, which are found to be

$$\tilde{N}_p^t = -\kappa^t P; \qquad \tilde{N}_p^b = -\kappa^b P \tag{7a}$$

where

$$\kappa^t = \frac{a_{11}^b f_t}{a_{11}^t f_b + a_{11}^b f_t}, \qquad \kappa^b = \frac{a_{11}^t f_b}{a_{11}^t f_b + a_{11}^b f_t} \tag{7b}$$

and $a_{11}^i = 1/E_1^i$ is the compliance constant of the corresponding face sheet ($i = t, b$).

The critical load for an asymmetric geometry and material configuration can be determined using the perturbation approach:

$$N^i(x) = N_p^i(x) + \xi N_s^i(x) \qquad (i = t, b, c) \tag{8a}$$

$$u_0^i(x) = u_p(x) + \xi u_{0s}^i(x) \qquad (i = t, b) \tag{8b}$$

$$w_0^i(x) = \xi w_{0s}^i(x), \qquad M^i(x) = \xi M_s^i(x), \qquad (i = t, b, c) \tag{8c}$$

$$u_0^c(x) = \xi u_{0s}^c(x), \qquad \phi_0^c(x) = \xi\phi_{0s}^c(x), \qquad V^c(x) = \xi V_s^c(x) \tag{8d}$$

The additional subscript p stands for primary, or the prebuckled state, while the additional subscript s stands for secondary, or the perturbed state, and ξ is an infinitesimally small quantity.

By considering Eq. (7) and substituting the displacements, the stress resultants on the face sheets, Eq. (8a), can be written as

$$N^i = -\kappa^i P + \xi(C_{11}^i f_i u_{0s,x}^i) \qquad (i = t, b) \tag{9}$$

We assume global buckling modes for the simply supported beam, as follows:

$$u_{0s}^{t,b,c} = U_0^{t,b,c}\cos\frac{\pi x}{a}; \qquad \phi_{0s}^c = \Phi_0^c \cos\frac{\pi x}{a} \tag{10a}$$

$$w_{0s}^{t,b,c} = W_0^{t,b,c}\sin\frac{n\pi x}{a} \tag{10b}$$

Substitution of the secondary terms into the buckled state equations leads to seven algebraic equations, and these are

$$\delta u_0^t\colon U_0^b\left(-\frac{7}{30c}C_{55}^c+\frac{c\pi^2}{35a^2}C_{11}^c\right)+U_{0n}^c\left(-\frac{4}{3c}C_{55}^c+\frac{2c\pi^2}{15a^2}C_{11}^c\right)+\Phi_0^c\left(-\frac{4}{5}C_{55}^c+\frac{2c^2\pi^2}{35a^2}C_{11}^c\right)+U_0^t\left(\frac{47}{30c}C_{55}^c+\frac{6c\pi^2}{35a^2}C_{11}^c+\frac{f_t\pi^2}{a^2}C_{11}^t\right)+W_0^b\left(-\frac{cf_b\pi^3}{70a^3}C_{11}^c-\eta_2^b\frac{\pi}{a}\right)+W_0^c\frac{\pi\beta_1}{a}+W_0^t\left(\frac{3cf_t\pi^3}{35a^3}C_{11}^c+\eta_3^t\frac{\pi}{a}\right)=0 \tag{11a}$$

$$\delta w_0^t\colon U_0^b\left(\frac{cf_t\pi^3}{70a^3}C_{11}^c+\eta_2^t\frac{\pi}{a}\right)+U_0^c\left(\frac{cf_t\pi^3}{15a^3}C_{11}^c-\eta_6^t\frac{\pi}{a}\right)+\Phi_0^c\left(\frac{c^2f_t\pi^3}{35a^3}C_{11}^c-\eta_4^t\frac{\pi}{a}\right)+U_0^t\left(\frac{3cf_t\pi^3}{35a^3}C_{11}^c+\frac{\pi}{a}\eta_3^t\right)+W_0^b\left(\frac{C_{33}^c}{6c}-\frac{cf_bf_t\pi^4}{140a^4}C_{11}^c-\beta_2\frac{\pi^2}{a^2}\right)+W_0^c\left(-\frac{4}{3c}C_{33}^c-\eta_7^t\frac{\pi^2}{a^2}\right)+W_0^t\left(-\kappa^tP\frac{\pi^2}{a^2}+\frac{7}{6c}C_{33}^c+\frac{3cf_t^2\pi^4}{70a^4}C_{11}^c+\frac{f_t^3\pi^4}{12a^4}C_{11}^t-\eta_8^t\frac{\pi^2}{a^2}\right)=0 \tag{11b}$$

$$\delta u_0^c\colon U_0^b\left(-\frac{4}{3c}C_{55}^c+\frac{2c\pi^2}{15a^2}C_{11}^c\right)+U_0^c\left(\frac{8}{3c}C_{55}^c+\frac{16c\pi^2}{15a^2}C_{11}^c\right)+U_0^t\left(-\frac{4}{3c}C_{55}^c+\frac{2c\pi^2}{15a^2}C_{11}^c\right)+W_0^b\left(\frac{\eta_6^b\pi}{a}-\frac{cf_b\pi^3}{15a^3}C_{11}^c\right)+W_0^t\left(\frac{cf_t\pi^3}{15a^3}C_{11}^c-\frac{\eta_6^t\pi}{a}\right)=0 \tag{11c}$$

$$\delta\phi_0^c\colon U_0^b\left(\frac{4}{5}C_{55}^c-\frac{2c^2\pi^2}{35a^2}C_{11}^c\right)+\Phi_0^c\left(\frac{8c}{5}C_{55}^c+\frac{16c^3\pi^2}{105a^2}C_{11}^c\right)+U_0^t\left(-\frac{4}{5}C_{55}^c+\frac{2c^2\pi^2}{35a^2}C_{11}^c\right)+W_0^b\left(\frac{c^2f_b\pi^3}{35a^3}C_{11}^c-\frac{\eta_4^b\pi}{a}\right)+W_{0n}^c\left(\frac{4c\beta_1\pi}{3a}\right)+W_0^t\left(\frac{c^2f_t\pi^3}{35a^3}C_{11}^c-\frac{\eta_4^t\pi}{a}\right)=0 \tag{11d}$$

$$\delta w_0^c\colon -U_0^b\left(\frac{\beta_1\pi}{a}\right)+\Phi_0^c\left(\frac{4c\beta_1\pi}{3a}\right)+U_0^t\left(\frac{\beta_1\pi}{a}\right)-W_0^b\left(\frac{4}{3c}C_{33}^c+\frac{\eta_7^b\pi^2}{a^2}\right)+W_0^c\left(\frac{8}{3c}C_{33}^c+\frac{16c\pi^2}{15a^2}C_{55}^c\right)-W_0^t\left(\frac{4}{3c}C_{33}^c+\frac{\eta_7^t\pi^2}{a^2}\right)=0 \tag{11e}$$

$$\delta u_0^b\colon U_0^b\left(\frac{47}{30c}C_{55}^c+\frac{6c\pi^2}{35a^2}C_{11}^c+f_b\frac{\pi^2}{a^2}C_{11}^b\right)+U_0^c\left(-\frac{4}{3c}C_{55}^c+\frac{2c\pi^2}{15a^2}C_{11}^c\right)+\Phi_0^c\left(\frac{4}{5}C_{55}^c-\frac{2c^2\pi^2}{35a^2}C_{11}^c\right)+U_0^t\left(-\frac{7}{30c}C_{55}^c+\frac{c\pi^2}{35a^2}C_{11}^c\right)+W_0^b\left(-\frac{3cf_b\pi^3}{35a^3}C_{11}^c-\eta_3^b\frac{\pi}{a}\right)-W_0^c\left(\frac{\pi}{a}\beta_1\right)+W_0^t\left(\frac{cf_t\pi^3}{70a^3}C_{11}^c+\eta_2^t\frac{\pi}{a}\right)=0 \tag{11f}$$

$$\delta w_0^b\colon U_0^b\left(-\frac{3cf_b\pi^3}{35a^3}C_{11}^c-\eta_3^b\frac{\pi}{a}\right)+U_0^c\left(-\frac{cf_b\pi^3}{15a^3}C_{11}^c-\eta_6^b\frac{\pi}{a}\right)+\Phi_{0n}^c\left(\frac{c^2f_b\pi^3}{35a^3}C_{11}^c-\eta_4^b\frac{\pi}{a}\right)+U_0^t\left(-\frac{cf_b\pi^3}{70a^3}C_{11}^c-\eta_2^b\frac{\pi}{a}\right)+W_0^b\left(-\kappa^bP\frac{\pi^2}{a^2}+\frac{7}{6c}C_{33}^c+\frac{3cf_b^2\pi^4}{70a^4}C_{11}^c+\frac{f_b^3\pi^4}{12a^4}C_{11}^b-\eta_8^b\frac{\pi^2}{a^2}\right)+W_0^c\left(-\frac{4}{3c}C_{33}^c-\eta_7^b\frac{\pi^2}{a^2}\right)+W_0^t\left(\frac{C_{33}^c}{6c}-\frac{cf_bf_t\pi^4}{140a^4}C_{11}^c-\beta_2\frac{\pi^2}{a^2}\right)=0 \tag{11g}$$

Notice that the loading P (eigenvalue) appears in the $\delta w_0 t$ Eq. (11b) in the term W_0^t and in the δw_0^b Eq. (11g) in the term W_0^b.

The equations can be cast in matrix form:

$$\left\{[K_{LC}]-\frac{\pi^2}{a^2}\lfloor G_a\rfloor[I]\right\}\{U\}=\{0\} \tag{12}$$

$[K_{LC}]$ is a 7×7 matrix involving material stiffnesses and sandwich dimensions, and each element is given in Appendix B. The subscript LC denotes that the sandwich system has linear strains in the core. Later in the paper another matrix, the K_{NLC} will represent additional terms that account for nonlinear axial strains in the core. The loading vector is represented by $\lfloor G_a\rfloor=\lfloor 0,0,0,0,\kappa^bP,0,\kappa^tP\rfloor$, if the equations of the system are written in the order of Eqs. (11a–11g), respectively. Seven unknown displacement amplitudes make up the vector $\{U\}=\lfloor U_0^b,U_0^c,\Phi_0^c,U_0^t,W_0^b,W_0^c,W_0^t\rfloor^T$. The critical load is determined by finding the value of P for which the system has a nontrivial solution, or finding P by zeroing the determinant:

$$\det\left\{[K_{LC}]-\frac{\pi^2}{a^2}\lfloor G_a\rfloor[I]\right\}=0 \tag{13}$$

B. Case (b): Uniform Strain Loading with Linear Axial Strains in the Core

In this case, both face sheets and the core have the same axial strain $\epsilon_{xxp}^i=\sigma_{xxp}^i/C_{11}^i$ for $i=t,b,c$, and imposing that the net stress resultant per unit width at each end is $-P$, gives

$$\tilde N^t=-\kappa^tP;\qquad \tilde N^b=-\kappa^bP;\qquad \tilde n^c=-\kappa^c\frac{P}{2c} \tag{14a}$$

where

$$\kappa^t=\frac{a_{11}^ba_{11}^cf_t}{a_{11}^ba_{11}^t2c+a_{11}^ba_{11}^cf_t+a_{11}^ta_{11}^cf_b},\qquad \kappa^b=\frac{a_{11}^ta_{11}^cf_b}{a_{11}^ba_{11}^t2c+a_{11}^ba_{11}^cf_t+a_{11}^ta_{11}^cf_b} \tag{14b}$$

$$\kappa^c=\frac{a_{11}^ta_{11}^b2c}{a_{11}^ba_{11}^t2c+a_{11}^ba_{11}^cf_t+a_{11}^ta_{11}^cf_b} \tag{14c}$$

and $a_{11}^i=1/E_1^i$ is the compliance of the top or bottom face or core ($i=t,b,c$).

When a uniform strain exists in the core, the face sheets have a nonzero transverse displacement at the primary state, which is due to the Poisson's effect on the core during compression (as opposed to the previous case). Thus, the top and bottom face sheets have primary state transverse displacements that are equal, yet opposite in direction, i.e., w_p and $-w_p$, respectively; furthermore they are constant along x. Moreover, when the loading is uniform strain, the axial displacement at the primary state in the face sheets and the core is the same, denoted by u_p. Therefore, in this case the displacements in the perturbation approach are

$$u_0^i(x)=u_p(x)+\xi u_{0s}^i(x)\qquad (i=t,b,c) \tag{15a}$$

$$\phi_0^c(x) = \xi\phi_{0s}^c(x) \tag{15b}$$

$$w_0^t(x) = w_p + \xi w_{0s}^t(x) \tag{15c}$$

$$w_0^c(x) = \xi w_{0s}^c(x) \tag{15d}$$

$$w_0^b(x) = -w_p + \xi w_{0s}^b(x) \tag{15e}$$

Since the axial stresses at the primary state are

$$\sigma_{xxp}^i = C_{11}^i u_{p,x}, \qquad i = t, b \quad \text{and} \quad \sigma_{xxp}^c = C_{11}^c u_{p,x} + C_{13}^c \frac{w_p}{c} \tag{16a}$$

the following relations hold true at the primary state:

$$C_{11}^t f^t u_{p,x} = -\kappa^t P; \qquad C_{11}^b f^b u_{p,x} = -\kappa^b P$$
$$2C_{13}^c w_p + 2cC_{11}^c u_{p,x} = -\kappa^c P \tag{16b}$$

These relationships are also confirmed by solving the prebuckling state equations.

Substituting these displacements into the EHSAPT governing equations again, leads to the same system of equations as for case (a), but this time the $\kappa^{t,b}$ are given by Eq. (14b) and include the contribution of the core.

Therefore, the critical load can be determined by solving the buckled state Eq. (11), which can be set in the form

$$\left\{[K_{LC}] - \frac{\pi^2}{a^2}\lfloor G_b \rfloor [I]\right\}\{U\} = \{0\} \tag{17a}$$

where $[K_{LC}]$ is the same as that given in case (a) because the core still has linear axial strains, but now $\lfloor G_b \rfloor = \lfloor 0, 0, 0, 0, \kappa^b P, 0, \kappa^t P \rfloor$ where the κ^is are those given in this section, Eq. (14b). Note that even though there is a distributed axial load on the core, $\tilde{n}^c = -\kappa^c P/(2c)$, it is not present in the loading vector because nonlinear axial strains in the core were neglected. Again, the critical load is determined by solving the value of P, which gives a nontrivial solution to the buckled state equations, i.e., by zeroing the determinant:

$$\det\left\{[K_{LC}] - \frac{\pi^2}{a^2}\lfloor G_b \rfloor [I]\right\} = 0 \tag{17b}$$

C. Case (c): Uniform Strain Loading with Nonlinear Axial Strains in the Core

If the nonlinear axial strain in the core is considered, the axial load appears in the "buckled state" equations for the core as well. The nonlinear axial strain for the core is

$$\epsilon_{xx}^c(x, z) = u_{,x}^c(x, z) + \frac{1}{2}[w_{,x}^c(x, z)]^2 \tag{18}$$

and involves many terms if expanded out in terms of the generalized coordinates. We would like to note that the solution procedure for this case becomes quite complicated because both primary and secondary generalized coordinates appear in the buckled state set of equations. Later in the paper, the results section will show that the extra work required to solve both sets of equations did not make significant gains in accuracy. We shall summarize the solution procedure for this case, which involves the perturbation approach with the same assumed deformation as in case (b), and neglecting higher-order terms of ξ. The resulting buckled state equations is

$$\left\{[K_{LC}] + [K_{NLC}] - \frac{\pi^2}{a^2}\lfloor G_c \rfloor [I]\right\}\{U\} = \{0\} \tag{19a}$$

where $[K_{LC}]$ is the same as in cases (a) and (c), and $[K_{NLC}]$ contains the additional terms that account for nonlinear axial strains in the core:

$$K_{NLC} = \frac{\pi^2}{a^2}(cC_{11}^c u_{p,x} + C_{13}^c w_p)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{15} & \frac{2}{15} & -\frac{1}{15} \\ 0 & 0 & 0 & 0 & \frac{2}{15} & -\frac{14}{15} & \frac{2}{15} \\ 0 & 0 & 0 & 0 & -\frac{1}{15} & \frac{2}{15} & \frac{4}{15} \end{bmatrix} \tag{19b}$$

Note that $[K_{NLC}]$ depends on the primary state displacements, in particular, on $u_{p,x}$, (the x-derivative of the uniform axial displacement) and w_p (the uniform transverse displacement of the top face sheet due to the Poisson's effect in the axially loaded core), see case (b). The solution to the primary state displacements can be obtained by solving the prebuckled-state equations and are

$$u_p = -\frac{\kappa^t P}{C_{11}^t}x = -\frac{\kappa^b P}{C_{11}^b}x = -\frac{\kappa^c P}{C_{11}^c}x \tag{19c}$$

$$w_p = -\frac{cC_{13}^c}{C_{33}^c}u_{p,x} \tag{19d}$$

Now that the nonlinear axial strain of the core is considered, not only the loading on the face sheets but also the loading on the core appears in the force vector:

$$G_c = [0, 0, 0, 0, \kappa^t P, \kappa^c P, \kappa^b P]^t \tag{19e}$$

Again, the critical load is determined by solving the value of P, which gives a nontrivial solution to the buckled state equations, i.e., by zeroing the determinant:

$$\det\left\{[K_{LC}] + [K_{NLC}] - \frac{\pi^2}{a^2}\lfloor G_c \rfloor [I]\right\} = 0 \tag{19f}$$

Finally, it should be noted that the solution procedure results in a usual eigenvalue problem and subsequently zeroing out a determinant. For cases (a) and (b), this results in a characteristic equation that is quadratic in P, and for case (c) it results in a cubic equation in P.

IV. Results for a Symmetric Sandwich Configuration

We consider a sandwich configuration with symmetric geometry ($f_t = f_b = f$) and same face sheet material, leading to the loading condition $\tilde{N}^t = \tilde{N}^b = -P/2$ on the top and bottom face sheets for case (a) (loading on face sheets) and $\tilde{N}^t = \tilde{N}^b = -\kappa^f P$ and $\tilde{n}^c = -\kappa^c P/(2c)$ for cases (b) and (c) (uniform strain, linear, and nonlinear core, respectively), where the κs are given in Eqs. (14b) and (14c).

Two material system sandwich configurations will be considered: 1) carbon/epoxy unidirectional faces with hexagonal glass/phenolic honeycomb, which represents a sandwich with a very soft core (axial stiffness of core very small compared with that of the face sheets, $E_1^c/E_1^f < 0.001$) and 2) e-glass/polyester unidirectional faces with balsa wood core, which represents a sandwich with a moderate core (E_1^c/E_1^f on the order of 0.01). The moduli and Poisson's ratios for these materials are given in Table 1.

The total thickness is considered constant at $h_{\text{tot}} = 2f + 2c = 30$ mm, the length over total thickness $a/h_{\text{tot}} = 30$, and we examine a range of face thicknesses defined by the ratio of face sheet thickness over total thickness, f/h_{tot}, between 0.02 and 0.20.

The results will be produced for 1) the simple sandwich buckling formula by Allen [6] (thick faces version), which has been proven to be the most accurate among the simple sandwich buckling formulas, and which considers the transverse shear effects of the core, 2) the HSAPT, which takes into account the core's transverse shear and also the core's transverse compressibility effects but neglects the core's axial stiffness effects, and 3) the present EHSAPT, which takes into

Table 1 Material properties, with moduli data in GPa

Material constants	Graphite epoxy face	E-glass polyester face	Balsa wood core	Glass-phenolic honeycomb core
E_1	181.0	40.0	0.671	0.032
E_2	10.3	10.0	0.158	0.032
E_3	10.3	10.0	7.72	0.300
G_{23}	5.96	3.5	0.312	0.048
G_{31}	7.17	4.5	0.312	0.048
G_{12}	7.17	4.5	0.200	0.013
ν_{32}	0.40	0.40	0.49	0.25
ν_{31}	0.016	0.26	0.23	0.25
ν_{12}	0.277	0.065	0.66	0.25

account all three effects, namely the core's transverse shear and transverse compressibility effects as well as the core's axial stiffness effects. The benchmark values are the critical loads from the elasticity solution (Kardomateas [5]). The global critical loads for the Allen [6] thick faces formula and the HSAPT are given in Appendix C.

The results are normalized with the Euler load P_{E0}:

$$P_{E0} = \frac{\pi^2}{a^2} 2\left[E_f \frac{f^3}{12} + E_f f\left(\frac{f}{2} + c\right)^2 + E_c \frac{c^3}{3}\right] \tag{20a}$$

Figure 2 shows the comparison of the theories to elasticity for the case of soft core and length ratio $a/h_{\text{tot}} = 30$, as an error %, calculated as

$$\text{Error } \% = \frac{P_{\text{cr,theory}} - P_{\text{cr,elasticity}}}{P_{\text{cr,elasticity}}} * 100 \tag{20b}$$

We can see that the errors are of the order of $\pm 0.5\%$, very small, i.e., for this sandwich configuration all predictions are very close to the elasticity. For this material system, the Allen thick faces formula, the HSAPT and the EHSAPT cases (a) and (c) are all conservative and give practically identical results for the entire range of face sheet thicknesses. On the contrary, the EHSAPT case (b) approach is less conservative and even becomes nonconservative for the very small ratios of f/h_{tot}. It should also be noted that the critical loads are significantly less than the Euler critical load, thus showing the importance of transverse shear in sandwich structures.

Figure 3 shows that for the moderate core sandwich and length ratio $a/h_{\text{tot}} = 30$, the theories diverge as the face sheet thickness becomes thinner compared with the overall thickness of the sandwich cross section. Allen's formula and HSAPT give almost identical results and are the most conservative and can be as much as 15% below the elasticity value. The EHSAPT cases (a) and (c) are the most accurate, within 1% of the elasticity value, and on the conservative side. The EHSAPT case (b) is quite nonconservative, and can be as much as 40% above the elasticity value, i.e., it is the most inaccurate. This result shows the importance of including the nonlinear axial strain in the core for the actual uniform strain loading solution. However, it is also remarkable that the simplified approach of case (a) is identical to the most complex EHSAPT approach that has been considered, case (c).

Figures 4a and 4b show the effect of length for the moderate core configuration, i.e., results for $a/h_{\text{tot}} = 20$ and 10, respectively. For these shorter beam configurations, the EHSAPT cases (a) and (c) are consistently close to the elasticity solution for the entire range of the face sheet thicknesses, and stay within about 1% error, i.e., the most accurate. The other theories all diverge from elasticity for small f/h_{tot}. The Allen [6] thick faces formula and the HSAPT are again identical and most conservative, and the EHSAPT case (b) is again nonconservative and most inaccurate. Moreover, as the beam length decreases, in all cases the predictions become somewhat less conservative. For the soft core configuration the EHSAPT, HSAPT, and Allen formula all predict practically the same critical load for all three length cases examined.

Thus, we can conclude that, when we deal with the critical load of sandwich structures, the present EHSAPT produces results very close to the elasticity for a wide range of cores, as opposed to the other theories of formulas, which seem to be accurate only when the core is very soft. It is important, however, how this theory is implemented, in the sense that this high accuracy is obtained for either case (a) or case (c), but not for case (b).

An argument that explains the apparent inaccuracy of case (b) can be made as follows: In case (b), loads are distributed to both the faces and the core, but the load vector G_b has only the stress resultants from the faces and does not have a contribution from the core, thus the loads of G_b would sum to a value less than the applied load P. On the contrary, in case (c), the load vector G_c has stress resultants from both faces and the core (because now nonlinear strains are considered in

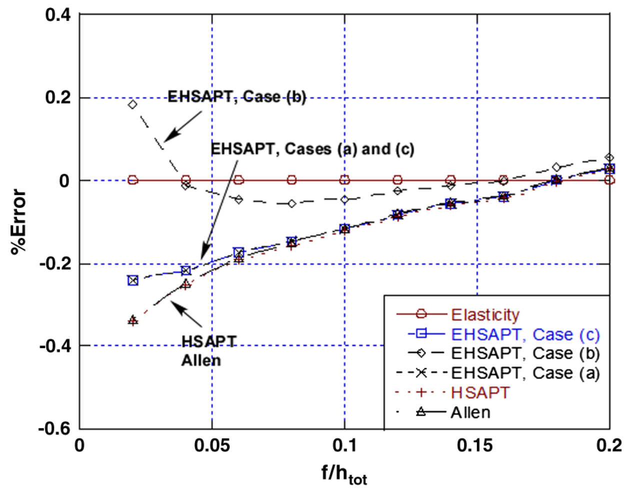

Fig. 2 Percent error (from elasticity) for the critical load of the various theories for the case of soft core and length $a = 30h_{\text{tot}}$.

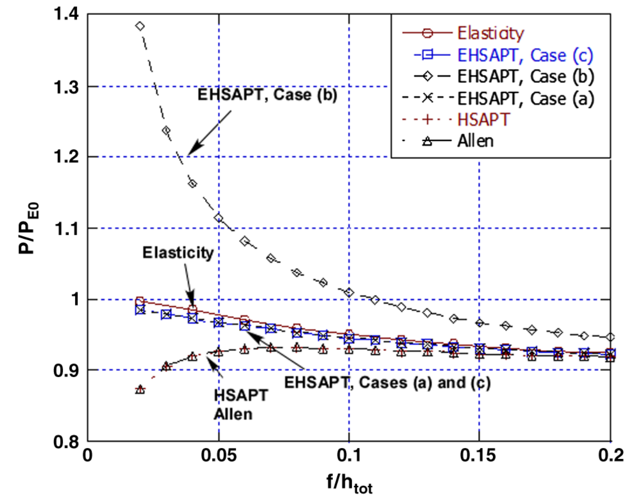

Fig. 3 Critical load (normalized with the Euler load) for the various theories for the case of moderate core and length $a = 30h_{\text{tot}}$.

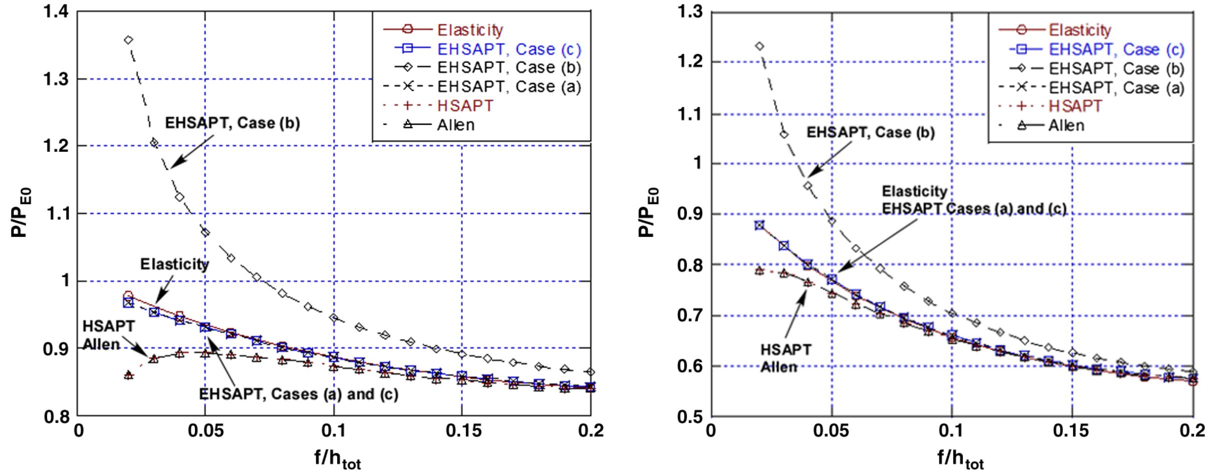

Fig. 4 Critical load (normalized with the Euler load) for the various theories for the case of moderate core and a) length $a = 20h_{\mathrm{tot}}$ and b) length $a = 10h_{\mathrm{tot}}$.

the core) and these stress resultants would sum to P. In case (a), loads were only applied to the faces, so although the load vector G_a contains only the stress resultants in the faces, these would again sum up to P.

Finally, a common observation in all these plots is that the Allen [6] thick faces formula and the HSAPT give almost identical predictions. In fact, it can be proven that the HSAPT critical load resuces to that of the Allen thick faces formula if only transverse shear effects are included (i.e., the HSAPT applied without the core's transverse compressibility effects). This derivation is outlined in Appendix C.

V. Conclusions

The governing equations for the EHSAPT, which takes into account the axial, transverse, and shear stiffness of the core were presented. The buckling equations were subsequently derived. Three different solution procedures were formulated using EHSAPT: case (a), in which the compressive loading is applied on the faces and the core strains are assumed linear; case (b), in which a uniform compressive strain is applied and the core strains are assumed linear; and case (c), in which there is again a uniform compressive strain through the thickness but now the core strains are assumed nonlinear. The perturbation approach was used for determining the critical load of a simply supported sandwich beam with a general asymmetric geometry and different face sheet materials. A simple case study of a symmetric geometry and same face sheet material sandwich configuration was used to compare the predictions of the simple sandwich buckling formula by Allen [6] (thick faces version), the HSAPT and the present EHSAPT. Sandwich configurations with a soft core and a moderate core were analyzed.

The following conclusions are drawn by comparing the critical loads from these different theories with the benchmark critical load predicted by elasticity:

1) The EHSAPT cases (a) and (c) are nearly identical for both the soft core and moderate core configurations.

2) For the soft core sandwich configurations ($E_1^c/E_1^f \leq 0.001$) all three theories (Allen [6] thick faces formula, HSAPT, and EHSAPT) predict the critical load within 1% of the critical load from elasticity.

3) For the moderate core sandwich configurations ($E_1^c/E_1^f \geq 0.01$), the EHSAPT cases (a) and (c) are consistently within about 1% of the critical load from elasticity. On the contrary, the Allen [6] thick faces formula, the HSAPT, and the EHSAPT case (b) diverge from elasticity for smaller f/h_{tot}. But the Allen thick faces formula and the HSAPT diverge to more conservative values whereas the EHSAPT Case (b) diverges to more nonconservative values for the smaller values of the ratio f/h_{tot} (i.e., thinner faces). The latter is also the least accurate and can be in significant error for these small f/h_{tot} ratios.

4) It is important how the compressive loading is applied, in the sense that loading the faces and assuming a linear core gives almost identical results to the most complex case of uniform strain loading and a nonlinear core. However, assuming uniform strain loading with a linear core gave quite inaccurate results for moderately stiff cores.

Appendix A: Governing Equations and Boundary Conditions

The variational principle (5) leads to the following governing ordinary differential equations and associated boundary conditions:

Top face sheet:

$$\begin{aligned}\delta u_0^t\colon\ & -N^t_{,x} - \left(\frac{4}{5}C_{55}^c + \frac{2c^2}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right)\phi_0^c \\ & - \left(\frac{7}{30c}C_{55}^c + \frac{c}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^b - \left(\frac{4}{3c}C_{55}^c + \frac{2c}{15}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^c \\ & + \left(\frac{47}{30c}C_{55}^c - \frac{6c}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^t - \left(\eta_2^b\frac{\partial}{\partial x} - \frac{cf_b}{70}C_{11}^c\frac{\partial^3}{\partial x^3}\right)w_0^b \\ & + \beta_1\frac{\partial w_0^c}{\partial x} + \left(\eta_3^t\frac{\partial}{\partial x} - \frac{3cf_t}{35}C_{11}^c\frac{\partial^3}{\partial x^3}\right)w_0^t = 0\end{aligned} \tag{A1a}$$

and

$$\begin{aligned}\delta w_0^t\colon\ & -N^t_{,x}w^t_{0,x} + M^t_{,xx} - N^t w^t_{0,xx} + \left(\eta_4^t\frac{\partial}{\partial x} + \frac{c^2 f_t}{35}C_{11}^c\frac{\partial^3}{\partial x^3}\right)\phi_0^c \\ & + \left(-\eta_2^t\frac{\partial}{\partial x} + \frac{cf_t}{70}C_{11}^c\frac{\partial^3}{\partial x^3}\right)u_0^b + \left(\eta_6^t\frac{\partial}{\partial x} + \frac{cf_t}{15}C_{11}^c\frac{\partial^3}{\partial x^3}\right)u_0^c \\ & + \left(-\eta_3^t\frac{\partial}{\partial x} + \frac{3cf_t}{35}C_{11}^c\frac{\partial^3}{\partial x^3}\right)u_0^t + \left(\frac{1}{6c}C_{33}^c + \beta_2\frac{\partial^2}{\partial x^2}\right. \\ & \left. - \frac{cf_bf_t}{140}C_{11}^c\frac{\partial^4}{\partial x^4}\right)w_0^b + \left(-\frac{4}{3c}C_{33}^c + \eta_7^t\frac{\partial^2}{\partial x^2}\right)w_0^c \\ & + \left(\frac{7}{6c}C_{33}^c + \eta_8^t\frac{\partial^2}{\partial x^2} + \frac{3cf_t^2}{70}C_{11}^c\frac{\partial^4}{\partial x^4}\right)w_0^t = 0\end{aligned} \tag{A1b}$$

Core:

$$\begin{aligned}\delta u_0^c\colon\ & -N^c_{,x} + \left(-\frac{4}{3c}C_{55}^c + \frac{c}{5}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^b \\ & + \left(\frac{8}{3c}C_{55}^c + \frac{4c}{15}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^c + \left(-\frac{4}{3c}C_{55}^c + \frac{c}{5}C_{11}^c\frac{\partial^2}{\partial x^2}\right)u_0^t \\ & - \left(\eta_{6a}^b\frac{\partial}{\partial x} + \frac{cf_b}{10}C_{11}^c\frac{\partial^3}{\partial x^3}\right)w_0^b + \left(\eta_{6a}^t\frac{\partial}{\partial x} + \frac{cf_t}{10}C_{11}^c\frac{\partial^3}{\partial x^3}\right)w_0^t = 0\end{aligned} \tag{A2a}$$

$$\delta\phi_0^c\colon\ V^c + M^c_{,x} + \left(\frac{8c}{5}C^c_{55} + \frac{4c^3}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right)\phi^c_0 + \left(\frac{9}{5}C^c_{55} - \frac{c^2}{7}C^c_{11}\frac{\partial^2}{\partial x^2}\right)u^b_0 + \left(-\frac{9}{5}C^c_{55} + \frac{c^2}{7}C^c_{11}\frac{\partial^2}{\partial x^2}\right)u^t_0 + \left(\frac{3}{2}\eta^b_{4a}\frac{\partial}{\partial x} + \frac{c^2 f_b}{14}C^c_{11}\frac{\partial^3}{\partial x^3}\right)w^b_0 - 2c\beta_1\frac{\partial w^c_0}{\partial x} + \left(\frac{3}{2}\eta^t_{4a}\frac{\partial}{\partial x} + \frac{c^2 f_t}{14}C^c_{11}\frac{\partial^3}{\partial x^3}\right)w^t_0 = 0 \tag{A2b}$$

and

$$\delta w_0^c\colon\ -V^c_{,x} - \frac{4c}{3}\beta_1\frac{\partial\phi^c_0}{\partial x} + (\beta_1 - C^c_{55})\left(\frac{\partial u^b_0}{\partial x} - \frac{\partial u^t_0}{\partial x}\right) - \left(\frac{4}{3c}C^c_{33} - \eta^b_{7a}\frac{\partial^2}{\partial x^2}\right)w^b_0 + \left(\frac{8}{3c}C^c_{33} + \frac{4c}{15}C^c_{55}\frac{\partial^2}{\partial x^2}\right)w^c_0 - \left(\frac{4}{3c}C^c_{33} - \eta^t_{7a}\frac{\partial^2}{\partial x^2}\right)w^t_0 = 0 \tag{A2c}$$

Bottom face sheet:

$$\delta u_0^b\colon\ -N^b_{,x} + \left(\frac{4}{5}C^c_{55} + \frac{2c^2}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right)\phi^c_0 + \left(\frac{47}{30c}C^c_{55} - \frac{6c}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right)u^b_0 - \left(\frac{4}{3c}C^c_{55} + \frac{2c}{15}C^c_{11}\frac{\partial^2}{\partial x^2}\right)u^c_0 - \left(\frac{7}{30c}C^c_{55} + \frac{c}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right)u^t_0 + \left(-\eta^b_3\frac{\partial}{\partial x} + \frac{3cf_b}{35}C^c_{11}\frac{\partial^3}{\partial x^3}\right)w^b_0 - \beta_1\frac{\partial w^c_0}{\partial x} + \left(\eta^t_2\frac{\partial}{\partial x} - \frac{cf_t}{70}C^c_{11}\frac{\partial^3}{\partial x^3}\right)w^t_0 = 0 \tag{A3a}$$

and

$$\delta w_0^b\colon\ -N^b_{,x}w^b_{0,x} + M^b_{,xx} - N^b w^b_{0,xx} + \left(\eta^b_4\frac{\partial}{\partial x} + \frac{c^2 f_b}{35}C^c_{11}\frac{\partial^3}{\partial x^3}\right)\phi^c_0 + \left(\eta^b_3\frac{\partial}{\partial x} - \frac{3cf_b}{35}C^c_{11}\frac{\partial^3}{\partial x^3}\right)u^b_0 - \left(\eta^b_6\frac{\partial}{\partial x} + \frac{cf_b}{15}C^c_{11}\frac{\partial^3}{\partial x^3}\right)u^c_0 + \left(\eta^b_2\frac{\partial}{\partial x} - \frac{cf_b}{70}C^c_{11}\frac{\partial^3}{\partial x^3}\right)u^t_0 + \left(\frac{7}{6c}C^c_{33} + \eta^b_8\frac{\partial^2}{\partial x^2} + \frac{3cf_b^2}{70}C^c_{11}\frac{\partial^4}{\partial x^4}\right)w^b_0 + \left(-\frac{4}{3c}C^c_{33} + \eta^b_7\frac{\partial^2}{\partial x^2}\right)w^c_0 + \left(\frac{1}{6c}C^c_{33} + \beta_2\frac{\partial^2}{\partial x^2} - \frac{cf_bf_t}{140}C^c_{11}\frac{\partial^4}{\partial x^4}\right)w^t_0 = 0 \tag{A3b}$$

The following constants, which were used in the governing Eqs. (A1–A3), are defined ($i = t, b$):

$$\eta^i_2 = \frac{1}{30}C^c_{13} + \left(\frac{1}{30} - \frac{7f_i}{60c}\right)C^c_{55} \tag{A4a}$$

$$\eta^i_3 = -\frac{11}{30}C^c_{13} + \left(\frac{19}{30} + \frac{47f_i}{60c}\right)C^c_{55}, \qquad \eta^i_4 = \frac{4c}{15}C^c_{13} + \left(\frac{4c}{15} + \frac{2f_i}{5}\right)C^c_{55} \tag{A4b}$$

$$\eta^i_6 = \frac{2}{3}C^c_{13} + \left(\frac{2}{3} + \frac{2f_i}{3c}\right)C^c_{55}, \qquad \eta^i_7 = -\frac{f_i}{5}C^c_{13} - \left(\frac{2c}{15} + \frac{f_i}{5}\right)C^c_{55} \tag{A4c}$$

$$\eta^i_8 = \frac{11f_i}{30}C^c_{13} - \left(\frac{4c}{15} + \frac{19f_i}{30} + \frac{47f_i^2}{120c}\right)C^c_{55}, \qquad \eta^i_{4a} = \eta^i_4 - \left(\frac{2c}{3} + f_i\right)C^c_{55} \tag{A4d}$$

$$\eta^i_{6a} = C^c_{13} - \eta^i_6, \qquad \eta^i_{7a} = \frac{2c + 3f_i}{6}C^c_{55} + \eta^i_7, \qquad \eta^i_{8a} = \frac{11f_i}{60}C^c_{13} - \eta^i_8 \tag{A4e}$$

and

$$\beta_1 = \frac{2}{5}(C^c_{13} + C^c_{55}) \tag{A4f}$$

$$\beta_2 = \frac{f_b + f_t}{60}C^c_{13} + \left(\frac{c}{15} + \frac{f_b + f_t}{60} - \frac{7f_bf_t}{120c}\right)C^c_{55} \tag{A4g}$$

At each end there are nine boundary conditions, three for each face sheet and three for the core. The corresponding boundary conditions at $x = 0$ and $x = a$, read as follows:

For the top face sheet:

1) Either $\delta u^t_0 = 0$ or

$$N^t + \left(\frac{2c^2}{35}C^c_{11}\right)\phi^c_{0,x} + \left(\frac{c}{35}C^c_{11}\right)u^b_{0,x} + \left(\frac{2c}{15}C^c_{11}\right)u^c_{0,x} + \left(\frac{6c}{35}C^c_{11}\right)u^t_{0,x} + \left(\frac{1}{30}C^c_{13} - \frac{cf_b}{70}C^c_{11}\frac{\partial^2}{\partial x^2}\right)w^b_0 - \frac{2}{5}C^c_{13}w^c_0 + \left(\frac{11}{30}C^c_{13} + \frac{3cf_t}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right)w^t_0 = \tilde{N}^t + \frac{\tilde{n}^c c}{3} \tag{A5a}$$

where $\tilde{N}^t$ is the end axial force per unit width at the top face. 2) Either $\delta w^t_0 = 0$ or

$$N^t w^t_{0,x} - M^t_{,x} - \left[\frac{2(2c + 3f_t)}{15}C^c_{55} + \frac{c^2 f_t}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right]\phi^c_0 + \left[\frac{(2c - 7f_t)}{60c}C^c_{55} - \frac{cf_t}{70}C^c_{11}\frac{\partial^2}{\partial x^2}\right]u^b_0 - \left[\frac{2(c + f_t)}{3c}C^c_{55} + \frac{cf_t}{15}C^c_{11}\frac{\partial^2}{\partial x^2}\right]u^c_0 + \left[\frac{(38c + 47f_t)}{60c}C^c_{55} - \frac{3cf_t}{35}C^c_{11}\frac{\partial^2}{\partial x^2}\right]u^t_0 + \left[\left(\frac{f_b}{60}C^c_{13} - \beta_2\right)\frac{\partial}{\partial x} + \frac{cf_bf_t}{140}C^c_{11}\frac{\partial^3}{\partial x^3}\right]w^b_0 - \eta^t_7\frac{\partial w^c_0}{\partial x} + \left[\eta^t_{8a}\frac{\partial}{\partial x} - \frac{3cf_t^2}{70}C^c_{11}\frac{\partial^3}{\partial x^3}\right]w^t_0 = 0 \tag{A5b}$$

3) Either $\delta w^t_{0,x} = 0$ or

$$M^t + \left(\frac{c^2 f_t}{35}C^c_{11}\right)\phi^c_{0,x} + \left(\frac{cf_t}{70}C^c_{11}\right)u^b_{0,x} + \left(\frac{cf_t}{15}C^c_{11}\right)u^c_{0,x} + \left(\frac{3cf_t}{35}C^c_{11}\right)u^t_{0,x} + \left(\frac{f_t}{60}C^c_{13} - \frac{cf_bf_t}{140}C^c_{11}\frac{\partial^2}{\partial x^2}\right)w^b_0 - \left(\frac{f_t}{5}C^c_{13}\right)w^c_0 + \left(\frac{11f_t}{60}C^c_{13} + \frac{3cf_t^2}{70}C^c_{11}\frac{\partial^2}{\partial x^2}\right)w^t_0 = \frac{\tilde{n}^c cf_t}{6} \tag{A5c}$$

where $\tilde{M}^t$ is the end moment per unit width at the top face (at the end $x = 0$ or $x = a$).

For the core:
1) Either $\delta u_0^c = 0$ or

$$N^c - \left(\frac{c}{5}C_{11}^c\right)u_{0,x}^b - \left(\frac{4c}{15}C_{11}^c\right)u_{0,x}^c - \left(\frac{c}{5}C_{11}^c\right)u_{0,x}^t + \left(\frac{1}{3}C_{13}^c + \frac{cf_b}{10}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^b - \left(\frac{1}{3}C_{13}^c + \frac{cf_t}{10}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^t = \frac{4\tilde{n}^c c}{3} \tag{A6a}$$

2) Either $\delta\phi_0^c = 0$ or

$$-M^c - \left(\frac{4c^3}{35}C_{11}^c\right)\phi_{0,x}^c + \left(\frac{c^2}{7}C_{11}^c\right)u_{0,x}^b - \left(\frac{c^2}{7}C_{11}^c\right)u_{0,x}^t - \left(\frac{2c}{5}C_{13}^c + \frac{c^2f_b}{14}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^b + \frac{4c}{5}C_{13}^c w_0^c - \left(\frac{2c}{5}C_{13}^c + \frac{c^2f_t}{14}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^t = 0 \tag{A6b}$$

3) Either $\delta w_0^c = 0$ or

$$V^c + \left[\frac{8c}{15}\phi_0^c + \frac{3}{5}u_0^b - \frac{3}{5}u_0^t - \frac{(2c+3f_b)}{10}w_{0,x}^b - \frac{4c}{15}w_{0,x}^c - \frac{(2c+3f_t)}{10}w_{0,x}^t\right]C_{55}^c = 0 \tag{A6c}$$

For the bottom face sheet:
1) Either $\delta u_0^b = 0$ or

$$N^b - \left(\frac{2c^2}{35}C_{11}^c\right)\phi_{0,x}^c + \left(\frac{6c}{35}C_{11}^c\right)u_{0,x}^b + \left(\frac{2c}{15}C_{11}^c\right)u_{0,x}^c + \left(\frac{c}{35}C_{11}^c\right)u_{0,x}^t - \left(\frac{11}{30}C_{13}^c + \frac{3cf_b}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^b + \frac{2}{5}C_{13}^c w_0^c + \left(-\frac{1}{30}C_{13}^c + \frac{cf_t}{70}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^t = \tilde{N}^b + \frac{\tilde{n}^c c}{3} \tag{A7a}$$

where $\tilde{N}^b$ is the end axial force per unit width at the bottom face;
2) Either $\delta w_0^b = 0$ or

$$N^b w_{0,x}^b - M_{,x}^b - \left[\frac{2(2c+3f_b)}{15}C_{55}^c + \frac{c^2f_b}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right]\phi_0^c + \left[-\frac{(38c+47f_b)}{60c}C_{55}^c + \frac{3cf_b}{35}C_{11}^c\frac{\partial^2}{\partial x^2}\right]u_0^b + \left[\frac{2(c+f_b)}{3c}C_{55}^c + \frac{cf_b}{15}C_{11}^c\frac{\partial^2}{\partial x^2}\right]u_0^c + \left[\frac{(-2c+7f_b)}{60c}C_{55}^c + \frac{cf_b}{70}C_{11}^c\frac{\partial^2}{\partial x^2}\right]u_0^t + \left(\eta_{8a}^b\frac{\partial}{\partial x} - \frac{3cf_b^2}{70}C_{11}^c\frac{\partial^3}{\partial x^3}\right)w_0^b - \left(\eta_7^b\frac{\partial}{\partial x}\right)w_0^c + \left[\left(\frac{f_t}{60}C_{13}^c - \beta_2\right)\frac{\partial}{\partial x} + \frac{cf_bf_t}{140}C_{11}^c\frac{\partial^3}{\partial x^3}\right]w_0^t = 0 \tag{A7b}$$

3) Either $\delta w_{0,x}^b = 0$ or

$$M^b + \left(\frac{c^2f_b}{35}C_{11}^c\right)\phi_{0,x}^c - \left(\frac{3cf_b}{35}C_{11}^c\right)u_{0,x}^b - \left(\frac{cf_b}{15}C_{11}^c\right)u_{0,x}^c - \left(\frac{cf_b}{70}C_{11}^c\right)u_{0,x}^t + \left(\frac{11f_b}{60}C_{13}^c + \frac{3cf_b^2}{70}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^b - \frac{f_b}{5}C_{13}^c w_0^c + \left(\frac{f_b}{60}C_{13}^c - \frac{cf_bf_t}{140}C_{11}^c\frac{\partial^2}{\partial x^2}\right)w_0^t = -\frac{\tilde{n}^c cf_b}{6} \tag{A7c}$$

In the above equations, the superscript ˜ denotes the known external boundary values.

Appendix B: Elements of the $[K_{LC}]$ Matrix

The $[K_{LC}]$ matrix is symmetric and has the following elements k_{ij} $i, j = 1, \ldots 7$:

$$k_{11} = \frac{47}{30c}C_{55}^c + \frac{6c\pi^2}{35a^2}C_{11}^c + C_{11}^b f_b\frac{\pi^2}{a^2}$$
$$k_{12} = -\frac{4}{3c}C_{55}^c + \frac{2c\pi^2}{15a^2}C_{11}^c \tag{B1a}$$

$$k_{13} = \frac{4}{5}C_{55}^c - \frac{2c^2\pi^2}{35a^2}C_{11}^c; \qquad k_{14} = -\frac{7}{30c}C_{55}^c + \frac{c\pi^2}{35a^2}C_{11}^c \tag{B1b}$$

$$k_{15} = -\frac{3cf_b\pi^3}{35a^3}C_{11}^c - \eta_3^b\frac{\pi}{a}; \qquad k_{16} = -\frac{\pi}{a}\beta_1$$
$$k_{17} = \frac{cf_t\pi^3}{70a^3}C_{11}^c + \eta_2^t\frac{\pi}{a} \tag{B1c}$$

$$k_{22} = \frac{8}{3c}C_{55}^c + \frac{16c\pi^2}{15a^2}C_{11}^c; \qquad k_{23} = 0; \qquad k_{24} = k_{12} \tag{B2a}$$

$$k_{25} = -\frac{cf_b\pi^3}{15a^3}C_{11}^c + \eta_6^b\frac{\pi}{a}; \qquad k_{26} = 0$$
$$k_{27} = \frac{cf_t\pi^3}{15a^3}C_{11}^c - \eta_6^t\frac{\pi}{a} \tag{B2b}$$

$$k_{33} = \frac{8c}{5}C_{55}^c + \frac{16c^3\pi^2}{105a^2}C_{11}^c; \qquad k_{34} = -\frac{4}{5}C_{55}^c + \frac{2c^2\pi^2}{35a^2}C_{11}^c \tag{B3a}$$

$$k_{35} = \frac{c^2f_b\pi^3}{35a^3}C_{11}^c - \eta_4^b\frac{\pi}{a}; \qquad k_{36} = \frac{4c\beta_1\pi}{3a}$$
$$k_{37} = \frac{c^2f_t\pi^3}{35a^3}C_{11}^c - \eta_4^t\frac{\pi}{a} \tag{B3b}$$

$$k_{44} = \frac{47}{30c}C_{55}^c + \frac{6c\pi^2}{35a^2}C_{11}^c + C_{11}^t f_t\frac{\pi^2}{a^2}$$
$$k_{45} = -\frac{cf_b\pi^3}{70a^3}C_{11}^c - \eta_2^b\frac{\pi}{a} \tag{B4a}$$

$$k_{46} = \beta_1\frac{\pi}{a}; \qquad k_{47} = \frac{3cf_t\pi^3}{35a^3}C_{11}^c + \eta_3^t\frac{\pi}{a} \tag{B4b}$$

$$k_{55} = \frac{7}{6c}C_{33}^c + \frac{3cf_b^2\pi^4}{70a^4}C_{11}^c + \frac{f_b^3\pi^4}{12a^4}C_{11}^b - \eta_8^b\frac{\pi^2}{a^2}$$
$$k_{56} = -\frac{4}{3c}C_{33}^c - \eta_7^b\frac{\pi^2}{a^2} \tag{B5a}$$

$$k_{57} = \frac{1}{6c}C_{33}^c - \frac{cf_bf_t\pi^4}{140a^4}C_{11}^c - \beta_2\frac{\pi^2}{a^2} \tag{B5b}$$

$$k_{66} = \frac{8}{3c}C_{33}^c + \frac{16c\pi^2}{15a^2}C_{55}^c; \qquad k_{67} = -\frac{4}{3c}C_{33}^c - \eta_7^t\frac{\pi^2}{a^2} \tag{B6}$$

$$k_{77} = \frac{7}{6c}C_{33}^c + \frac{3cf_t^2\pi^4}{70a^4}C_{11}^c + \frac{f_t^3\pi^4}{12a^4}C_{11}^t - \eta_8^t\frac{\pi^2}{a^2} \tag{B7}$$

Appendix C: Critical Load from HSAPT and Allen's Buckling Formula

The sandwich buckling formula of Allen [6] (thick faces version) considers the shear stress in the core, and neglects the axial and transverse stiffnesses of the core. The critical load for global buckling from this formula is given in Allen for a symmetric configuration as

$$P_{\mathrm{cr,Allen}} = P_{E2}\left[\frac{1+\frac{P_{Ef}}{P_c}-\frac{P_{Ef}^2}{P_cP_{E2}}}{1+\frac{P_{E2}}{P_c}-\frac{P_{Ef}}{P_c}}\right] \tag{C1a}$$

where

$$P_{E2} = E_{f1}\frac{\pi^2}{a^2}\left[\frac{f^3}{6}+\frac{f(2c+f)^2}{2}\right] \tag{C1b}$$

$$P_{Ef} = E_{f1}\frac{\pi^2}{a^2}\frac{f^3}{6}; \qquad P_c = G_{c31}\frac{(2c+f)^2}{2c} \tag{C1c}$$

i.e., P_{E2} represents the Euler load of the sandwich column in the absence of core shear strain with the bending stiffness of the core ignored, but with local bending stiffness of the faces included; P_{Ef} represents the sum of the Euler loads of the two faces when they buckle as independent struts (i.e., when the core is absent), and P_c is the contribution to the buckling load due to shear.

The critical load from the HSAPT is found from solving for the load P in the governing equation for the nontrivial solution (Frostig and Baruch [3]):

$$P_{\mathrm{cr,HSAPT}} = \frac{2\pi^2[(EA)(EI)(2c)g_1\pi^2 + 6E_3^cG_{31}^cg_2a^2]}{(EA)G_{31}^c(2c)^3\pi^4 + 12(EA)E_3^c(2c)\pi^2a^2 + 24E_3^cG_{31}^ca^4} \tag{C2a}$$

where EA and EI are, respectively, the axial and bending stiffnesses per unit width of a sandwich beam that is geometrically uniform along the span, i.e.,

$$EA = E_1^f f, \qquad EI = \frac{E_1^f f^3}{12} \tag{C2b}$$

and we have defined

$$g_1 = G_{31}^c(2c)^2\frac{\pi^2}{a^2} + 12E_3^c$$
$$g_2 = (EA)f^2 + 2(EA)(2c)f + (EA)(2c)^2 + 4(EI) \tag{C2c}$$

This original formulation of the critical global buckling load of HSAPT can be algebraically manipulated by making use of the Allen [6] thick parameters above to appear in the following form:

$$P_{\mathrm{cr,HSAPT}} = P_{E2}\left\{\frac{1+\left[1+\frac{(2c)^2G_{31}^c\pi^2}{12E_3^ca^2}\right]\left(\frac{P_{Ef}}{P_c}-\frac{P_{Ef}^2}{P_cP_{E2}}\right)}{1+\left[1+\frac{(2c)^2G_{31}^c\pi^2}{12E_3^ca^2}\right]\frac{(P_{E2}-P_{Ef})}{P_c}}\right\} \tag{C3}$$

Thus, when E_3^c goes to infinity (an incompressible core) the $P_{\mathrm{cr,HSAPT}}$ approaches the $P_{\mathrm{cr,Allen}}$ formula.

Acknowledgments

The financial support of the Office of Naval Research, grant no. N00014-07-10373, and the interest and encouragement of the Grant Monitor Y. D. S. Rajapakse are both gratefully acknowledged.

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