JOURNAL OF
SANDWICH
STRUCTURES
&MATERIALS

Original Article

Journal of Sandwich Structures and Materials
15(4) 412–428
© The Author(s) 2013
Reprints and permissions:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/1099636213485520
jsm.sagepub.com
SAGE

Free vibration of unidirectional sandwich panels, Part II: Incompressible core

CN Phan[1], Y Frostig[2] and GA Kardomateas[1]

Abstract

The free vibration response of a unidirectional sandwich panel with an incompressible core using shear deformable and layered models is presented. The models considered include the layered model, ordinary sandwich panel theory that uses a zig-zag in-plane displacements distribution in addition to the shear deformations and the first and high-order shear deformable theories that adopt an approach which replaces the layered sandwich panel with a single layer with equivalent properties. The mathematical formulations use Hamilton's principle to present the general equations of motion and the specific mass and stiffness matrices for a simply-supported panel. The results of the models are compared with the closed-form solution of the 2D elasticity equations of motion and finite element results of ADINA. The numerical comparison is described in terms of eigenfrequencies and eigenmodes of displacements and stresses and reveal a good correlation at the lower modes only.

Keywords

Sandwich construction, incompressible core, layered panel, shear deformation, free vibrations, eigenfrequencies, eigenmodes, computational models

Introduction

Sandwich panels are starting to emerge as main and secondary carrying members in various industries such as aerospace, mechanical, and civil engineering for structural applications and as such may be subjected to static and dynamics types of

[1]School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA, USA
[2]Faculty of Civil and Environmental Engineering, Technion-Israel Institute of Technology, Haifa, Israel

Corresponding author:
Y Frostig, Faculty of Civil and Environmental Engineering, Technion-Israel Institute of Technology, Haifa, 32000, Israel.
Email: cvrfros@techunix.technion.ac.il

loading schemes. A typical panel is usually made of two face sheets: metallic or composite laminated that are interconnected with either a very stiff incompressible core in the vertical direction such as metallic honeycomb, or solid light material such as balsa wood, or a compressible one, flexible in the vertical direction such as a low-strength light or dense foam. The kinematic relations adopted in the computational models for the analysis of such panels are in tandem with the properties of the core and to mention a few: the first-order shear deformable theory (FOSDT) andthe high-order shear deformable theory (HOSDT) that uses an equivalent single layer (ESL) models, and the ordinary sandwich panel theory (OSPT), which uses a layered structure with zig-zag distribution for the in-plane displacement. Such models are appropriate for the analysis of panels with a metallic honeycomb core. However, one of the typical errors involved in the analysis of such panels is the use of the aforementioned models for the analysis of panels where the core is compliant, especially in dynamic problems. In general, the eigenmodes of a panel consist of overall bending modes as well as through-the-thickness modes, which cannot be detected when using a core that is infinitely stiff in the vertical direction. Hence, one of the goals of the article is to identify the accuracy of the incompressible model through comparison with accurate solution such as the elasticity one, see Part I.

The classical works regarding sandwich panels are dedicated to panels with metallic honeycomb cores that are infinitely stiff in the vertical direction and very flexible in the in-plane direction. Such cores are assumed to be incompressible, their section planes remain linear or take a 'zig-zag' shape under static and dynamic loads and their in-plane rigidity is negligible. These works already appear in textbooks and reviews, and to mention a few: Allen [1], Plantema [2], Zankert [3], Vinson [4] and a thorough review on sandwich panels by Noor et al. [5]. The general approach adopted for the analysis of sandwich panels, which are actually layered structures, is the use of equivalent solid plate theories, such as the first order with shear deformations by Mindlin [6], the high-order approach by Reddy [7,8] and higher order theories by Kant and Mallikarjuna [9], Senthilnathan et al. [10] and Kant and Swaminathan [11]. In addition there are various finite element (FE) approaches that are based on Reddy's high-order theories [12,13]; or based on a 'zig-zag' displacement pattern through the thickness of the panel [14], and also using Mindlin plate theory with linearly varying shear stresses and uniform vertical normal stresses through the thickness of the panel which contradicts equilibrium within the core [15]. Most of the aforementioned theories and numerical approaches based on FEs assume that the height of the core remains unchanged, i.e. incompressible, and the boundary conditions for the upper and the lower face sheets are identical at the same edge and located at centroid of panel, which in many cases contradict real plate supports. These assumptions are correct as long as the core is incompressible.

The equations of motion presented ahead assume that the sandwich panel which consists of a core between two face sheets to be elastic, linear with small displacements and its face sheets to have in-plane and flexural rigidity and negligible shear.

The core is incompressible with shear resistance and negligible in-plane and flexural rigidity and its interfaces with the face sheets fulfill the conditions of full bond.

This article consists of a mathematical formulation and a numerical investigation. The mathematical part presents the equations of motion in terms of partial differential equation along with the appropriate mass and stiffness matrices of a simply-supported sandwich panel. A numerical comparison with the compressible core models, elasticity (see part I) and FE results is presented and discussed. Finally, a summary is presented and conclusions are drawn.

Mathematical formulation

The equations of motion for the various models have been derived through Hamilton's principle, which uses the first variation of the kinetic and internal energy as follows

Kinetic energy:

$$\delta T = \int_{t_1}^{t_2} \left(\int_{V_t} \rho_t u_{t,t} \delta u_{t,t} + \rho_t w_{t,t} \delta w_{t,t} \mathrm{d}v + \int_{V_b} \rho_b u_{b,t} \delta u_{b,t} + \rho_b w_{b,t} \delta w_{b,t} \mathrm{d}v \right.$$
$$\left. + \int_{V_c} \rho_c u_{c,t} \delta u_{c,t} + \rho_c w_{c,t} \delta w_{c,t} \mathrm{d}v \right) \mathrm{d}t \qquad (1)$$

where $\rho_j (j=t,b,c)$ is the mass density of the upper and lower face sheets and the core respectively; $u_{j,t}$ and $w_{j,t}(j=t,b,c)$ are the velocities in the longitudinal and vertical direction respectively of the various constituents of the sandwich panel; $f_{,t} = \frac{\partial f}{\partial t}$ is the first derivative of the function f with respect to the time coordinate; $V_j(j=t,b,c)$ is the volume of upper and lower face sheets and core, respectively and $\mathrm{d}v$ is the volume of a differential segment.

Internal potential energy:

$$\delta U = \int_{V_t} \sigma_{xxt} \delta \varepsilon_{xxt} + \alpha_f \tau_{xzt} \delta \gamma_{xzt} \mathrm{d}v + \int_{V_b} \sigma_{xxb} \delta \varepsilon_{xxb} + \alpha_f \tau_{xzb} \delta \gamma_{xzb} \mathrm{d}v$$
$$+ \int_{V_c} \lambda_c \sigma_{xxc} \delta \varepsilon_{xxc} + \alpha_c \tau_{xzc} \delta \gamma_{xzc} + \beta_c \sigma_{zzc} \delta \varepsilon_{zzc} \mathrm{d}v \qquad (2)$$

where σ_{xxj} and ε_{xxj} $(i=x$ and $j=t,b)$ are the longitudinal normal stresses and strains in the upper and the lower face sheet respectively; τ_{xzj} and γ_{xzj} $(j=t,b)$ are the vertical shear stress and angle respectively at the various face sheets; τ_{xzc} and γ_{xzc} are the vertical shear stresses and shear strains in the core on the longitudinal and transverse faces of the core; σ_{zzc} and ε_{zzc} are the vertical normal stresses and strains in the vertical direction of the core geometry respectively. The sign convention for stresses and displacements are provided in Figure 1. The values

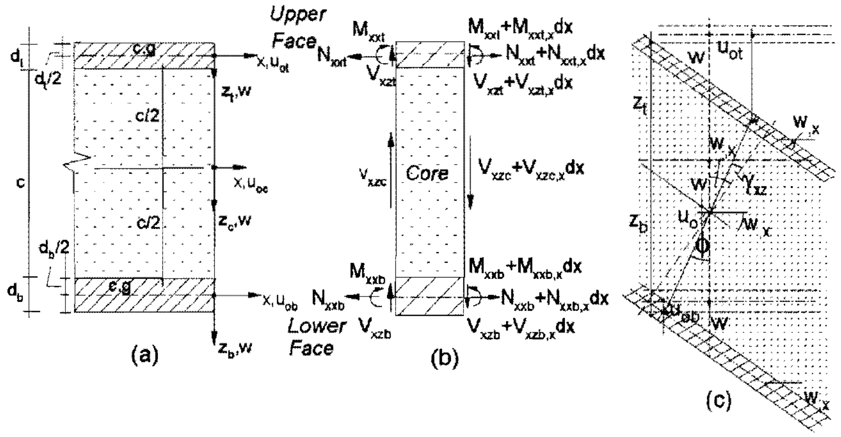

Figure 1. Coordinate system, stress resultants and displacement pattern for the OSPT model: (a) coordinate system; (b) stress resultants; (c) zig-zag displacement pattern. OSPT: ordinary sandwich panel theory.

for the α, β and λ terms are: OSPT $\alpha_c = 1$, $\alpha_f = \lambda_c = \beta_c = 0$; while for the FOSDT and HOSDT [7]) they are $\alpha_f = \alpha_c = 1$, $\lambda_c = 1$, $\beta_c = 0$.

The mathematical formulation starts with the layered-wise model (OSPT) followed briefly by the two ESLs of the first and the high-order shear deformable models. It briefly presents the equations of motion and their solutions for the case of a simply-supported sandwich panel.

OSPT Model

The ordinary sandwich panel (OSPT) is a layered-wise model with an incompressible core. It assumes a piecewise linear, zig-zag, in-plane displacement patterns and a uniform vertical displacement through the depth of the panel, which reflects that the core is incompressible (Figure 1). In addition, the core is assumed to be of an anti-plane which means that the core is incompressible, the section plane of the core remains plane after deformation, and the in-plane rigidity of the panel is neglected while its shear rigidity is considered [16]. Hence, the displacements for the face sheets ($j = t,b$) and the core (Figure 1(c)), along with their strains read

$$
\begin{aligned}
u_j(x, z_j, t) &= u_{oj}(x, t) - z_j \frac{\partial}{\partial x} w_j(x, t), \quad w_j(x, z, t) = w(x, t) \\
\varepsilon_{xxj}(x, z_j, t) &= \frac{\partial}{\partial x} u_{oj}(x, t) - z_j \frac{\partial^2}{\partial x^2} w(x, t) \\
u_c(x, z_c, t) &= u_o(x, t) - \phi(x, t) z_c, \quad w_c(x, z_c, t) = w(x, t) \\
\varepsilon_{xxc}(x, z_c, t) &= \frac{\partial}{\partial x} u_o(x, t) - \left(\frac{\partial}{\partial x} \phi(x, t)\right) z_c, \quad \gamma_c(x, z_c, t) = -\phi(x, t) + \frac{\partial}{\partial x} w(x, t)
\end{aligned} \tag{3}
$$

where $u_{oj}(j=t,b)$ are the in-plane displacements at the face sheets, u_o and ϕ are the in-plane displacements of the centroid of the core and the rotation of the section plane, respectively (Figure 1(c)).

The in-plane displacements of the face sheets are determined through the requirement of full bond at the various face–core interfaces in the longitudinal direction and they read $(j=t,b)$

$$u_{oj} = u_o + (-1)^{kt}\phi z_{cj} + 1/2(-1)^{kb} d_j w_x(x,t) \tag{4}$$

where at the upper interface $j=t$, $kt=1$ and $kb=0$ and at the lower interface $j=b$, $kt=0$ and $kb=1$; $z_{cj}=c/2$ $(j=t,b)$ and $w_x = \frac{\partial}{\partial x} w_j(x,t)$. For sign convention and the stress resultants see Figure 1(a) and (b). Notice that the strains, after using equation (4) are described by only three unknowns, u_{oc}, ϕ, and w, similar to the number of variables in the ESL models, see ahead.

The equations of motion have been derived using the strains (equations (3) and (4)) that are substituted into the variations of the kinetic and potential energies (equations (1) and (2)) with the following factors: $\alpha_c=1$, $\alpha_f=\lambda_c=\beta_c=0$, and into the Hamilton's principle. Here, the stress resultants involved are the in-plane stress resultants and vertical shear stress resultants and the bending moments of the face sheets in addition to the vertical shear stress resultants in the core (Figure 1(a) and (b)). Thus the equations of motion for isotropic face sheets core read

$$\begin{aligned}
&\left(-\frac{1}{2}EA_t d_t + \frac{1}{2}EA_b d_b\right) w_{xxx} + (-EA_b z_{cb} + EA_t z_{ct})\phi_{xx} + (-EA_b - EA_t)u_{oxx} \\
&+ \left(\frac{1}{2}M_t d_t - \frac{1}{2}M_b d_b\right) w_{xtt} + (M_b z_{cb} - M_t z_{ct} + Q_{mc})\phi_{tt} + (M_t + M_c + M_b)u_{ott} = 0 \\
&(-4I_{mb} - 4I_{mt})w_{xxtt} + \left(\frac{1}{2}M_t d_t z_{ct} + \frac{1}{2}M_b d_b z_{cb}\right)\phi_{xtt} + \left(\frac{1}{2}M_b d_b - \frac{1}{2}M_t d_t\right) u_{oxtt} \\
&+ (M_t + M_c + M_b)w_{tt} + \left(EI_b + EI_t + \frac{1}{4}EA_t d_t^2 + \frac{1}{4}EA_b d_b^2\right) w_{xxxx} \\
&+ \left(-\frac{1}{2}EA_t z_{ct} d_t - \frac{1}{2}EA_b z_{cb} d_b\right)\phi_{xxx} + \left(\frac{1}{2}EA_t d_t - \frac{1}{2}EA_b d_b\right) u_{oxxx} \\
&- w_{xx} GA_c - \phi_x GA_c = 0 \\
&\left(-\frac{1}{2}M_b d_b z_{cb} - \frac{1}{2}M_t d_t z_{ct}\right) w_{xtt} + \left(M_b z_{cb}^2 + M_t z_{ct}^2 + I_{mc}\right)\phi_{tt} \\
&+ (M_b z_{cb} - M_t z_{ct} + Q_{mc})u_{ott} + \left(\frac{1}{2}EA_b z_{cb} d_b + \frac{1}{2}EA_t z_{ct} d_t\right) w_{xxx} \\
&+ \left(-EA_b z_{cb}^2 - EA_t z_{ct}^2\right)\phi_{xx} + (-EA_b z_{cb} + EA_t z_{ct})u_{oxx} + w_x GA_c + \phi GA_c = 0
\end{aligned} \tag{5}$$

where the subscripts x and t denotes a single or multiple derivatives of the dependent variables with these variables; M_j, Q_{mj}, I_{mj} $(j=t,c,b)$ are the mass, first moment,

moment of inertia mass for the face sheets and the core sections, respectively; EA_j and EI_j ($j=t,b$) are in-plane rigidity and moment and high-order moments of the upper and the lower face respectively; GA_c is the shear rigidity of the core and they read

$$
\begin{aligned}
M_j &= \int_{-1/2d_j}^{1/2d_j} \rho_j b_w \mathrm{d}z_j, \quad I_{mj} = \int_{-1/2d_j}^{1/2d_j} \rho_j z_j b_w \mathrm{d}z_j \\
EA_j &= \int_{-1/2d_j}^{1/2d_j} E_j b_w \mathrm{d}z_j, \quad EI_j = \int_{-1/2d_j}^{1/2d_j} E_j z_j b_w \mathrm{d}z_j \quad (j=t,b) \\
M_c &= \int_{-z_{ct}}^{z_{cb}} \rho_c b_w \mathrm{d}z, \quad I_{mc} = \int_{-z_{ct}}^{z_{cb}} \rho_c z_c b_w \mathrm{d}z, GA_c = G_c c b_w
\end{aligned} \tag{6}
$$

where E_j ($j=t,b$) are the modulus of elasticity of the face sheets and G_c is the shear modulus of the core.

In the case of a simply-supported panel a closed-form solution in terms of trigonometric functions exists and it reads

$$
u_o = \sum_{m=1}^{N} C_{uo,m} \cos(\alpha_m x) e^{I\omega_m t}, \quad \phi = \sum_{m=1}^{N} C_{\phi,m} \cos(\alpha_m x) \grave{e}^{I\omega_m t}, \quad w = \sum_{m=1}^{N} C_{w,m} \sin(\alpha_m x) e^{I\omega_m t} \tag{7}
$$

where $C_{k,m}$ ($k=u_o, \phi, w$) are the constants of the solution that construct the eigenmodes, $\alpha_m = m\pi/L$ where m is the half wave number, L is the length of the panel and N is the number of half waves; $I=\sqrt{-1}$, ω_m is the eigenfrequency of the m wave number and t is the time coordinate and x is the longitudinal coordinate. Notice that for each half wave number there are three modes of longitudinal and vertical displacements and a rotation. Hence through substitution of this solution into the equations of motions, equation (5) yields

$$
(M_m - \lambda_m K_m) C_m = 0 \tag{8}
$$

where $\lambda_m = \omega_m^2$; $\mathbf{C}_m^T = [\mathbf{C}_{uo,m}, \mathbf{C}_{\phi,m}, \mathbf{C}_{w,m}]$, and $\mathbf{0}$ is a zero vector of length three and $\boldsymbol{M}_m$ and $\boldsymbol{K}_m$ are the mass and the stiffness matrices respectively, and they read

$$
\begin{aligned}
M_m = \Bigg[&\left[M_{t'} + M_c + M_{b'}, \left(\frac{1}{2} M_t d_t - \frac{1}{2} M_b d_b\right) \alpha_{m'} Q_{mc} - z_{ct} M_t + z_{cb} Mb \right], \\
&\left[\left(\frac{1}{2} M_t d_t - \frac{1}{2} M_b d_b\right) \alpha_{m'} (4\mathrm{I}_{mt} + 4\mathrm{I}_{mb}) \alpha_m^2 + M_t + M_b \right. \\
&\left. + M_{c'} \left(-\frac{1}{2} M_b d_b z_{cb} - \frac{1}{2} M_t d_t z_{ct} \right) \alpha_m \right], \\
&\left[Q_{mc} - z_{ct} M_t + z_{cb} M_{b'} \left(-\frac{1}{2} M_b d_b z_{cb} - \frac{1}{2} M_t d_t z_{ct} \right) \alpha_{m'} z_{cb}^2 M_b + \mathrm{I}_{mc} + z_{cb}^2 M_t \right] \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
K_m = \Bigg[& \left[(EA_b + EA_t)\alpha_{m'}^2 \left(\frac{1}{2} EA_t d_t - \frac{1}{2} EA_b d_b \right) \alpha_{m'}^3 (-EA_t z_{ct} + EA_b z_{cb}) \alpha_m^2 \right], \\
& \left[\left(\frac{1}{2} EA_t d_t - \frac{1}{2} EA_b d_b \right) \alpha_{m'}^3 \left(EI_b + EI_t + \frac{1}{4} EA_t d_t^2 + \frac{1}{4} EA_b d_b^2 \right) \alpha_m^4 \right. \\
& \left. + \alpha_m^2 GA_{c'} \left(-\frac{1}{2} EA_t z_{ct} d_t - \frac{1}{2} EA_b z_{cb} d_b \right) \alpha_m^3 + \alpha_m GA_c \right], \\
& \left[(-EA_t z_{ct} + EA_b z_{cb}) \alpha_{m'}^3 \left(-\frac{1}{2} EA_t z_{ct} d_t - \frac{1}{2} EAb^z cb^d b \right) \alpha_m^3 \right. \\
& \left. + \alpha_m GA_{c'} \left(EA_b z_{cb}^2 + EA_t z_{ct}^2 \right) \alpha_m^3 + GA_c \right] \Bigg]
\end{aligned} \tag{9}
$$

FOSDT Model

The first-order shear deformable models for sandwich panels is based on Reissner–Mindlin approach [6], and is presented here with some modification due to the core. It assumes that the section plane of the panel, although layered, is linear but not perpendicular to the centroid line after deformation. In addition, the core here is incompressible in the vertical direction and therefore the vertical displacement is assumed to be identical through the depth of the panel. Thus the displacements and the strains read

$$
\begin{aligned}
& u(x,z,t) = u_o(x,t) - z\phi(x,t), w(x,z,t) = w(x,t) \\
& \varepsilon_{xxj}(x,z,t) = \frac{\partial}{\partial x} u(x,z,t) = \frac{\partial}{\partial x} u_o(x,t) - z \frac{\partial}{\partial x} \phi(x,t)' \\
& \gamma_{xz}(x,z,t) = \frac{\partial}{\partial z} u(x,z,t) + \frac{\partial}{\partial x} w(x,t) = -\phi(x,t) + \frac{\partial}{\partial x} w(x,t)
\end{aligned} \tag{10}
$$

where u_o, w and ϕ are the in-plane displacement at the centroid of the panel, the vertical displacement, and rotation of section, respectively. Sign convention, stress resultants, and displacement patterns are given in Figure 2.

The equations of motion are derived substituting the strains (equations (10)), into the potential energy (equation (2)) with the following factors: $\alpha_f = \alpha_c = 1$, $\lambda_c = 1$, $\beta_c = 0$, and the kinetic energy (equation (1)) into Hamilton's principle. Here, it is assumed that the velocities have the same distributions as the displacements. Hence, the equations of motion read

$$
\begin{aligned}
-GA_c(\phi_x + w_{xx}) + Mw_{tt} = 0, \quad & - Q_m \phi_{tt} - u_{oxx} EA_{tot} + Mu_{ott} = 0, \\
& -u_{ott} Q_m + I_m \phi_{tt} - EI_{tot} \phi_{xx} + GA_c(\phi + w_x) = 0
\end{aligned}
$$

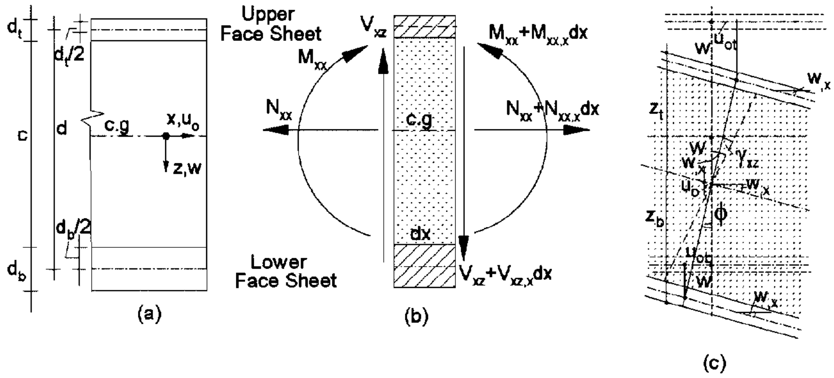

Figure 2. FOSDT displacement pattern through depth of panel.
FOSDT: first-order shear deformable theory.

where

$$M = \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} \rho_j b_w \mathrm{d}z\right), \quad Q_m = \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} \rho_j b_w z \mathrm{d}z\right),$$

$$I_m = \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} \rho_j b_w z^2 \mathrm{d}z\right)$$

$$EA_{tot} = \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} E_j b_w \mathrm{d}z\right), \quad EI_{tot} = \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} E_j b_w z^2 \mathrm{d}z\right)$$

$$GA = G_{eq} h b_w \quad \text{with} \quad G_{eq} = \left[\frac{G_c h}{c}, \frac{h}{\left(\frac{d_t}{G_t} + \frac{c}{G_c} + \frac{d_b}{G_b}\right)}\right] \tag{11}$$

And M, Q_m, and I_m are the sectional mass, first moment of mass, and moment of inertia mass of the of the panel, respectively; $1 = t$, $2 = c$, $3 = b$ indices refer to the designated face sheets and core; z_{jk} ($j = 1,2,3$ and $k = t,b$) are the upper and the lower vertical coordinates of the layer j; b_w is the width of the panel, EA_{tot} and EI_{tot} are the in-plane and flexural rigidity of the panel, respectively; GA is the shear rigidity of the panel; G_{eq} is the equivalent shear modulus of the panel; G_j ($j = t,c,b$) is the shear moduli of the face sheets and the core. The equivalent shear rigidity in this case is based on equivalence of the shear energy of the layered panel, two face sheets and a core, with that of a single layer panel. The classical equivalent shear modulus by Allen [1] where $GA_{eq} = \frac{c+\frac{d_t}{2}+\frac{d_b}{2}}{c} G_c A_c$, Here, A_c is the area of the core, which yields eigenfrequencies that are much lower than those of the elasticity solution.

The solution of the dynamic equations for a simply-supported panel is the same as in the OSPT case (equations (7)), which yields an eigenvalue problem (equations (8)). Hence the mass and stiffness matrices read

$$M_m = \begin{bmatrix} M & 0 & -Q_m \\ 0 & M & 0 \\ -Q_m & 0 & \mathrm{I}_m \end{bmatrix}, \quad K_m = \begin{bmatrix} EA_{tot} & 0 & 0 \\ 0 & GA\,\alpha_m^2 & GA\,\alpha_m \\ 0 & GA\,\alpha_m\,EI_{tot} & \alpha_m^2 + GA \end{bmatrix} \tag{12}$$

HOSDT Model

The high-order shear deformable model for sandwich panels presented briefly ahead is based on Reddy's high-order approach [7], with some modification. It uses an in-plane displacement distribution through the entire depth of the panel that is cubic and it is based on the assumption that the upper and the lower outer fibers of the panel are free of shear stresses (Figure 3). Hence the displacements and the strains here read

$$u(x,z,t) = u_o(x,t) + \left(z - 4/3\,\frac{z^3}{h^2}\right)\phi(x,t) - 4/3\,\frac{z^3\,\frac{\partial}{\partial x}w(x,t)}{h^2}, \quad w(x,z,t) = w(x,t)$$

$$\varepsilon_{xxj}(x,z,t) = \frac{\partial}{\partial x}u_o(x,t) + \left(z - 4/3\,\frac{z^3}{h^2}\right)\frac{\partial}{\partial x}\phi(x,t) - 4/3\,\frac{z^3\,\frac{\partial^2}{\partial x^2}w(x,t)}{h^2}$$

$$\gamma_{xz}(x,z,t) = \left(1 - 4\,\frac{z^2}{h^2}\right)\phi(x,t) - 4\,\frac{z^2\,\frac{\partial}{\partial x}w(x,t)}{h^2} + \frac{\partial}{\partial x}w(x,t) \tag{13}$$

The sign convention of the coordinates and the ordinary stress resultants is the same as that of the FOSDT model (Figure 2(a) and (b)). Here, the number of unknowns is the same as for the OSPT and the FOSDT models.

The appropriate equations of motion are derived by substituting the strains, equations (13) into the potential energy with the following factors: $\alpha_f = \alpha_c = 1$, $\lambda_c = 1$, $\beta_c = 0$ and the kinetic energy into Hamilton's principle (equations (1) and (2)). Hence, they read

$$\left(-\frac{4}{3}\frac{I3_m}{h^2} + Qm\right)\phi_{tt} + Mu_{ott} + \frac{4}{3}\frac{EI3_{tot}\phi_{xx}}{h^2} - u_{oxx}EA_{tot} + \frac{4}{3}\frac{w_{xxx}EI3_{tot}}{h^2} = 0$$

$$\left(-\frac{16}{9}\frac{I6_m}{h^4} + \frac{4}{3}\frac{I4_m}{h^2}\right)\phi_{xtt} + \frac{4}{3}\frac{u_{oxtt}I3_m}{h^2} + Mw_{tt} - \frac{16}{9}\frac{I6_m w_{xxtt}}{h^4}$$

$$+ \left(\frac{4GA2_{tot}}{h^2} + \frac{4\left(GA2_{tot} - \frac{4GA4_{tot}}{h^2}\right)}{h^2} - GA_{tot}\right)w_{xx}$$

$$
\begin{aligned}
&+\left(\frac{4GA2_{tot}}{h^2}+\frac{4\left(GA2_{tot}-\frac{4GA4_{tot}}{h^2}\right)}{h^2}-GA_{tot}\right)\phi_x \\
&-\frac{4}{3}\frac{\left(EI4_{tot}-\frac{4}{3}\frac{EI6_{tot}}{h^2}\right)\phi_{xxx}}{h^2}-\frac{4}{3}\frac{EI3_{tot}u_{oxxx}}{h^2}+\frac{16}{9}\frac{w_{xxxx}EI6_{tot}}{h^4}=0 \\
&\left(-\frac{8}{3}\frac{I4_m}{h^2}+\frac{16}{9}\frac{I6_m}{h^4}+I2_m\right)\phi_{tt}+\left(-\frac{4}{3}\frac{I3_m}{h^2}+Q_m\right)u_{ott} \\
&+\left(\frac{16}{9}\frac{I6_m}{h^4}-\frac{4}{3}\frac{I4_m}{h^2}\right)w_{xtt}+\left(-\frac{4GA2_{tot}}{h^2}-\frac{4\left(GA2_{tot}-\frac{4GA4_{tot}}{h^2}\right)}{h^2}+GA_{tot}\right)\phi \\
&+\left(\frac{4}{3}\frac{EI4_{tot}-\frac{4}{3}\frac{EI6_{tot}}{h^2}}{h^2}-EI_{tot}+\frac{4}{3}\frac{EI4_{tot}}{h^2}\right)\phi_{xx}+\frac{4}{3}\frac{EI3_{tot}u_{oxx}}{h^2} \\
&+\left(-\frac{4GA2_{tot}}{h^2}-\frac{4\left(GA2_{tot}-\frac{4GA4_{tot}}{h^2}\right)}{h^2}+GA_{tot}\right)w_x \\
&+\left(\frac{4}{3}\frac{EI4_{tot}}{h^2}-\frac{16}{9}\frac{EI6_{tot}}{h^4}\right)w_{xxx}=0
\end{aligned}
\tag{14}
$$

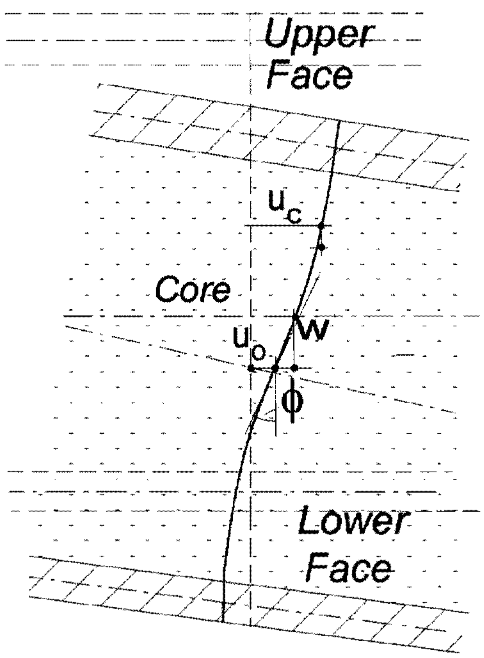

Figure 3. Displacement pattern of the HOSDT model through depth of panel.
HOSDT: high-order shear deformable theory.

where M, Q_m, Ik_m $(k=2\ldots6)$ are the sectional mass, first moment of inertia, and high-order moment of inertia mass of the panel, respectively; EA_{tot} is the in-plane sectional rigidity, EIk_{tot} $(k=2\ldots6)$ are the high-order moments of the section, GA_{tot} is the shear sectional rigidity, and GIk_{tot} $(k=2\ldots4)$ are the moments of the shear rigidity of the section. The mass, flexural, and the shear high-order moments read

$$
\begin{aligned}
Ik_m &= \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} \rho_j b_w z^k \mathrm{d}z\right)\\
EIk_{tot} &= \sum_{j=1}^{3}\left(\int_{z_{jt}}^{z_{jb}} E_j b_w z^k \mathrm{d}z\right), GA_{tot} = G_{eq} c b_w \quad with \quad G_{eq} = \frac{G_c(c + \mathrm{d}_t/2 + \mathrm{d}_b/2)}{c}\\
GIk_{tot} &= \int_{-z_t}^{z_b} G_{eq} b_w z^k \mathrm{d}z
\end{aligned}
\tag{15}
$$

The j number designate the face sheet and the core as follows: $j=1=t, j=2=c$, $j=3=b$ and z_{jk} $(k=t,b)$ are the upper and the lower vertical coordinates of the layer j.

The solution of the dynamic equations for a simply-supported panel is the same as in the OSPT case (equations (7) and (8)). Hence the mass and stiffness matrices read

$$
M_m = \begin{bmatrix}
M & -\frac{4}{3}\frac{13_m\alpha_m}{h^2} & -\frac{4}{3}\frac{13_m}{h^2} + Q_m \\
-\frac{4}{3}\frac{13_m\alpha_m}{h^2} & \frac{16}{9}\frac{\alpha_m^2 16_m}{h^4} + M & \frac{16}{9}\frac{\alpha_m 16_m}{h^4} - \frac{4}{3}\frac{\alpha_m 14_m}{h^2} \\
-\frac{4}{3}\frac{13_m}{h^2} + Q_m & \frac{16}{9}\frac{\alpha_m 16_m}{h^4} - \frac{4}{3}\frac{\alpha_m 14_m}{h^2} & -\frac{8}{3}\frac{14_m}{h^2} + \frac{16}{9}\frac{16_m}{h^4} + 12_m
\end{bmatrix}
$$

$$
\begin{aligned}
K_m = \Bigg[&\left[EA_{tot}\alpha_{m'}^2, -\frac{4}{3}\frac{\alpha_m^3 EI3_{tot}}{h^2}, -\frac{4}{3}\frac{EI3_{tot}\alpha_m^2}{h^2}\right],\\
&\Bigg[-\frac{4}{3}\frac{\alpha_m^3 EI3_{tot}}{h^2}, \frac{16}{9}\frac{\alpha_m^4 EI6_{tot}}{h^4} + \frac{16\,\alpha_m^2 GI4_{tot}}{h^4} + \alpha_m^2 GA_{tot} - \frac{8\alpha_m^2 GI2_{tot}}{h^2}, \alpha_m GA_{tot}\\
&-\frac{8\alpha_m GI2_{tot}}{h^2} + \frac{16\alpha_m GI4_{tot}}{h^4} + \frac{16}{9}\frac{\alpha_m^3 GI6_{tot}}{h^4} - \frac{4}{3}\frac{\alpha_m^3 EI4_{tot}}{h^2}\Bigg],\\
&\Bigg[-\frac{4}{3}\frac{EI3_{tot}\alpha_m^2}{h^2}, \alpha_m GA_{tot} - \frac{8\alpha_m G12_{tot}}{h^4} + \frac{16\alpha_m G14_{tot}}{h^4} + \frac{16}{9}\frac{\alpha_m^3 E16_{tot}}{h^4}\\
&\Bigg[-\frac{4}{3}\frac{\alpha_m^3 EI_{tot}}{h^2}, -\frac{8}{3}\frac{\alpha_m^3 EI4_{tot}}{h^2} + \frac{16}{9}\frac{\alpha_m^3 EI6_{tot}}{h^4} - \frac{8\,GI2_{tot}}{h^2}\\
&+\frac{16\,GI4_{tot}}{h^4} + GA_{tot} + EI_{tot}\alpha_m^2\Bigg]\Bigg]
\end{aligned}
\tag{16}
$$

Numerical study

The numerical study discusses the free vibration of a simply-supported sandwich panel with an incompressible core. The data adopted is sandwich panel used in the experimental blast investigation of Gardner et al. [17] but assuming that the core incompressible where the core is uniform through depth and of low and high density. The results include the eigenfrequencies for the first and the second half wave numbers and the eigenmodes for the first half wave number. In addition, the results for the compressible cores are also presented for comparison (for details see Part I of this study).

The investigated panel consists of face sheets made of E-Glass vinyl-ester laminated composite with a quasi-isotropic layup, $[0/45/90/-45]_s$ with density of $1800\,kg/m^3$ and an equivalent modulus of elasticity of 13,600 MPa. The foam core, A300, is Corecell^{TM} A-series styrene acrylonitrile (SAN) foams with density of $58.5\,kg/m^3$ and elasticity modulus of 32 MPa, a Poisson's ratio of 0.25 and a shear modulus of 12.8 MPa. The geometry is presented in Figure 4.

The eigenfrequencies are presented in a non-dimensional form with respect to the corresponding eigenfrequency of a unidirectional panel with only the identical flexural rigidity. The results in Table 1 include: a light core of A300

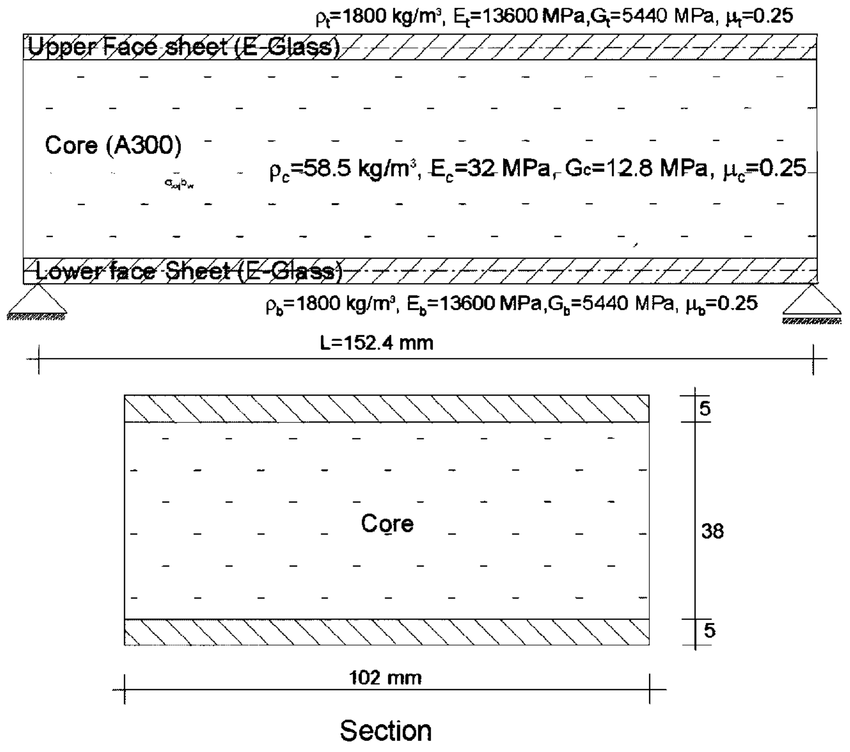

Figure 4. Layout of a sandwich panel and a typical section [17].

($\rho = 58.5\text{kg/m}^3, E_c = 32\text{MPa}, G_c = 12.8\text{MPa}, \mu_c = 0.25$),a heavy core of A800 ($\rho = 150\text{kg/m}^3, E_c = 117\text{MPa}, G_c = 46.8\text{MPa}, \mu_c = 0.25$) and the results for the second half wave numberwith a light core. The eigenmodes of the first wave number along the panel and through the depth of the panel appear in Figures 5 and 6, respectively.

The results in Table 1 consist of only three eigenfrequencies for the incompressible core models for the first two half-wave number. In addition, there are five columns with the compressible (High-Order models, Elasticity model results, see Part I), and the FE model (ADINA) ones.The number of values in each computational model corresponds to the number of unknowns in the formulation. The results reveal that, for the eigenfrequency of the first mode in the first two half wave numbers yielded higher values as compared with the compressible models and the elasticity solution. In addition, the models yield totally inaccurate results, almost identical for the higher modes in both cases of the two half wave numbers. The discrepancies are larger for the second wave number.

The eigenmodes along the panel are presented in Figure 5, and through the depth of the panel in Figure 6. They have been determined through normalization with respect to the largest value in the eigenvector of the results while in the elasticity model they have been normalized with respect to the largest displacement through its depth. In addition, they have been also normalized with respect to the sign of the corresponding mode of the elasticity model.

Figure 5 consists of modes due to displacements only of all models as well as the elasticity since this model has only displacement variables. The results describe the in-plane and vertical displacements at the centroid of the panelfor the various

Table 1. Non-dimensional eigenfrequencies of modes for first and the second half-waves for light (L, A300) and heavy (H, A800) core. *See Part I.

Computation model									
Mode no	Wave no.	FOSDT	HOSDT	OSPT	HSAPT* (Mixed)	HSAPT* (Displ.)	EHSAPT*	Elasticity*	FE ADINA
1	N=1(L)	0.167	0.165	0.165	0.163	0.164	0.164	0.164	0.163
	N=1(H)	0.308	0.296	0.287	0.285	0.286	0.286	0.286	
	N=2(L)	0.085	0.094	0.101	0.099	0.100	0.100	0.099	
2	N=1(L)	2.251	2.251	2.251	0.569	0.567	0.575	0.574	0.574
	N=1(H)	2.251	2.251	2.251	1.136	1.138	1.127	1.134	
	N=2(L)	1.126	1.126	1.126	0.157	0.156	0.156	0.156	
3	N=1(L)	2.383	2.362	2.366	2.251	1.576	1.704	1.691	1.704
	N=1(H)	2.612,	2.555	2.560	2.252	1.824	1.982	1.980	
	N=2(L)	1.179	1.170	1.173	1.126	0.411	0.529	0.332	

FOSDT: first-order shear deformable theory; HOSDT: high-order shear deformable theory; OSPT: ordinary sandwich panel theory; HSAPT: high-order sandwich panel theory; EHSAPT: extended high-order sandwich panel theory.

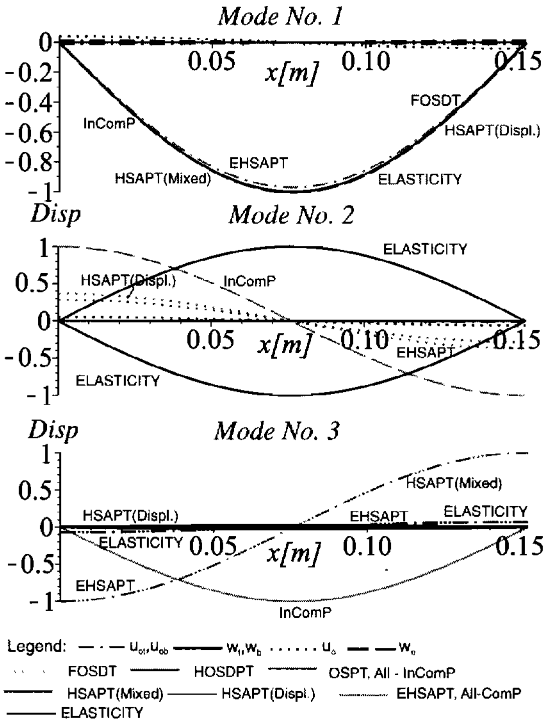

Figure 5. Three eigenmodes of displacements of the first half-wave number along panel for various computational models and the elasticity solution.
FOSDT: first-order shear deformable theory; HOSDT: high-order shear deformable theory; OSPT: ordinary sandwich panel theory; HSAPT: high-order sandwich panel theory; EHSAPT: extended high-order sandwich panel theory.

incompressible models while the other high-order models include the displacements of the face sheets and centroid of core in addition. For more details see Part I. The first mode corresponds to a pure bending mode where the two face sheet move in tandem and is identical for all models. The second mode is rotation of the section without in-plane displacements for the incompressible models and a pumping type where the two face sheet move in opposite directions for the compressible models. The third mode is a longitudinal movement of the face sheets that correlates well with all models although the corresponding eigenfrequencies are quite different.

The modes through the depth of the panel appear in Figure 6 for the first three modes that correspond to the first half wave number. Here the normalization is with respect to the extreme value of the longitudinal or the vertical displacements in all models. In the first mode the distribution of the longitudinal displacement is in the form of a zig-zag curve except for the HOSDPT and the vertical displacements changes within the depth of the core. The correlation for all models is satisfactory. A satisfactory correlation exists also for the vertical displacement only of the second mode. The correlation in the third mode is less satisfactory.

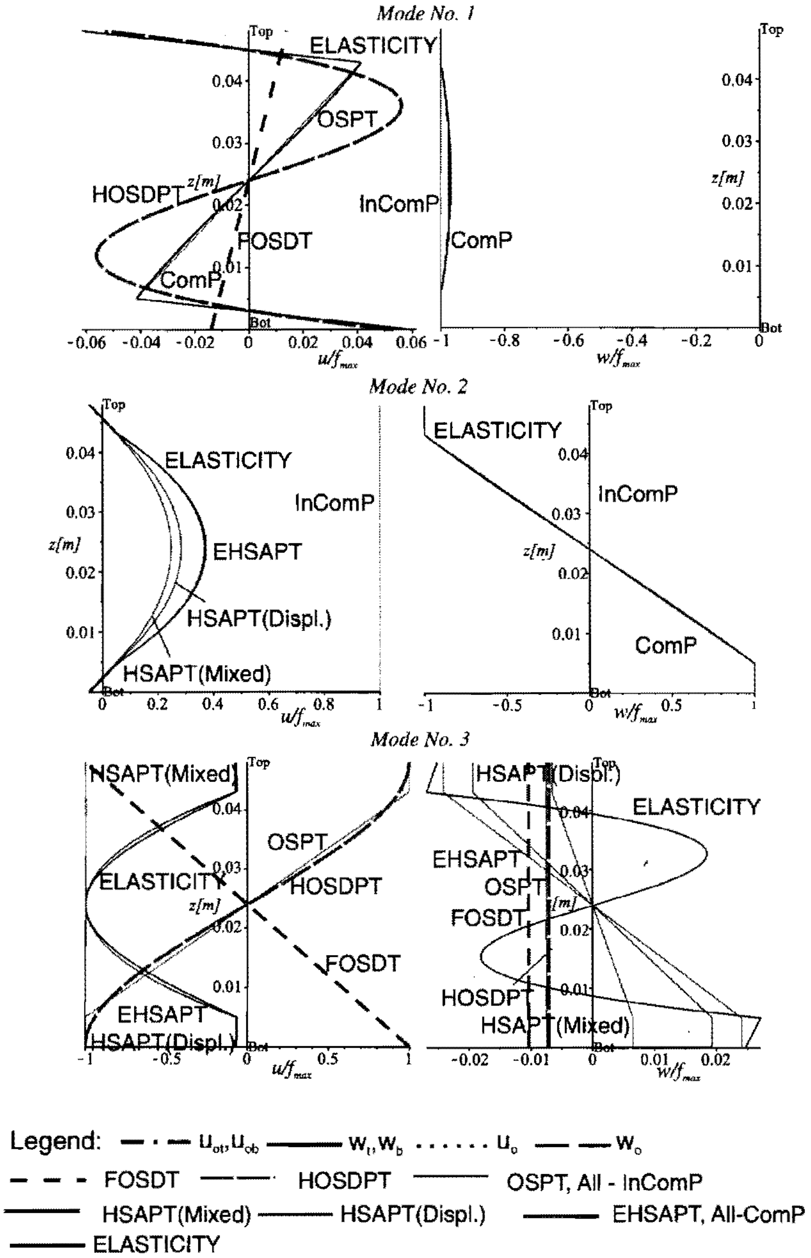

Figure 6. Three eigenmodes of stresses of the first half-wave number through depth of panel for various computational models and the elasticity solution.
FOSDT: first-order shear deformable theory; HOSDT: high-order shear deformable theory; OSPT: ordinary sandwich panel theory; HSAPT: high-order sandwich panel theory; EHSAPT: extended high-order sandwich panel theory.

Summary and conclusions

In this article, a numerical study based on a rigorous systematic analysis of the problem of free vibrations of sandwich panels with incompressible core of layered-wise core model to ESLs was presented. The model formulations are based on Hamilton's principle with appropriate kinematic relations of small deformations. It includes three types of computational models with an incompressible core: the ordinary sandwich model (OSPT) that uses a zig-zag in-plane displacement pattern through its depth; the first-order shear deformable model by Mindlin–Reissner (FOSDT) and the high-order shear deformable model by Reddy that uses a third-order polynomial distribution (HOSDT) replacing the layered construction with an ESL. The benchmark is the closed-form solution of the elasticity model for the case of isotropic or orthotropic simply-supported sandwich panel. The equations of motion that consists of ODEs for all models are valid for any type of layout of the sandwich panel and to any boundary conditions. In all models the mass and the stiffness matrices have been derived for the particular case of a simply-supported panel of any construction of the sandwich panel for comparison with the elasticity solution.

The numerical study uses a particular sandwich panel setup that has been used for blast response in the University of Rhode Island [17] with some modification. Here, the study looked into the response of a light and a heavy core, eigenfrequencies and modes of first and second half-wave numbers and comparison with elasticity and FE results. The results reveal that the first mode can be detected accurately by all models while the higher ones can be detected correctly but not for the higher ones. Part I results show that the HSAPT (displacement) and the EHSAPT models yield accurate results in the higher modes also. The introduction of the heavy core with larger moduli of elasticity and shear and specific weight follows the same trends. The correlations with eigenmodes longitudinally and through the depth of the panel are similar in the first mode for all models. They are quite different from the elasticity solution for the higher modes.

The comparison between the various computational models and the elasticity solution reveals that the EHSAPT and the HSAPT with the displacement formulation yield accurate results in terms of eigenfrequencies and eigenmodes, see Part I. Hence, in the case of a sandwich panel with a general construction layout and general boundary conditions the incompressible models yields in accurate results for the higher modes. Thus, when layered panels are involved the HSAPT or EHSAPT formulation should be used for a general layout and boundary conditions.

Acknowledgement

The finite elements runs of ADINA have been conducted by Prof. I Sheinman of Technion. His assistance is gratefully acknowledged.

Conflict of interest

None declared.

Funding

The research has been supported by the Office of Naval Research [Grant N00014-11-1-0597] and the Ashtrom Engineering Company that supports the professorship chair of Prof. Frostig. The ONR program manager was Dr Yapa DS Rajapakse. His interest, encouragements and the financial support received are gratefully acknowledged.

References

1. Allen HG. *Analysis and design of structural sandwich panels*. London: Pergamon Press, 1969.
2. Plantema FJ. *Sandwich construction*. New York: John Wiley and Sons, 1966.
3. Zenkert D. *An introduction to sandwich construction*. London: Chameleon Press Ltd., 1995.
4. Vinson JR. *The behavior of sandwich structures of isotropic and composite materials*. Lancaster: Technomic Publishing Co. Inc, 1999.
5. Noor AK, Burton WS and Bert CW. Computational models for sandwich panels and shells. *Appl Mech Rev* 1966; 49: 155–199.
6. Mindlin RD. Influence of transverse shear deformation on the bending of classical plates. *Trans ASME J Appl Mech* 1951; 8: 18–31.
7. Reddy JN. *Energy and variational methods in applied mechanics*. New York: John Wiley and Sons, Inc., 1984.
8. Reddy JN. A review of refined theories of laminated composite plate. *Shock Vib Digest* 1993; 22(7): 3–17.
9. Kant T and Mallikarjuna. A high-order theory for free vibration of unsymmetrically laminated composite and sandwich plates-finite element evaluation. *Comput Struct* 1989; 32(5): 1125–1132.
10. Senthilnathan NR, Lim SP, Lee KH, et al. Vibration of laminated orthotropic plates using a simplified higher-order displacement theory. *Compos Struct* 1988; 10: 211–229.
11. Kant T and Swaminathan K. Analytical solution for free vibrations for laminated composite and sandwich plates based on a higher-order refined theory. *Compos Struct* 2001; 53: 73–85.
12. Meunier M and Shenoi RA. Dynamic analysis of composite sandwich plates with damping modelled using high-order shear deformation theory. *Compos Struct* 2001; 54: 243–254.
13. Nayak AK, Moy SSJ and Shenoi RA. Free vibration analysis of composite sandwich plates based Reddy's higher-order theory. *Compos Part B: Eng* 2002; 33: 505–519.
14. Bardell NS, Dundson JM and Langley RS. Free vibration analysis of coplanar sandwich panels. *Compos Struct* 1997; 38(1-4): 463–475.
15. Lee LJ and Fan YJ. Bending and vibration analysis of composite sandwich plates. *Comput Struct* 1996; 60(1): 103–112.
16. Frostig Y. Classical and high-order computational models in the analysis of modern sandwich panels. *Compos Part B: Eng* 2003; 34(1): 83–100.
17. Gardner N, Wang E, Kumar P, et al. Blast mitigation in a sandwich composite using graded core and polyurea interlayer. *Exp Mech* 2012; 52(2): 119–133.