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Effect of an Elastic Foundation on the Buckling and Postbuckling of Delaminated Composites Under Compressive Loads

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Introduction

Consider a composite boxed beam filled with a soft elastic medium such as foam or a sandwich beam consisting of two fiber-reinforced sheets separated by a low stiffness core. A bending load on these structures is equivalent to a compressive force on one face and a tensile force on the other; furthermore, a delamination may be present on the compressively loaded composite face. In those cases the composite face rests on an elastic "foundation" which imposes reaction forces on the beam that are proportional to the deflection of the "foundation."

Studies of the delamination problem have been undertaken for the usual case without the elastic foundation (e.g., Chai et al., 1981; Yin et al., 1986; Bottega and Maewal, 1983). In a recent study by Kardomateas and Schmueser (1987) the perturbation technique was used to obtain an analytical expression for the initial postbuckling deflections. This solution is extended here for the case of a beam/plate on an elastic foundation. Analytical solutions for the critical load and the initial postbuckling behavior will be derived.

Analysis

The configuration consists of a homogeneous, orthotropic beam-plate of thickness T, length L and unit width, containing a single delamination of length $\ell = 2a$ and at depth H from the top surface of the plate. The plate has a permanently attached Winkler-type elastic foundation. Over the delamination region, the laminate consists of two parts, the part above the delamination, of thickness H, referred to as the "upper" part, and the part below the delamination, of thickness $T-H$, referred to as the "lower" part. The remaining laminate outside the delamination interval and of thickness T is referred to as the "base" laminate. Local coordinate systems with the origin at the left end of each part are assumed. These parts have a common section referred to as the "interface section." The corresponding axial and shearing forces and moments at this section for the different parts are denoted by P_i, V_i, M_i.

Although the differential equation for the deflections of the

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upper delaminated layer have the usual form (e.g., Kardomateas and Schmueser, 1987), for the lower part and the base plate there is an additional term due to the elastic foundation. In terms of the modulus of the foundation, β:

$$D_i \frac{d^4 y_i}{dx^4} + P_i \frac{d^2 y_i}{dx^2} = -\beta y_i, \tag{1}$$

where $D_i = E_1 t_i^3/[12(1-\nu_{13}\nu_{31})]$ are the bending stiffnesses (E_1 is the modulus of elasticity in the axial $\equiv 1$ direction; ν_{13}, ν_{31} are Poisson's ratio's where 3 is the in-plane transverse direction and t_i is the thickness of the corresponding part). A condition of common deflection ζ at the interface section should be satisfied:

$$y_u|_{x=0,\ell} = y_l|_{x=0,\ell} = y_b|_{x=\ell_0} = \zeta. \tag{2}$$

Force and moment equilibrium at this section give:

$$P_u + P_l = P_b = P;\ V_u + V_l = V_b;\ M_u + M_l + P_u\left(\frac{T-H}{2}\right) - P_l\left(\frac{H}{2}\right) = M_b. \tag{3}$$

Furthermore, the shortening due to the deflections of the upper and lower parts should be geometrically compatible, which is expressed as:

$$(1-\nu_{13}\nu_{31})\frac{P_u\ell}{A_u E_1} - (1-\nu_{13}\nu_{31})\frac{P_l\ell}{A_l E_1} + \frac{1}{2}\int_0^\ell y_u'^2 dx - \frac{1}{2}\int_0^\ell y_l'^2 dx = T y_u'|_{x=0}, \tag{4}$$

where A_u, A_l are the cross-sectional areas of the upper and lower part.

Buckling. The deflection and load quantities at each part are developed into ascending perturbation series with respect to the angle at the interface section, ϕ.

$$y_i(x) = \phi y_{i,1}(x) + \phi^2 y_{i,2}(x) + \ldots;\ P_i = P_{i,0} + \phi P_{i,1} + \phi^2 P_{i,2} + \ldots \tag{5}$$

$$M_i = \phi M_{i,1} + \phi^2 M_{i,2} + \ldots;\ V_i = \phi V_{i,1} + \phi^2 V_{i,2} + \ldots$$

By this definition, at the interface section:

$$y'_{i,1} = 1;\ y'_{i,2} = y'_{i,3} = \ldots = 0. \tag{6}$$

Substituting equations (5) into the differential equation and equating like powers of ϕ leads to a set of linear differential equations and boundary conditions for each part. In the first approximation the terms in the first power of o are equated. For the resulting equation, assuming a solution of the form $y = e^{\gamma x}$ gives for γ either a purely imaginary number or a complex one with a real and an imaginary part, rendering the solution in terms of trigonometric only or a combination of trigonometric and hyperbolic functions depending on the magnitude of the modulus of the foundation.

Define

$$k_{i,0}^2 = P_{i,0}/D_i;\ \lambda_i = \beta/D_i. \tag{7}$$

The solution for the lower part for $k_{l,0}^4 > 4\lambda_l$ is given as:

$$y_{l,1} = \sum_{j=1,2} c_{1j}\cos(\gamma_j a - \gamma_j x);$$

$$\gamma_{1,2} = \sqrt{|(-k_{l,0}^2 \pm \sqrt{k_{l,0}^4 - 4\lambda_l})/2|}. \tag{8}$$

For $k_{l,0}^4 < 4\lambda_l$, the solution to the first order equation is

$$y_{l,1} = c_{11}\cosh(\gamma_1 a - \gamma_1 x)\cos(\gamma_2 a - \gamma_2 x) + c_{12}\sinh(\gamma_1 a - \gamma_1 x)\sin(\gamma_2 a - \gamma_2 x), \tag{9}$$

where γ_1 and γ_2 are defined as follows

$$r = \sqrt{\lambda_l};\ \theta = \arccos|-k_{l,0}^2/(2r)|;\ \gamma_1 = \sqrt{r}\cos(\theta/2);\ \gamma_2 = \sqrt{r}\sin(\theta/2). \tag{10}$$

The constants c_{1j} are found from equations (6) in terms of the common deflection ζ_1 from equation (2). The first order end shear can be expressed in the form

$$V_{l,1} = -D_l(y'''_{l,1} + k_{l,0}^2 y'_{l,1})|_{x=0} = V^c_{l,1} + \zeta_1 V^f_{l,1}, \tag{11}$$

and the first order end moment is

$$M_{l,1} = -D_l y''_{l,1}|_{x=0} = M^c_{l,1} + \zeta_1 M^f_{l,1}, \tag{12}$$

where the quantities $V^c_{l,1}$, $V^f_{l,1}$, $M^c_{l,1}$, $M^f_{l,1}$ are given in terms of γ_j. Analogous quantities are found for the base plate (subscript b).

The condition of shear equilibrium (3), written for the first order terms, produces an expression for ζ_1 (note that $V_{u,1} = 0$),

$$\zeta_1 = (V^c_{l,1} - V^c_{b,1})/(V^f_{b,1} - V^f_{l,1}). \tag{13}$$

Writing the associated moment equilibrium and compatibility equations (3), (4), for the terms in ϕ^1, substituting the end moments from equation (12) and the quantity ζ_1 from equation (13), and eliminating the quantity $P_{l,1}H/2 - P_{u,1}(T-H)/2$, gives an equation for the critical load, P_0 (characteristic equation), as follows

$$D_u k_{u,0}\cot k_{u,0}a + M^c_{l,1} - M^c_{b,1} + (V^c_{l,1} - V^c_{b,1})(M^f_{l,1} - M^f_{b,1})/(V^f_{b,1} - V^f_{l,1}) = -TE_1 H(T-H)/[4a(1-\nu_{13}\nu_{31})]. \tag{14}$$

Postbuckling. When the terms in ϕ^2 are equated, the following differential equation for the lower part and the base plate is obtained from equation (1):

$$D_l y^{(4)}_{l,2} + P_{l,0} y''_{l,2} + \beta y_{l,2} = -P_{l,1} y''_{l,1}. \tag{15}$$

For $k_{l,0}^4 > 4\lambda_l$, with the definition (8) for γ_j, the solution for the lower part is:

$$y_{l,2} = \sum_{j=1,2} c_{2j}\frac{P_{l,1}}{D_l}\cos(\gamma_j a - \gamma_j x) + b_{2j}\frac{P_{l,1}}{D_l}(x-a)\sin(\gamma_j a - \gamma_j x);\ b_{2j} = \frac{(-1)^j c_{1j}\gamma_j}{2(\gamma_2^2 - \gamma_1^2)}. \tag{16}$$

For $k_{l,0}^4 < 4\lambda_l$, the solution to equation (15), and with the definition (10) for γ_j, is found to be:

$$\begin{aligned} y_{l,2} = {} & c_{21}\frac{P_{l,1}}{D_l}\cosh(\gamma_1 a - \gamma_1 x)\cos(\gamma_2 a - \gamma_2 x) \\ & + c_{22}\frac{P_{l,1}}{D_l}\sinh(\gamma_1 a - \gamma_1 x)\sin(\gamma_2 a - \gamma_2 x) + \\ & + b_{21}\frac{P_{l,1}}{D_l}(x-a)\cosh(\gamma_1 a - \gamma_1 x)\sin(\gamma_2 a - \gamma_2 x) \\ & + b_{22}\frac{P_{l,1}}{D_l}(x-a)\sinh(\gamma_1 a - \gamma_1 x)\cos(\gamma_2 a - \gamma_2 x), \end{aligned}$$

$$b_{2j} = \frac{(\gamma_2^2 - \gamma_1^2)[c_{12}\gamma_{3-j} + (-1)^j c_{11}\gamma_j] + 2\gamma_1\gamma_2[c_{11}\gamma_{3-j} + (-1)^{3-j} c_{12}\gamma_j]}{8\gamma_1\gamma_2(\gamma_1^2 + \gamma_2^2)}. \tag{17}$$

The constants c_{2j} can be found from equation (6) in terms of ζ_2. Furthermore, the second order end shear can be expressed as:

$$V_{l,2} = -D_l(y'''_{l,2} + k_{l,0}^2 y'_{l,2})|_{x=0} = P_{l,1}V^g_{l,2} + \zeta_2 V^f_{l,2}, \tag{18}$$

and the second order moment is written in the form

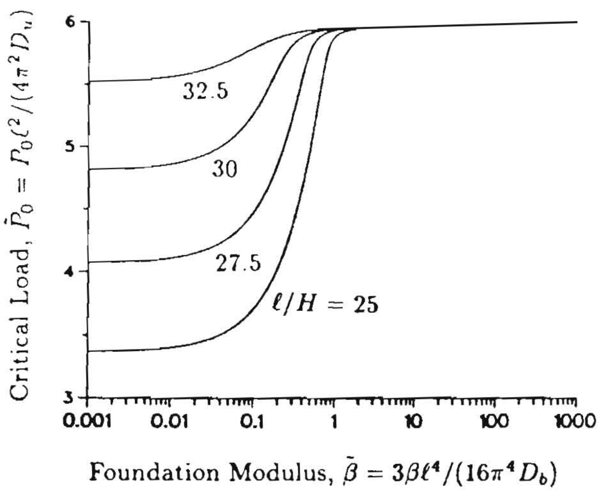

Fig. 1 Critical (buckling) force versus foundation modulus for a set of delamination lengths

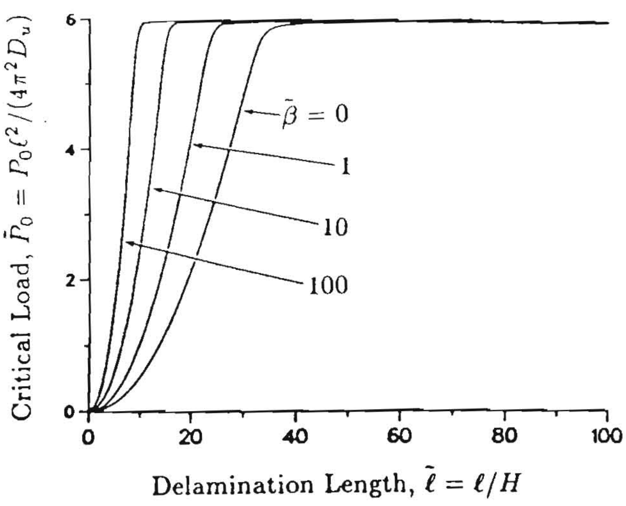

Fig. 2 Critical load versus delamination length for a set of foundation moduli

$$M_{l,2} = -D_l y''_{l,2}|_{x=0} = P_{l,1} M^g_{l,2} + \zeta_2 M^f_{l,2}, \tag{19}$$

where the quantities $V^g_{l,2}$, $V^f_{l,2}$, $M^g_{l,2}$, $M^f_{l,2}$ are given in terms of γ_j. Analogous quantities are found for the base plate (subscript b). An additional quantity needed to solve the problem is the shortening due to the first order deflections, which apears when equation (4) is written for the terms in ϕ^2, $S_l = 1/2 \int_0^\ell y'^2_{l,1} dx$, and which is already known since the first order solution has been obtained.

Now the shear condition (3) allows determining ζ_2 in terms of the first order (yet unknown) forces (note that $V_{u,2} = P_{u,1}$) as follows:

$$\zeta_2 = \frac{V^g_{b,2} - 1}{V^f_{l,2} - V^f_{b,2}} P_{u,1} + \frac{V^g_{b,2} - V^g_{l,2}}{V^f_{l,2} - V^f_{b,2}} P_{l,1}. \tag{20}$$

Writing the moment equilibrium equation (3) and the geometric compatibility equation (4) for the terms in ϕ^2, expressing the second order moments at the interface from equation (19), in terms of the (yet undetermined) first order end forces, taking into account equation (20), and eliminating the quantity $P_{l,2}H/2 - P_{u,2}(T-H)/2$ gives the following equation for the first order forces $P_{u,1}$ and $P_{l,1}$:

$$\left[M^g_{u,2} - M^g_{b,2} + \frac{V^g_{b,2} - 1}{V^f_{l,2} - V^f_{b,2}} (M^f_{l,2} - M^f_{b,2})\right] P_{u,1}$$
$$+ \left[M^g_{l,2} - M^g_{b,2} + \frac{V^g_{b,2} - V^g_{l,2}}{V^f_{l,2} - V^f_{b,2}} (M^f_{l,2} - M^f_{b,2})\right] P_{l,1} =$$
$$= \left(\frac{2k_{u,0}a - \sin 2k_{u,0}a}{4k_{u,0}\sin^2 k_{u,0}a} - S_l\right) \frac{E_1 H(T-H)}{4a(1-\nu_{13}\nu_{31})}. \tag{21}$$

The second equation needed for finding $P_{u,1}$, $P_{l,1}$ is the first order moment equilibrium equation at the interface, namely,

$$P_{l,1}H/2 - P_{u,1}(T-H)/2 = F_0(P_{u,0}, P_{l,0}), \tag{22}$$

where F_0 is the left-hand side of equation (14) which depends only on the zero order quantities. The above system of linear equations allows finding $P_{u,1}$ and $P_{l,1}$ and hence the first order applied end force $P_1 = P_{u,1} + P_{l,1}$. The solution to higher order terms can be obtained in a similar fashion.

The postcritical characteristics are studied next. For this purpose we use the expression derived by Yin and Wang (1984) for the energy release rate of a one-dimensional delamination in terms of the axial forces and bending moments acting across the various cross sections adjacent to the tip of the delamination (these quantities are directly determined from the above postbuckling solution).

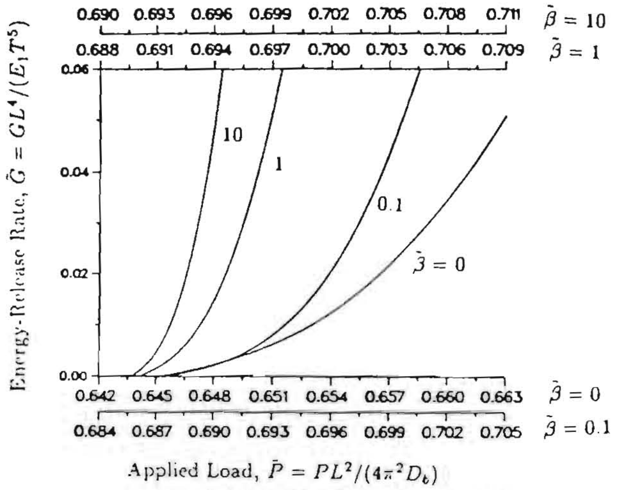

Fig. 3 **Strain energy release rate versus applied compressive force during the initial postbuckling stage for a delamination length $\ell/H = 40$ and a set of foundation moduli. The different load scales are of the same length and correspond to the different initial buckling loads.**

Discussion of Results

Numerical examples are presented for the case of a clamped-clamped plate with $H/T = 1/6$, $L/H = 200$. Figure 1 shows the variation of the critical load, normalized with respect to the Euler load for the delaminated layer, $\bar{P}_0 = P_0\ell^2/(4\pi^2 D_u)$, with the foundation modulus which is normalized as $\bar{\beta} = 3\beta\ell^4/(16\pi^4 D_b)$ (on a logarithmic scale). As $\bar{\beta}$ increases, the critical load increases, the effect being bigger on the smaller delaminations. Notice that there is a small range of values for the foundation modulus, for which the critical load undergoes a rather significant increase. Two limiting cases are the "global" buckling, characterized by buckling of the composite beam-plate as a whole, and "local" buckling, characterized by deflections of only the delaminated layer, the rest of the plate remaining flat (typical of long and thin delaminations). Figure 2 shows the variation of the critical load with delamination length for a set of values for $\bar{\beta}$. As $\bar{\beta}$ increases, the curves are shifted to the left, indicating the attainment of loads similar in magnitude to the local buckling ones for smaller delamination lengths.

The variation during the initial postbuckling stage of the normalized strain energy release rate, $\bar{G} = G/(E_1 T^5/L^4)$, and applied load normalized with respect to the Euler load for the entire beam with no elastic foundation, $\bar{P} = PL^2/(4\pi^2 D_b)$, is plotted in Fig. 3 for the case of delamination length $\ell/H=40$ and for a set of values of the foundation modulus. Since the critical load changes with $\bar{\beta}$, different scales (of the same length) are used on the load axis. The important thing to observe is that the curves are steeper for a larger $\bar{\beta}$. This increased slope means that delamination growth will occur sooner and that there will be potentially more energy absorbed since the energy released per unit applied load is larger.

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An Approximate Solution of the Axisymmetric von Karman Equations for a Point-Loaded Circular Plate

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Introduction

A closed-form solution of the axisymmetric von Karman equations or their equivalent for a point-loaded circular plate continues to evade researchers. Various kinds of approximate solutions now exist. See Chia (1980). The most popular of these (Vol'mir, 1956 or Timoshenko and Woinowsky-Krieger, 1959) assumes that the *shape* of the plate does not change from that predicted by linear theory even when deflections are large. The load required to produce a given deflection, however, becomes higher as membrane effects become important. Various series-form solutions have also been employed, including perturbation solutions in terms of load (Stippes and Hausrath, 1952; Cherepy, 1960) or central deflection (Chien and Yeh, 1954; Schmidt, 1968). Frakes and Simmonds (1985) used the symbol manipulating program MACSYMA to generate asymptotic solutions, the convergence of which was then improved using Aitken-Shanks transformations. Berger (1955) proposed a modification to the von Karman equations which led to numerous new plate solutions. The axisymmetric point-loaded plate problem was solved by Basuli (1961) using Berger's approach. A numerical technique developed by Brodland (1987) does not rely on the simplifications inherent in the von Karman equations, is valid for arbitrarily large strains and rotations, and thus allows a highly accurate reference solution to be calculated. Unfortunately, those of the above solutions which are for clamped plates have drawbacks which limit their usefulness. Many involve long and complicated mathematical expressions, some produce spurious results when loads are high, and others require considerable computation.

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The approximate solution presented here consists of general equations which are easy to use and which compare well with other solutions including the previously unpublished numerical results of Brodland. The transverse deflection is given by an assumed expression which contains a single parameter associated with plate shape. This parameter, β, is determined by minimizing a shear-related residual. Thus, the present analysis takes advantage of the ease with which assumed-form solutions can be used, while ensuring that the *shape change* which occurs with increasing load can be accommodated.

Analysis

Consider a thin, clamped plate of radius a and thickness h made from an isotropic, linearly elastic material (Young's modulus E, Poisson's ratio ν) and subjected to a central point load p. When transverse deflections w are of the order of the plate thickness, deformation is governed by the Kirchhoff nonlinear plate theory as embodied in the two, fourth-order von Karman equations. When deformations are axisymmetric, it is sometimes convenient to use the mathematically equivalent set of three equations given by Way (1934). These equations of axial equilibrium, radial equilibrium and compatibilty can be written, respectively, in dimensionless form as

$$\frac{d}{dR}\left[\frac{1}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right)\right]=\frac{P}{2\pi R}+S_r^m\frac{dW}{dR}$$

$$\frac{d}{dR}(R\,S_r^m)-S_t^m=0 \tag{1}$$

$$R\frac{d}{dR}(S_r^m+S_t^m)+6(1-\nu^2)\left(\frac{dW}{dR}\right)^2=0$$

where dimensionless radial position, transverse deflection and centric load are given by

$$R=\frac{r}{a},\ W=\frac{w}{h},\ P=\frac{pa^2}{Dh}, \tag{2}$$

dimensionless radial and circumferential (hoop) stresses are

$$S_r^m=\frac{\sigma_r^m a^2 h}{D},\ S_t^m=\frac{\sigma_t^m a^2 h}{D}, \tag{3}$$

and

$$D=\frac{Eh^3}{12(1-\nu^2)}. \tag{4}$$

For a clamped plate, the associated boundary conditions representing zero dimensionless transverse deflection, slope, and radial deflection at the clamped edge are

$$W_{|R=1}=0,\ \frac{dW}{dR}{}_{|R=1}=0 \text{ and } U_{|R=1}=0 \tag{5}$$

where

$$U=\frac{ua}{h^2}. \tag{6}$$

The expression chosen for the transverse deflection is

$$W(R)=\frac{W_o}{\beta-2}(2R^\beta-\beta R^2+\beta-2) \tag{7}$$

where W_o is the center deflection and β is a parameter which affects the plate shape. A related form was used by Nadai (1925) to analyze uniformly-loaded plates. The expression given in equation (7) satisfies the boundary conditions $(5)_1$ and $(5)_2$ associated with the transverse deflection W for all values of W_o and β. In addition, it is easy to show using l'Hopital's rule that it approaches the well known linear solution as $\beta\rightarrow 2$; i.e.,