Theory of Elasticity of Filament Wound Anisotropic Ellipsoids With Specialization to Torsion of Orthotropic Bars

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The problem of determining the state of stress in a homogeneous ellipsoidal body (as can be produced by filament winding on a mandrel of elliptical cross-section) is formulated. The material is assumed to possess curvilinear anisotropy, referred to the coordinate system that is inherent to the geometry of the filament wound body. The differential equations that govern the stress functions are derived for the general anisotropy and loading cases. Then the results for the special case of torsion of an orthotropic body are derived.

Introduction

The theory of elasticity of an anisotropic body has been developed up to now for the cases of rectilinear (referred to a cartesian coordinate system) or cylindrical (referred to a polar coordinate system) anisotropy (Lekhnitskii, 1963). The latter one is the underlying theory for studying the mechanics of filament wound circular cylinders (see e.g., Sherrer, 1967; Pagano and Halpin, 1968).

In fact, filament winding has been employed up to now in constructing mostly fiber reinforced composite circular cylinders. This manufacturing process certainly can be applied to parts of other cross-sectional shapes. Such a case is an elliptical cross-section with potential applications in components of automotive suspension systems. Therefore, there is a need for predicting the distribution of stress that these parts experience in service. Obviously, interlaminar shear constitutes a major limitation since the shear strength is limited by the bonding between the layers. By proper design, the values of interlaminar shear can be minimized. A formulation of the mechanics of those parts can also be useful in modelling the complex filament winding process.

Consider the ellipsoid of Fig. 1, produced by filament winding on an elliptical mandrel. It is obvious that one cannot neglect the inherent geometry of the curved layers in considering the elasticity. We should choose a system of curvilinear coordinates so that the coordinate directions at each point coincide with the directions which are equivalent in the sense of the elastic properties. Such a system in this case is the ($\hat{\xi}$, $\hat{\eta}$, $\hat{z}$), where $\hat{\xi}$ is the direction normal to the layers (thicknesswise direction), $\hat{\eta}$ is the tangent to the periphery and $\hat{z}$ is along the ellipsoid axis. In this homogeneous anisotropic body all the directions $\hat{\xi}$, $\hat{\eta}$, $\hat{z}$ which are drawn through different points are equivalent with respect to the elastic properties. We shall refer to this type of curvilinear anisotropy as "elliptical anisotropy." It should be noted that since the body that is produced by filament winding on the elliptical mandrel has constant thickness, the outer bounding surface is *not* a true ellipse, as opposed to the internal one; in the limit of infinite thickness the outer contour is a circle. This is also a unique feature of the present problem shared in general by filament wound bodies.

In this work the mechanics of an elliptically anisotropic body will be developed. First, let us assume that: (1) the body possesses elliptical anisotropy of the most general kind; (2) the stresses act on the planes normal to the ellipsoid axis and do not vary along the generator; and (3) there are no body forces. After deriving the governing equations, the solution is produced for the special case of torsion of an orthotropic ellipsoid. In developing the theory, formulas from the differential geometry are used (see, for example do Carmo, 1976) to express the unit vectors, element of area, etc., in terms of the curvilinear coordinates.

Formulation

Consider an ellipse with semiaxes $c \cosh a$ and $c \sinh a$ (foci at $x = \pm c$). This is the cross-section of the mandrel. On a cartesian coordinate system, the filament wound body is defined in terms of the curvilinear coordinates ξ and η (Fig. 1),

$$x = c \cosh a \cos \eta + \xi \frac{c}{h} \sinh a \cos \eta \qquad (1a)$$

$$x = c \sinh a \sin \eta + \xi \frac{c}{h} \cosh a \sin \eta \qquad (1b)$$

where

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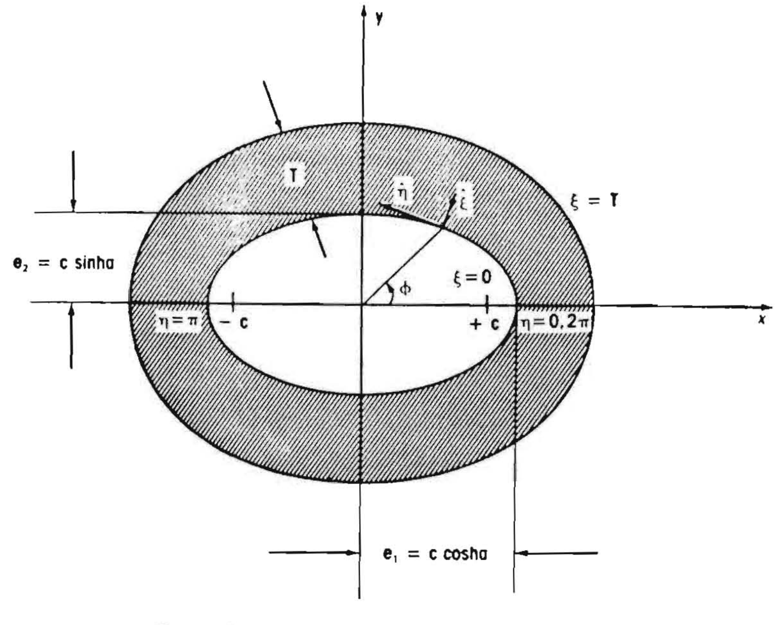

Fig. 1 Cross-section of the filament wound ellipsoid

$$h = h(\eta) = c\sqrt{\sinh^2 a + \sin^2 \eta} \tag{2}$$

Different values of ξ correspond to different layers on the elliptical mandrel. So ξ varies from 0 to T, where T is the total thickness of the body. On any of these layers ξ is constant and η varies through a range of 2π. The above equations describe the geometry in accordance with the features of the filament wound body (notice again that because of the constant thickness the outer bounding surface is not a true ellipse, as opposed to the internal one).

The components of stresses are assumed to be functions only of ξ and η. We shall deduce equations which enable us to determine the stresses and displacements in the general case under consideration.

Denote

$$q = \left[\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right]^{1/2} = h + \xi\,\frac{c^2 \sinh 2a}{2h^2} \tag{3}$$

The equations of equilibrium of a body in which the stresses do not depend on z will have the form:

$$q\sigma_{\xi\xi,\xi} + \tau_{\xi\eta,\eta} + \frac{\partial q}{\partial \xi}(\sigma_{\xi\xi} - \sigma_{\eta\eta}) = 0 \tag{4a}$$

$$\sigma_{\eta\eta,\eta} + q\tau_{\xi\eta,\xi} + 2\,\frac{\partial q}{\partial \xi}\,\tau_{\xi\eta} = 0 \tag{4b}$$

$$(q\tau_{\xi z}),\xi + \tau_{\eta z,\eta} = 0 \tag{4c}$$

The stresses and strains will be referred to the curvilinear coordinate system ($\hat{\xi}$, $\hat{\eta}$, $\hat{z}$). In this way, the infinitesimally small elements of the body bounded by three pairs of corresponding coordinate surfaces will be identical. The stress-strain equations are given by

$$\epsilon_{\xi\xi} = a_{11}\sigma_{\xi\xi} + a_{12}\sigma_{\eta\eta} + a_{13}\sigma_{zz} + a_{14}\tau_{\eta z} + a_{15}\tau_{\xi z} + a_{16}\tau_{\xi\eta} \tag{5a}$$

$$\epsilon_{\eta\eta} = a_{12}\sigma_{\xi\xi} + a_{22}\sigma_{\eta\eta} + \ldots + a_{26}\tau_{\xi\eta} \tag{5b}$$

$$\ldots$$

$$\gamma_{\xi\eta} = a_{16}\sigma_{\xi\xi} + a_{26}\sigma_{\eta\eta} + \ldots + a_{66}\tau_{\xi\eta} \tag{5f}$$

The right sides of the last six equations (5) are functions only of ξ and η. Here a_{ij} are the elastic constants of a body with elliptical anisotropy and the strains $\epsilon_{\xi\xi}$, $\epsilon_{\eta\eta}$, . . ., $\gamma_{\xi\eta}$ are connected with the components of displacement by the following formulas:

$$\epsilon_{\xi\xi} = u_{\xi,\xi} \tag{6a}$$

$$\epsilon_{\eta\eta} = \frac{1}{q}u_{\eta,\eta} + \frac{1}{q}\,\frac{\partial q}{\partial \xi}u_\xi \tag{6b}$$

$$\epsilon_{zz} = u_{z,z} \tag{6c}$$

$$\gamma_{\eta z} = u_{\eta,z} + \frac{1}{q}u_{z,\eta} \tag{6d}$$

$$\gamma_{\xi z} = u_{\xi,z} + u_{z,\xi} \tag{6e}$$

$$\gamma_{\xi\eta} = \frac{1}{q}u_{\xi,\eta} + u_{\eta,\xi} - \frac{1}{q}\,\frac{\partial q}{\partial \xi}u_\eta \tag{6f}$$

Let us introduce the notation

$$D = a_{13}\sigma_{\xi\xi} + a_{23}\sigma_{\eta\eta} + a_{33}\sigma_{zz} + a_{34}\tau_{\eta z} + a_{35}\tau_{\xi z} + a_{36}\tau_{\xi\eta} \tag{7}$$

Integrating the third, fourth, and fifth equations of the strain-displacement equations (6), we obtain the following:

$$u_z = zD + W_0(\xi,\eta), \tag{8a}$$

$$u_\eta = -\frac{z^2}{2}\,\frac{1}{q}\,\frac{\partial D}{\partial \eta} + z\left(a_{14}\sigma_{\xi\xi} + a_{24}\sigma_{\eta\eta} + \ldots + a_{46}\tau_{\xi\eta} - \frac{1}{q}\,\frac{\partial W_0}{\partial \eta}\right) + V_0\,(\xi,\eta), \tag{8b}$$

$$u_\xi = -\frac{z^2}{2}\,\frac{\partial D}{\partial \xi} + z\left(a_{15}\sigma_{\xi\xi} + a_{25}\sigma_{\eta\eta} + \ldots + a_{56}\tau_{\xi\eta} - \frac{\partial W_0}{\partial \xi}\right) + U_0\,(\xi,\eta). \tag{8c}$$

where U_0, V_0, W_0 are arbitrary functions of ξ and η which appear as a result of integration with respect to z. It is further required that the functions (8) satisfy the remaining three equations of the strain-displacement equations (6). By substituting the expressions (8) for u_ξ, u_η, u_z and (5) for the strains into the remaining strain-displacement equations and equating equal powers of z yields the following equations for D:

$$\frac{\partial^2 D}{\partial \xi^2} = 0, \tag{9a}$$

$$\frac{\partial^2 D}{\partial \eta^2} + q\,\frac{\partial q}{\partial \xi}\,\frac{\partial D}{\partial \xi} - \frac{1}{q}\,\frac{\partial q}{\partial \eta}\,\frac{\partial D}{\partial \eta} = 0, \tag{9b}$$

$$\frac{\partial^2 D}{\partial \xi \partial \eta} - \frac{1}{q} \frac{\partial q}{\partial \xi} \frac{\partial D}{\partial \eta} = 0. \tag{9c}$$

and for U_0, V_0:

$$\frac{\partial U_0}{\partial \xi} = a_{11}\sigma_{\xi\xi} + a_{12}\sigma_{\eta\eta} + \ldots + a_{16}\tau_{\xi\eta} \tag{10a}$$

$$\frac{\partial V_0}{\partial \eta} + \frac{\partial q}{\partial \xi} U_0 = q\,(a_{12}\sigma_{\xi\xi} + a_{22}\sigma_{\eta\eta} + \ldots + a_{26}\tau_{\xi\eta}) \tag{10b}$$

$$\frac{\partial U_0}{\partial \eta} + q\frac{\partial V_0}{\partial \xi} - \frac{\partial q}{\partial \xi} V_0 = q\,(a_{16}\sigma_{\xi\xi} + a_{26}\sigma_{\eta\eta} + \ldots + a_{66}\tau_{\xi\eta}) \tag{10c}$$

as well as the following equations for W_0:

$$\frac{\partial}{\partial \xi}\left(a_{15}\sigma_{\xi\xi} + a_{25}\sigma_{\eta\eta} + \ldots + a_{56}\tau_{\xi\eta} - \frac{\partial W_0}{\partial \xi}\right) = 0 \tag{11a}$$

$$\frac{\partial}{\partial \eta}\left(a_{14}\sigma_{\xi\xi} + a_{24}\sigma_{\eta\eta} + \ldots + a_{46}\tau_{\xi\eta} - \frac{1}{q}\frac{\partial W_0}{\partial \eta}\right) + \frac{\partial q}{\partial \xi}\left(a_{15}\sigma_{\xi\xi} + a_{25}\sigma_{\eta\eta} + \ldots + a_{56}\tau_{\xi\eta} - \frac{\partial W_0}{\partial \xi}\right) = 0 \tag{11b}$$

$$\frac{\partial}{\partial \eta}\left(a_{15}\sigma_{\xi\xi} + a_{25}\sigma_{\eta\eta} + \ldots + a_{56}\tau_{\xi\eta} - \frac{\partial W_0}{\partial \xi}\right) + q\frac{\partial}{\partial \xi}\left(a_{14}\sigma_{\xi\xi} + a_{24}\sigma_{\eta\eta} + \ldots + a_{46}\tau_{\xi\eta} - \frac{1}{q}\frac{\partial W_0}{\partial \eta}\right) - \frac{\partial q}{\partial \xi}\left(a_{14}\sigma_{\xi\xi} + a_{24}\sigma_{\eta\eta} + \ldots + a_{46}\tau_{\xi\eta} - \frac{1}{q}\frac{\partial W_0}{\partial \eta}\right) = 0 \tag{11c}$$

The solution to (9) is found to be:

$$D = a_{33}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B \times \left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] \tag{12}$$

and consequently, from (8*a*), (6*c*), (5*c*),

$$\sigma_{zz} = A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B \times \left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C - \frac{1}{a_{33}}(a_{13}\sigma_{\xi\xi} + a_{23}\sigma_{\eta\eta} + a_{34}\tau_{\eta z} + a_{35}\tau_{\xi z} + a_{36}\tau_{\xi\eta}) \tag{13}$$

where A, B, C, are arbitrary constants. Let us introduce the new functions U, V, W which represent the displacements accompanied by deformation. We shall also introduce the constants v_{0x}, v_{0y}, v_{0z}, ω_x, ω_y, ω_z which characterize the rigid body translation and rotation about the cartesian coordinate system:

$$U_0 = U + \frac{c}{h}(v_{0x}\sinh a\cos\eta + v_{0y}\cosh a\sin\eta) + \omega_z \frac{c^2\sin 2\eta}{2h} \tag{14a}$$

$$V_0 = V + \frac{c}{h}(-v_{0x}\cosh a\sin\eta + v_{0y}\sinh a\cos\eta) + \omega_z\left(\xi + \frac{c^2\sinh 2a}{2h}\right) \tag{14b}$$

$$W_0 = W + c\omega_x\sin\eta\left(\sinh a + \frac{\xi}{h}\cosh a\right) - c\omega_y\cos\eta\left(\cosh a + \frac{\xi}{h}\sinh a\right) + v_{0z} \tag{14c}$$

Then, from (10), (13), and (14) we obtain the equations which connect the functions U, V with the components of stresses. In terms of the reduced elastic constants β_{ij}, defined by:

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \quad i, j = 1,2,4,5,6 \tag{15}$$

these equations are:

$$\frac{\partial U}{\partial \xi} = \beta_{11}\sigma_{\xi\xi} + \beta_{12}\sigma_{\eta\eta} + \beta_{14}\tau_{\eta z} + \beta_{15}\tau_{\xi z} + \beta_{16}\tau_{\xi\eta} + a_{13}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B \times \left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] \tag{16a}$$

$$\frac{\partial V}{\partial \eta} + \frac{\partial q}{\partial \xi}U = q\,(\beta_{12}\sigma_{\xi\xi} + \beta_{22}\sigma_{\eta\eta} + \beta_{24}\tau_{\eta z} + \beta_{25}\tau_{\xi z} + \beta_{26}\tau_{\xi\eta}) + qa_{23}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right) \times \cos\eta + B\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] \tag{16b}$$

$$\frac{\partial U}{\partial \eta} + q\frac{\partial V}{\partial \xi} - \frac{\partial q}{\partial \xi}V = q\,(\beta_{16}\sigma_{\xi\xi} + \beta_{26}\sigma_{\eta\eta} + \beta_{46}\tau_{\eta z} + \beta_{56}\tau_{\xi z} + \beta_{66}\tau_{\xi\eta}) + qa_{36}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right) \times \cos\eta + B\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] \tag{16c}$$

Equations (11) hold if we formally substitute W for W_0 and the solution to the resulting equations is found to be:

$$a_{15}\sigma_{\xi\xi} + a_{25}\sigma_{\eta\eta} + \ldots + a_{56}\tau_{\xi\eta} - \frac{\partial W}{\partial \xi} = \bar{\theta}\frac{\partial h}{\partial \eta} \tag{17a}$$

$$a_{14}\sigma_{\xi\xi} + a_{24}\sigma_{\eta\eta} + \ldots + a_{46}\tau_{\xi\eta} - \frac{1}{q}\frac{\partial W}{\partial \eta} = \bar{\theta}\left(\xi + h\frac{\partial q}{\partial \xi}\right) \tag{17b}$$

where $\bar{\theta}$ is a constant associated with the amount of twisting. Taking into account (13) and (15), the above equations (17) yield the expressions between W and the components of stresses:

$$\frac{\partial W}{\partial \xi} = \beta_{15}\sigma_{\xi\xi} + \beta_{25}\sigma_{\eta\eta} + \beta_{45}\tau_{\eta z} + \beta_{55}\tau_{\xi z} + \beta_{56}\tau_{\xi\eta} + a_{35}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B \times \left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] - \bar{\theta}\frac{\partial h}{\partial \eta} \tag{18a}$$

$$\frac{1}{q}\frac{\partial W}{\partial \eta} = \beta_{14}\sigma_{\xi\xi} + \beta_{24}\sigma_{\eta\eta} + \beta_{44}\tau_{\eta z} + \beta_{45}\tau_{\xi z} + \beta_{46}\tau_{\xi\eta} + a_{34}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B \times \left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] - \bar{\theta}\left(\xi + h\frac{\partial q}{\partial \xi}\right) \tag{18b}$$

By using (8), (14), (12), and (18) we obtain the general expressions for the displacements which correspond to stresses depending only on the two coordinates ξ and η

$$u_\xi = U(\xi,\eta) - \frac{z^2}{2}\frac{1}{h} a_{33} (A \sinh a \cos\eta + B \cosh a \sin\eta) +$$

$$+ z\bar{\theta}\frac{\partial h}{\partial\eta} - z\frac{c}{h}(\omega_x \cosh a \sin\eta - \omega_y \sinh a \cos\eta) +$$

$$+ \frac{c}{h}(v_{0x} \sinh a \cos\eta + v_{0y} \cosh a \sin\eta) + \frac{\partial h}{\partial\eta}\omega_z \qquad (19a)$$

$$u_\eta = V(\xi,\eta) + \frac{z^2}{2}\frac{1}{h} a_{33} (A \cosh a \sin\eta - B \sinh a \cos\eta) +$$

$$+ z\bar{\theta}\left(\xi + h\frac{\partial q}{\partial\xi}\right)$$

$$- z\frac{c}{h}(\omega_x \sinh a \cos\eta + \omega_y \cosh a \sin\eta)$$

$$+ \frac{c}{h}(-v_{0x} \cosh a \sin\eta + v_{0y} \sinh a \cos\eta)$$

$$+ \left(\xi + h\frac{\partial q}{\partial\xi}\right)\omega_z \qquad (19b)$$

$$u_z = W(\xi,\eta) + z a_{33}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B\right.$$

$$\left.\times\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right] +$$

$$\xi\frac{c}{h}(\omega_x \cosh a \sin\eta - \omega_y \sinh a \cos\eta)$$

$$+ c(\omega_x \sinh a \sin\eta - \omega_y \cosh a \cos\eta) + v_{0z} \qquad (19c)$$

We can satisfy the equations of equilibrium (4) by introducing the two stress functions $F(\xi,\eta)$ and $\Psi(\xi,\eta)$:

$$\sigma_{\xi\xi} = \frac{1}{q^2}F_{,\eta\eta} + \frac{1}{q}\frac{\partial q}{\partial\xi}F_{,\xi} - \frac{1}{q^3}\frac{\partial q}{\partial\eta}F_{,\eta} \qquad (20a)$$

$$\sigma_{\eta\eta} = F_{,\xi\xi} \qquad (20b)$$

$$\tau_{\xi\eta} = -\frac{1}{q}F_{,\xi\eta} + \frac{1}{q^2}\frac{\partial q}{\partial\xi}F_{,\eta} \qquad (20c)$$

$$\tau_{\xi z} = \frac{1}{q}\Psi_{,\eta},\quad \tau_{\eta z} = -\Psi_{,\xi} \qquad (20d)$$

Eliminating U, V from (16) and W from (18), gives the following system of differential equations which the stress functions F and Ψ must satisfy:

$$L_4^I F + L_3^I \Psi = \frac{1}{h}\left(\frac{3\xi}{h}\frac{\partial q}{\partial\xi}\frac{\partial h}{\partial\eta} + \frac{\partial q}{\partial\eta}\right)[(a_{13}B - a_{36}A)$$

$$\times \sinh a \cos\eta - (a_{13}A + a_{36}B)\cosh a \sin\eta] -$$

$$-\frac{3}{q}\frac{\partial q}{\partial\xi}\frac{\partial h}{\partial\eta} a_{36}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B\right.$$

$$\left.\times\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right]$$

$$+ \frac{q}{h}\frac{\partial q}{\partial\xi}\{[a_{36}B - (2a_{23} - a_{13})A]\sinh a \cos\eta - [a_{36}A$$

$$+ (2a_{23} - a_{13})B]\cosh a \sin\eta\}$$

$$+ \frac{q}{h}[(a_{13}B - a_{36}A)\sinh a \sin\eta + (a_{13}A + a_{36}B)\cosh a \cos\eta]$$

$$(21a)$$

$$L_3^{II} F + L_2^{II}\Psi = \frac{\partial q}{\partial\xi} a_{34}\left[A\left(\cosh a + \frac{\xi}{h}\sinh a\right)\cos\eta + B\right.$$

$$\left.\times\left(\sinh a + \frac{\xi}{h}\cosh a\right)\sin\eta + C\right]$$

$$+ \frac{q}{h}[(a_{35}A + a_{34}B)\cosh a \sin\eta$$

$$-(a_{35}B - a_{34}A)\sinh a \cos\eta] - \bar{\theta}2q \qquad (21b)$$

L_2^{II}, L_3^I, L_3^{II}, L_4^I are the differential operators of the second, third, and fourth orders which have the form:

$$L_4^I = q^2\beta_{22}\frac{\partial^4}{\partial\xi^4} + \frac{1}{q^2}\beta_{11}\frac{\partial^4}{\partial\eta^4} + q\frac{\partial q}{\partial\xi}2\beta_{22}\frac{\partial^3}{\partial\xi^3}$$

$$+ \frac{1}{q^3}\left(q\frac{\partial q}{\partial\xi}2\beta_{16} - \frac{\partial q}{\partial\eta}6\beta_{11}\right)\frac{\partial^3}{\partial\eta^3} +$$

$$+ \frac{1}{q}\frac{\partial q}{\partial\xi}\left[\frac{\partial h}{\partial\eta}3(\beta_{16} + \beta_{26}) - q\frac{\partial q}{\partial\xi}\beta_{11}\right]\frac{\partial^2}{\partial\xi^2}$$

$$-\frac{1}{q^3}\left\{\frac{\partial q}{\partial\xi}\left(\frac{\partial q}{\partial\eta} + 2\frac{\partial h}{\partial\eta}\right)6\beta_{16}\right.$$

$$-q\left(\frac{\partial q}{\partial\xi}\right)^2(2\beta_{12} + \beta_{66}) + \left[4\frac{\partial^2 q}{\partial\eta^2}\right.$$

$$\left.\left.-\frac{15}{q}\left(\frac{\partial q}{\partial\eta}\right)^2 - 2q\left(\frac{\partial q}{\partial\xi}\right)^2\right]\beta_{11}\right\}\frac{\partial^2}{\partial\eta^2} +$$

$$+ \frac{1}{q^3}\frac{\partial q}{\partial\xi}\left\{3\frac{\partial h}{\partial\eta}\left(2\frac{\partial q}{\partial\eta} + 3\frac{\partial h}{\partial\eta}\right)\right.$$

$$\left.-q\left[3\frac{\partial^2 q}{\partial\eta^2} - q\left(\frac{\partial q}{\partial\xi}\right)^2\right]\right\}\beta_{11}\frac{\partial}{\partial\xi} +$$

$$+ \frac{1}{q^4}\left\{\left[3\frac{\partial q}{\partial\xi}\left(\frac{\partial h}{\partial\eta} + \frac{\partial q}{\partial\eta}\right)\left(\frac{\partial q}{\partial\eta} + 3\frac{\partial h}{\partial\eta}\right) -\right.\right.$$

$$\left.-q\frac{\partial q}{\partial\xi}\left(\frac{\partial^2 q}{\partial\eta^2} + 3\frac{\partial^2 h}{\partial\eta^2} - q\left(\frac{\partial q}{\partial\xi}\right)^2\right)\right]2\beta_{16} +$$

$$+ q\left(\frac{\partial q}{\partial\xi}\right)^2\left[q\frac{\partial q}{\partial\xi}2\beta_{26} - \left(\frac{\partial q}{\partial\eta} + 6\frac{\partial h}{\partial\eta}\right)(2\beta_{12} + \beta_{66})\right] -$$

$$-\left[q\frac{\partial^3 q}{\partial\eta^3} - 10\frac{\partial^2 q}{\partial\eta^2}\frac{\partial q}{\partial\eta} + \frac{15}{q}\left(\frac{\partial q}{\partial\eta}\right)^3\right.$$

$$\left.\left.+ q\left(\frac{\partial q}{\partial\xi}\right)^2\left(2\frac{\partial q}{\partial\eta} + 3\frac{\partial h}{\partial\eta}\right)\right]\beta_{11}\right\}\frac{\partial}{\partial\eta} -$$

$$-q2\beta_{26}\frac{\partial^4}{\partial\xi^3\partial\eta} + (2\beta_{12} + \beta_{66})\frac{\partial^4}{\partial\xi^2\partial\eta^2} - \frac{1}{q}2\beta_{16}\frac{\partial^4}{\partial\xi\partial\eta^3} -$$

$$-\frac{1}{q}\frac{\partial q}{\partial\eta}(2\beta_{12} + \beta_{66})\frac{\partial^3}{\partial\xi^2\partial\eta} + \frac{1}{q^2}\left[\frac{\partial q}{\partial\eta}6\beta_{16}\right.$$

$$\left.-q\frac{\partial q}{\partial\xi}(2\beta_{12} + \beta_{66})\right]\frac{\partial^3}{\partial\xi\partial\eta^2} +$$

$$+ \frac{1}{q^2}\left\{\left[\frac{\partial^2 q}{\partial\eta^2} - \frac{3}{q}\left(\frac{\partial q}{\partial\eta}\right)^2 - q\left(\frac{\partial q}{\partial\xi}\right)^2\right]2\beta_{16} +\right.$$

$$+ \frac{\partial q}{\partial\xi}\left(\frac{\partial q}{\partial\eta} + 3\frac{\partial h}{\partial\eta}\right)(2\beta_{12} + \beta_{66}) - q\left(\frac{\partial q}{\partial\xi}\right)^2 2\beta_{26}$$

$$-6\frac{\partial q}{\partial \xi}\frac{\partial h}{\partial \eta}\beta_{11}\Big\}\frac{\partial^2}{\partial\xi\partial\eta} \tag{22a}$$

$$L_3^I=-q^2\beta_{24}\frac{\partial^3}{\partial\xi^3}+\frac{1}{q}\beta_{15}\frac{\partial^3}{\partial\eta^3}+q\frac{\partial q}{\partial\xi}(\beta_{14}-2\beta_{24})\frac{\partial^2}{\partial\xi^2}$$

$$-\frac{1}{q^2}\frac{\partial q}{\partial\eta}3\beta_{15}\frac{\partial^2}{\partial\eta^2}-\frac{1}{q}\frac{\partial q}{\partial\xi}\frac{\partial h}{\partial\eta}3\beta_{46}\frac{\partial}{\partial\xi}-$$

$$-\frac{1}{q^2}\left[\frac{\partial^2 q}{\partial\eta^2}-\frac{3}{q}\left(\frac{\partial q}{\partial\eta}\right)^2\right.$$

$$\left.-q\left(\frac{\partial q}{\partial\xi}\right)^2\right]\beta_{15}\frac{\partial}{\partial\eta}+q(\beta_{25}+\beta_{46})\frac{\partial^3}{\partial\xi^2\partial\eta}-$$

$$-(\beta_{14}+\beta_{56})\frac{\partial^3}{\partial\xi\partial\eta^2}+\frac{1}{q}\left[\frac{\partial q}{\partial\eta}(\beta_{14}+\beta_{56})\right.$$

$$\left.+q\frac{\partial q}{\partial\xi}(\beta_{46}-\beta_{15})\right]\frac{\partial^2}{\partial\xi\partial\eta} \tag{22b}$$

$$L_3^{II}=-q\beta_{24}\frac{\partial^3}{\partial\xi^3}+\frac{1}{q^2}\beta_{15}\frac{\partial^3}{\partial\eta^3}-\frac{\partial q}{\partial\xi}(\beta_{14}+\beta_{24})\frac{\partial^2}{\partial\xi^2}+$$

$$+\frac{1}{q^3}\left[q\frac{\partial q}{\partial\xi}(\beta_{14}+\beta_{56})-3\frac{\partial q}{\partial\eta}\beta_{15}\right]\frac{\partial^2}{\partial\eta^2}$$

$$-\frac{1}{q^2}\frac{\partial q}{\partial\xi}\frac{\partial h}{\partial\eta}3\beta_{15}\frac{\partial}{\partial\xi}-$$

$$-\frac{1}{q^3}\left\{\frac{\partial q}{\partial\xi}\left(\frac{\partial q}{\partial\eta}+3\frac{\partial h}{\partial\eta}\right)(\beta_{14}+\beta_{56})\right.$$

$$\left.-q\left(\frac{\partial q}{\partial\xi}\right)^2\beta_{46}+\left[\frac{\partial^2 q}{\partial\eta^2}-\frac{3}{q}\left(\frac{\partial q}{\partial\eta}\right)^2\right]\beta_{15}\right\}\frac{\partial}{\partial\eta}+$$

$$+(\beta_{25}+\beta_{46})\frac{\partial^3}{\partial\xi^2\partial\eta}-\frac{1}{q}(\beta_{14}+\beta_{56})\frac{\partial^3}{\partial\xi\partial\eta^2}+$$

$$+\frac{1}{q^2}\left[\frac{\partial q}{\partial\eta}(\beta_{14}+\beta_{56})+q\frac{\partial q}{\partial\xi}(\beta_{15}-\beta_{46})\right]\frac{\partial^2}{\partial\xi\partial\eta} \tag{22c}$$

$$L_2^{II}=q\beta_{44}\frac{\partial^2}{\partial\xi^2}+\frac{1}{q}\beta_{55}\frac{\partial^2}{\partial\eta^2}+\frac{\partial q}{\partial\xi}\beta_{44}\frac{\partial}{\partial\xi}$$

$$-\frac{1}{q^2}\frac{\partial q}{\partial\eta}\beta_{55}\frac{\partial}{\partial\eta}-2\beta_{45}\frac{\partial^2}{\partial\xi\partial\eta} \tag{22d}$$

The components of stresses and displacements inside the body must be continuous and single-valued functions of the coordinates. For their determination, we must find the stress functions which satisfy equations (21) and the boundary conditions, which will next be considered.

We assume that the external tractions have components on the $\hat{\xi}$, $\hat{\eta}$ directions, denoted by Ξ_n, H_n, respectively. Then the conditions on the contour bounding the cross section (outmost and innermost layers) can be written in the following form:

$$\sigma_{\xi\xi}\cos(\hat{n},\hat{\xi})+\tau_{\xi\eta}\cos(\hat{n},\hat{\eta})=\Xi_n \tag{23a}$$

$$\tau_{\xi\eta}\cos(\hat{n},\hat{\xi})+\sigma_{\eta\eta}\cos(\hat{n},\hat{\eta})=H_n \tag{23b}$$

$$\tau_{\xi z}\cos(\hat{n},\hat{\xi})+\tau_{\eta z}\cos(\hat{n},\hat{\eta})=0 \tag{23c}$$

where $\hat{n}$ is in the direction of the exterior normal (being along the direction of $\hat{\xi}$). By integrating (23c) with respect to the arc length, $ds=qd\eta$, from a certain initial point ($s=0$) to the variable point s, and taking into account (20d), we find that the stress function Ψ is constant on the boundary:

$$\Psi=c_i \tag{24}$$

where c_i are constants corresponding to each of the boundary contours (for a simply connected region we could just set this constant equal to zero).

Now, let us consider the conditions at the ends of an ellipsoid of finite length. The conditions at the ends have the form:

$$\iint\left(\frac{\partial x}{\partial\xi}\tau_{\xi z}+\frac{1}{q}\frac{\partial x}{\partial\eta}\tau_{\eta z}\right)dS=0,\ \iint\sigma_z y dS=M_1 \tag{25a}$$

$$\iint\left(\frac{\partial y}{\partial\xi}\tau_{\xi z}+\frac{1}{q}\frac{\partial y}{\partial\eta}\tau_{\eta z}\right)dS=0,\ \iint\sigma_z x dS=M_2 \tag{25b}$$

$$\iint\left(\sigma_z dS=P_z,\right.$$

$$\iint\left[\left(\xi+h\frac{\partial q}{\partial\xi}\right)\tau_{\eta z}+\frac{\partial h}{\partial\eta}\tau_{\xi z}\right]dS=M_t \tag{25c}$$

where the integrals are taken over the entire area of the cross-section (notice that $dS=qd\xi d\eta$). Since the stresses do not depend on z, these conditions exist not only at the ends, but also in any cross-section. In the above expressions for the end loading, P_z is the resultant axial force, M_1, M_2 are the bending moments about the x, y axes, respectively, and M_t is the twisting moment.

The first equations of (25a) and (25b), which express the condition of zero resultant forces along the x and y axes, are satisfied identically. Indeed, using the equilibrium condition (4c),

$$\iint\left(\tau_{\xi z}\frac{\partial x}{\partial\xi}+\tau_{\eta z}\frac{1}{q}\frac{\partial x}{\partial\eta}\right)dS$$

$$=\iint\left\{\tau_{\xi z}\frac{\partial x}{\partial\xi}+\tau_{\eta z}\frac{1}{q}\frac{\partial x}{\partial\eta}+\frac{x}{q}[(q\tau_{\xi z})_{,\xi}+\tau_{\eta z,\eta}]\right\}dS$$

$$=\iint[(xq\tau_{\xi z})_{,\xi}+(x\tau_{\eta z})_{,\eta}]\frac{1}{q}dS$$

By transforming the double integral into a contour integral using the divergence theorem in curvilinear coordinates (contour γ), and apply (23c), we obtain:

$$\iint\left(\tau_{\xi z}\frac{\partial x}{\partial\xi}+\tau_{\eta z}\frac{1}{q}\frac{\partial x}{\partial\eta}\right)dS$$

$$=\int_\gamma x[\tau_{\xi z}\cos(\hat{n},\hat{\xi})+\tau_{\eta z}\cos(\hat{n},\hat{\eta})]\,ds=0$$

It can be proved in the same way that the first integral in (25b) is also equal to zero. Therefore, by substituting (1) and (13) in (25), we find that for a body with a bounded cross-section on the ends (and in any cross section) the following conditions must be satisfied:

$$CS-\frac{1}{a_{33}}\iint(a_{13}\sigma_{\xi\xi}+a_{23}\sigma_{\eta\eta}+a_{34}\tau_{\eta z}+a_{35}\tau_{\xi z}$$

$$+a_{36}\tau_{\xi\eta})\,qd\xi d\eta=P_z \tag{26a}$$

$$AI_1-\frac{1}{a_{33}}\iint(a_{13}\sigma_{\xi\xi}+a_{23}\sigma_{\eta\eta}+\ldots+a_{36}\tau_{\xi\eta})c$$

$$\times\left(\sinh a\sin\eta+\frac{\xi}{h}\cosh a\sin\eta\right)qd\xi d\eta=M_1 \tag{26b}$$

$$BI_2-\frac{1}{a_{33}}\iint(a_{13}\sigma_{\xi\xi}+a_{23}\sigma_{\eta\eta}+\ldots+a_{36}\tau_{\xi\eta})c$$

$$\times\left(\cosh a \cos\eta + \frac{\xi}{h}\sinh a\cos\eta\right) q d\xi d\eta = M_2 \qquad (26c)$$

$$\iint\left[\left(\xi + h\frac{\partial q}{\partial \xi}\right)\tau_{\eta z} + \frac{\partial h}{\partial \eta}\tau_{\xi z}\right] q d\xi d\eta = M_t \qquad (26d)$$

where S is the cross-sectional area and I_1, I_2 are the principal moments of interia with respect to the x and y axes which are assumed to pass through the centroid of the section (Fig. 1).

The stress functions F and Ψ which satisfy the equations (21) and the boundary conditions on the cross-section (23) will contain the four arbitrary constants $\bar{\theta}$, A, B, C. These are in turn found from the equations (26) for the end loading.

Torsion of an Orthotropic Body

If a plane of elastic symmetry normal to the z-axis exists at each point, then the coefficients a_{14}, a_{24}, a_{34}, a_{46}, a_{15}, a_{25}, a_{35}, a_{56} in the stress-strain equations (5) are equal to zero. If there are two planes of elastic symmetry (along the normal and the tangent to the periphery of each layer), then in addition to the above, the coefficients a_{16}, a_{26}, a_{36}, a_{45}, are also equal to zero. The latter is the case of an orthotropic body with elliptical anisotropy. As a consequence, in this case:

$$\beta_{14} = \beta_{24} = \beta_{45} = \beta_{46} = \beta_{15} = \beta_{25} = \beta_{56} = 0 \qquad (27)$$

The system (21) reduces to two equations: One equation involves only the function F; the other equation involves only the function Ψ.

In this case of an orthotropic body it may be more convenient to use the engineering elastic characteristics, which are denoted for the Young's moduli as E_ξ, E_η, E_z, for the Poisson's ratios as $\nu_{\xi\eta}$, $\nu_{\xi z}$, $\nu_{z\xi}$, $\nu_{z\eta}$, $\nu_{\eta z}$, and for the shear moduli as $G_{\xi\eta}$, $G_{\eta z}$, $G_{\xi z}$. The reduced elastic constants are expressed as:

$$\beta_{11} = \frac{1-\nu_{\xi z}\nu_{z\xi}}{E_\xi},\ \beta_{12} = -\frac{\nu_{\xi\eta}+\nu_{\xi z}\nu_{z\eta}}{E_\xi},\ \beta_{22} = \frac{1-\nu_{\eta z}\nu_{z\eta}}{E_\eta}, \qquad (28a)$$

$$\beta_{66} = \frac{1}{G_{\xi\eta}},\ \beta_{44} = \frac{1}{G_{\eta z}},\ \beta_{55} = \frac{1}{G_{\xi z}}. \qquad (28b)$$

The other β_{ij} are either zero or do not enter into the above formulas.

Let us consider the particular problem of torsion of a filament wound orthotropic ellipsoidal bar.

The equation for the stress function Ψ becomes:

$$q\beta_{44}\Psi_{,\xi\xi} + \frac{1}{q}\beta_{55}\Psi_{,\eta\eta} + \frac{\partial q}{\partial \xi}\beta_{44}\Psi_{,\xi}$$

$$-\frac{1}{q^2}\frac{\partial q}{\partial \eta}\beta_{55}\Psi_{,\eta} = -2\bar{\theta}q \qquad (29)$$

The constant $\bar{\theta}$ is the scaling factor that represents the magnitude of the twisting moment M_t, and thus can be found from the end condition (26d). To eliminate this constant from (30), set

$$\Psi(\xi,\eta) = \bar{\theta}\psi(\xi,\eta) \qquad (30)$$

Equation (29) then becomes

$$q\beta_{44}\psi_{,\xi\xi} + \frac{1}{q}\beta_{55}\psi_{,\eta\eta} + \frac{\partial q}{\partial \xi}\beta_{44}\psi_{,\xi}$$

$$-\frac{1}{q^2}\frac{\partial q}{\partial \eta}\beta_{55}\psi_{,\eta} = -2q \qquad (31)$$

This equation is solved in conjunction with the boundary conditions (23c) which are equivalent to enforcing a constant value of ψ on the boundary. Notice, however, that the domain is multiply connected. So the boundary condition (24) is expressed in detail:

$$\psi = c_1 \text{ at } \xi = 0,\ \psi = c_2 \text{ at } \xi = T. \qquad (32)$$

The value of ψ on the outer boundary can be chosen arbitrarily to be zero, $c_2 = 0$. The other constant, c_1, is yet unspecified. For its determination we use the requirement that the displacement u_z be single valued. Thus, if we make a circuit around the contour of the inner boundary, the following condition should be satisfied:

$$\int_0^{2\pi}\left(\frac{\partial u_z}{\partial \eta}\right)_{\xi=0} d\eta = 0 \qquad (33)$$

By (19c) the above equation is reduced to

$$\int_0^{2\pi}\left(\frac{\partial W}{\partial \eta}\right)_{\xi=0} d\eta = 0 \qquad (34)$$

Substituting (18b) for $\partial W/\partial\eta$, we obtain

$$\int_0^{2\pi}\left[h\beta_{44}(\tau_{\eta z})_{\xi=0} - \bar{\theta}\frac{c^2\sinh 2a}{2}\right] d\eta = 0 \qquad (35)$$

By using (20) and (30) the following condition is obtained for the function ψ, that allows determining the constant $\psi|_{\xi=0} = c_1$:

$$\int_0^{2\pi} h(\psi_{,\xi})_{\xi=0}\, d\eta = -\frac{c^2\pi\sinh 2a}{\beta_{44}} \qquad (36)$$

The above equation serves as another condition on ψ which must be satisfied to ensure that the displacements are single valued. Notice also that (33) and (34) hold for general anisotropy.

Having obtained $\psi(\xi,\eta)$, the constant $\bar{\theta}$ is found from the end condition (26d). In terms of the rigidity C_R defined by

$$C_R = \iint\left[-\left(\xi + h\frac{\partial q}{\partial \xi}\right)q\psi_{,\xi} + \frac{\partial h}{\partial \eta}\psi_{,\eta}\right] d\xi d\eta \qquad (37)$$

we obtain by using (20d) and (30),

$$\bar{\theta} = M_t/C_R \qquad (38)$$

Now, let us look into the physical meaning of the constant $\bar{\theta}$. If we assume pure torsion, i.e., $F=0$, then from (26b, c) we get $A = B = 0$. Then from (19a, b), it is seen that $\bar{\theta}z$ is the angle of twisting. Therefore, $\bar{\theta}$ measures the twist angle per unit length.

Finally, modelling the geometry as indicated previously, allows a direct usage of the engineering properties of the material, as these can be determined by independent tests. For example, for a unidirectional composite that is laid up at 0 deg, the η direction is the direction of the fibers, the z direction is the one at 90 deg to the fibers on the plane of the composite sheet and the ξ direction is the out-of-plane normal direction. In this manner, the elastic characteristics of the composite layers can be used directly in the above formulation.

Results and Discussion

The finite difference method was used to solve (31) and (36), together with the condition $\psi\ |_{\xi=T}=0$ (e.g., Ketter and Prawel, 1969). Two important points should be noted. First, the structure of equation (31) shows that the domain can be reduced by symmetry conditions to one quarter of the elliptical cross-section, i.e., for $0 \le \eta \le \pi/2$. Then, the values of the function ψ at all other points are given for $\pi/2 \le \eta \le \pi$ by $\psi(\xi,\eta) = \psi(\xi, \pi-\eta)$; for $\pi \le \eta \le 3\pi/2$ by $\psi(\xi,\eta) = \psi(\xi, \eta-\pi)$; and for $3\pi/2 \le \eta \le 2\pi$ by $\psi(\xi,\eta) = \psi(\xi, 2\pi-\eta)$. However, this symmetry will not exist if β_{45} is not zero, i.e., if only one plane of elastic symmetry normal to the z-axis exists, in which case, equation (31) should be modified

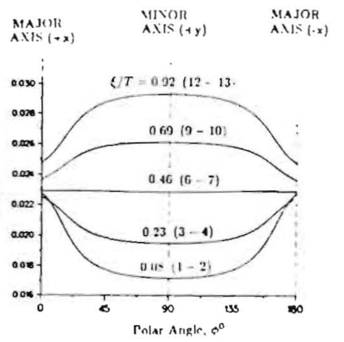

Fig. 2 Angular distribution of the shear stress $\tau_{\eta z}$ for a set of layer interfaces in a 13-ply orthotropic bar (layer numbers denoted in parentheses)

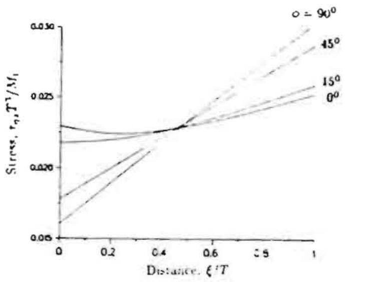

Fig. 3 Thicknesswise distribution (along ξ) of the shear stress $\tau_{\eta z}$

to include the extra term (this case arises, for example, when an orthotropic composite sheet is laid up at a different lay up angle.) In the latter case the domain can be reduced by symmetry to only one half of the cross-section, i.e., for $0 \leq \eta \leq \pi$. Then, for $\pi \leq \eta \leq 2\pi$, $\psi(\xi, \eta) = \psi(\xi, \eta - \pi)$. Second, due to local nature of the finite-difference operators, the coefficient matrix is highly banded. By storing only those terms of the coefficient matrix which fall within the bandwidth, a simple Gauss elimination scheme can be used to solve the relevant system of linear equations with maximum economy and, therefore, find the values of the unknowns ψ_{ij} at the interior points plus the unknown value c_1 on the inner boundary $\xi = 0$.

Employing the elasticity formulation and the solution technique discussed earlier, the distribution of stress was obtained for material properties typical of a glass fiber/epoxy composite (Lekhnitskii, 1963): G_{12} = 5,700 MN/m^2, G_{23} = 5,000 MN/m^2, where the direction 1 is along the fibers, 2 is the in-plane normal to the fibers and 3 the out-of-plane normal direction. If this material is laid up at 90 deg, then $1 \equiv \eta$, $2 \equiv z$, $3 \equiv \xi$. As an example case, 13 layers of 1-mm thickness each are wound on an elliptical mandrel with major semiaxis e_1 = 35 mm and minor one e_2 = 20 mm.

Figure 2 shows the angular distribution of the shearing stress $\tau_{\eta z}$ at certain distances ξ across the thickness, which correspond to layer interfaces as given in parentheses. For a distance ξ below a particular value (in this case below $\xi/T = 0.462$), this interlaminar shear has the maximum at the major axis and the minimum at the minor one. The exact opposite happens for layers that are above this threshold value. In fact, the largest interlaminar shear stress $\tau_{\eta z}$ occurs at the outermost interface (between layers 12 and 13) and at the minor axis. Furthermore, Fig. 3 shows the variation of this stress with the thicknesswise distance ξ. Near the major axis the change in $\tau_{\eta z}$ between the innermost and the outermost layers is relatively small and the curve is seen to be highly nonlinear. On the contrary, near the minor axis the variation can be characterized as linear and of relatively large magnitude. Also, the curves have a common intersection point at the threshold value of ξ/T = 0.462, where here is practically no angular variation.

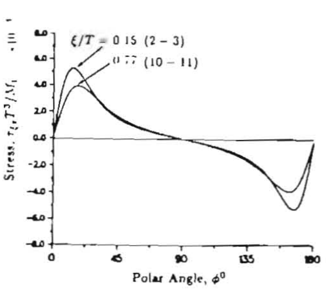

Fig. 4 Angular distribution of the shear stress $\tau_{\xi z}$ for two-layer interfaces in a 13-ply bar (layer numbers denoted in parentheses)

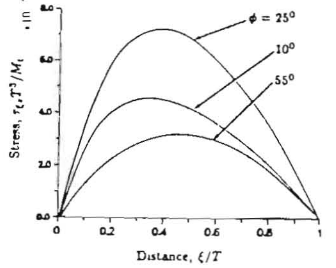

Fig. 5 Thicknesswise distribution (along ξ) of the shear stress $\tau_{\xi z}$

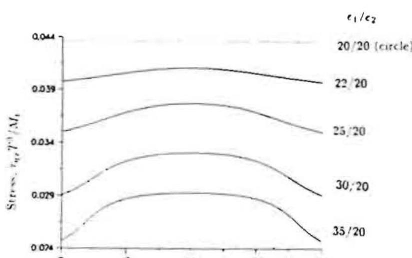

Fig. 6 Effect of a change in the major to minor axis ratio of the mandrel on the angular distribution of the shear stress $\tau_{\eta z}$ at the outermost layer interface. The dashed curve is the solution for a circular cross-section.

The distribution of the stress $\tau_{\xi z}$ with the polar angle ϕ is shown in Fig. 4 for two distinct values of ξ. It is seen that this component of stress, which is two orders of magnitude less than the stress $\tau_{\eta z}$, peaks at about ϕ = 25 deg and it follows the same pattern for all points thicknesswise. The distribution of $\tau_{\xi z}$ across the thickness is illustrated in Fig. 5. The maximum value is shifted outwards for points closer to the minor axis, so that for $\phi = 10$ deg, the peak occurs at about $\xi/T = 0.3$, whereas for $\phi = 55$ deg, the peak is at about ξ/T = 0.5. Notice that $\tau_{\xi z}$ is zero at all points on both the minor and major axis, as well at all points on the inner and outer bounding surface.

Now, let us keep the minor axis of the mandrel at 20 mm while the major axis is reduced until it becomes equal to the minor one (the limit of a circular section). The effect of this variation in the major to minor axis ratio on the angular distribution of the maximum interlaminar shear stress $\tau_{\eta z}$ (between layers 12-13) is shown in Fig. 6. As the elliptical mandrel approaches a circle, the distribution becomes flatter. The solution is seen to converge to that constant value (since for a circular section there is only radial dependence) that was given for the solution of the torsion of cylindrically orthotropic rods by Lekhnitskii (1963).

Conclusion

We have presented a formulation of the theory of elasticity of filament wound anisotropic ellipsoids. The anisotropy in this case is curvilinear, defined with respect to the directions that are equivalent in the sense of the elastic properties, and which are dictated by the regularities of the body. The differential equations for the stresses have been derived for a material possessing anisotropy of the most general kind. Furthermore, the solution for the special case of torsion of an orthotropic body was obtained and results for the angular and thicknesswise distribution of the shear stresses were presented.

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