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Transient Thermal Stresses in Cylindrically Orthotropic Composite Tubes

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A solution is given for the stresses and displacements in an orthotropic, hollow circular cylinder, due to an imposed constant temperature on the one surface and heat convection into a medium of a different constant temperature at the other surface. Temperature-independent material properties are assumed and a displacement approach is used. Results for the variation of stresses with time and through the thickness are presented.

Introduction

An understanding of thermally-induced stresses in anisotropic bodies is essential for a comprehensive study of their response due to an exposure to a temperature field, which may in turn occur in service or during the manufacturing stages. For example, during the curing stages of filament wound bodies, thermal stresses may be induced from the heat buildup and cooling process. The level of these stresses may well exceed the ultimate strength.

Composite tubes, which can be produced by filament winding on a cylindrical mandrel, have useful applications in such parts as automotive suspension components, landing gears, and launch tubes. Considerable work has been done on the stress field due to mechanical loading (e.g., Lehknitskii, 1963; Sherrer, 1967; Pagano, 1972). Less literature is devoted to studies of thermally-induced stresses. To this extent, formulations and solutions for the thermal stresses in orthotropic cylinders have been presented, for example, by Kalam and Tauchert (1978) due to a steady-state plane temperature distribution, and Hyer and Cooper (1986) due to a steady-state circumferential temperature gradient. The plane thermal-stress problem of a thin circular disc of orthotropic material was considered by Parida and Das (1972). Thermal effects on the microstructure level were analyzed by Avery and Herakovich (1986), by considering an orthotropic fiber in an isotropic matrix under a uniform temperature change.

In this work the problem of transient (time-dependent) thermal stresses in a hollow orthotropic circular cylinder is treated. It is assumed that one surface of the cylinder is at a constant temperature T_0, and at the other there is heat convection into a medium at the reference temperature. The insight provided by this analysis may prove helpful in such instances as choosing curing cycle conditions. The material properties are assumed temperature-independent and a displacement approach is used. It is also assumed that the stresses act on the planes normal to the cylinder axis and do not vary along the generator and that there are no body forces. Numerical results are presented for the variation of the stresses and displacements with time and through the thickness.

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Mathematical Formulation

Consider a hollow cylinder of inner and outer radius r_1 and r_2, respectively. We denote by r the radial, θ the circumferential, and z the axial coordinate (Fig. 1). The cylinder is assumed to have zero initial temperature. For $t>0$, the boundary $r=r_1$ is kept at temperature T_0 and at $r=r_2$ there is convection into a medium at the reference (zero) temperature. Although the reference temperature is taken as zero, the analysis would be valid for any nonzero value (this is discussed further in the results section).

The thermal problem consists of the heat conduction equation

$$K\left(\frac{\partial^2 T(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,t)}{\partial r}\right)=\frac{\partial T(r,t)}{\partial t}(r_1<r<r_2,t>0), \tag{1a}$$

and the initial and boundary conditions

$$T(r,t=0)=0 \quad \text{at} \quad r_1\le r\le r_2, \tag{1b}$$

$$T(r_1,t)=T_0(t>0), \tag{1c}$$

$$\left.\frac{\partial T(r,t)}{\partial r}\right|_{r=r_2}+hT(r_2,t)=0(t>0), \tag{1d}$$

where K is the thermal diffusivity of the composite in the r direction, and h is the ratio of the convective heat-transfer coefficient of the composite tube and the surrounding medium, and the thermal conductivity of the composite in the r direction. The temperature distribution $T(r,t)$ can be found in Carslaw and Jaeger (1959) in terms of the Bessel functions of the first and second kind J_n and Y_n (note that as the range r

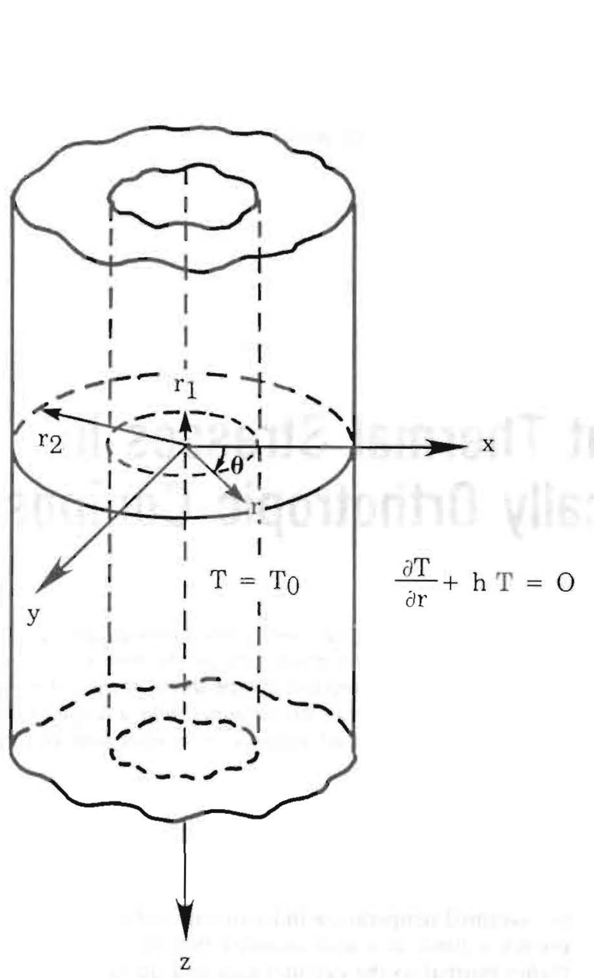

Fig. 1 Definition of the geometry

does not extend to the origin, Bessel functions of the second kind are not excluded, as opposed to the solid cylinder case). It is given in the form

$$T(r,t)=d_1+d_2\ln(r/r_2)$$

$$+\sum_{n=1}^{\infty} e^{-Ka_n^2 t}[d_{4n}J_0(ra_n)+d_{5n}Y_0(ra_n)], \tag{2}$$

where $\pm a_n$ are the roots (all real and simple) of:

$$[xY_1(r_2x)-hY_0(r_2x)]J_0(r_1x)-[xJ_1(r_2x)-hJ_0(r_2x)]Y_0(r_1x)=0. \tag{3}$$

The constants d_i are given in Appendix I. Since there is only radial dependence of the temperature field, the hoop displacements are zero and the stresses and strains are independent of θ. Therefore, for the orthotropic body, the thermoelastic stress-strain relations are

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{\theta z}\\ \tau_{rz}\\ \tau_{r\theta}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\ C_{12}&C_{22}&C_{23}&0&0&0\\ C_{13}&C_{23}&C_{33}&0&0&0\\ 0&0&0&C_{44}&0&0\\ 0&0&0&0&C_{55}&0\\ 0&0&0&0&0&C_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{rr}-\alpha_r\Delta T\\ \epsilon_{\theta\theta}-\alpha_\theta\Delta T\\ \epsilon_{zz}-\alpha_z\Delta T\\ \gamma_{\theta z}\\ \gamma_{rz}\\ \gamma_{r\theta}\end{bmatrix}, \tag{4}$$

where C_{ij} are the elastic constants and α_i the thermal expansion coefficients (we have used the notation $1\equiv r$, $2\equiv\theta$, $3\equiv z$).

Since the temperature does not depend on the axial coordinate, we can assume that the stresses are independent of z. In addition to the constitutive equations (4), the elastic response of the cylinder must satisfy the equilibrium equations:

$$\sigma_{rr,r}+(\sigma_{rr}-\sigma_{\theta\theta})/r=0, \tag{5a}$$

$$\tau_{r\theta,r}+2\tau_{r\theta}/r=0;\quad r^{-1}(r\tau_{rz})_{,r}=0. \tag{5b}$$

For the problem without the thermal effects the expressions for the displacement field were derived by Lehknitskii (1963). A similar procedure was followed and lead to the general solution for the displacements in this thermoelastic problem (see also Hyer and Cooper, 1986). Due to the symmetry of the problem, only rigid body translation and rotation contribute to the θ component of the displacement field and the strains and stresses do not depend on θ. Furthermore, there are no twisting strains. Therefore, the displacements have the form:

$$u_r=U(r,t)+z(\omega_y\cos\theta-\omega_x\sin\theta)+v_{0x}\cos\theta+v_{0y}\sin\theta, \tag{6a}$$

$$u_\theta=-z(\omega_x r\cos\theta+\omega_y r\sin\theta)-v_{0x}\sin\theta+v_{0y}\cos\theta+\omega_z r, \tag{6b}$$

$$u_z=zC(t)+\omega_x r\sin\theta-\omega_y r\cos\theta+v_{0z}. \tag{6c}$$

In the above expressions, the function $U(r,t)$ represents the radial displacements accompanied by deformation, and the constants v_{0x}, v_{0y}, v_{0z}, ω_x, ω_y, ω_z characterize the rigid body translation and rotation about the cartesian coordinate system. The parameter C is time-dependent and is found from the boundary conditions, as discussed later.

The strains are now expressed in terms of the displacement U:

$$\epsilon_{rr}=\frac{\partial U(r,t)}{\partial r};\ \epsilon_{\theta\theta}=\frac{U(r,t)}{r};\ \epsilon_{zz}=C(t). \tag{7a}$$

$$\gamma_{\theta z}=\gamma_{rz}=\gamma_{r\theta}=0 \tag{7b}$$

Substituting (4) and (7) into (5a) yields the following differential equation for the displacement field $U(r,t)$:

$$C_{11}\left(\frac{\partial^2 U(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial U(r,t)}{\partial r}\right)-\frac{C_{22}}{r^2}U(r,t)$$

$$=q_1\frac{\partial T(r,t)}{\partial r}+q_2\frac{T(r,t)}{r}+(C_{23}-C_{13})\frac{C(t)}{r}, \tag{8}$$

where

$$q_1=C_{11}\alpha_r+C_{12}\alpha_\theta+C_{13}\alpha_z, \tag{9a}$$

$$q_2=(C_{11}-C_{12})\alpha_r+(C_{12}-C_{22})\alpha_\theta+(C_{13}-C_{23})\alpha_z. \tag{9b}$$

The parameter $C(t)$ is now written in the form

$$C(t)=c_0+\sum_{n=1}^{\infty}c_n e^{-Ka_n^2 t}. \tag{10}$$

To solve equation (8), set

$$U(r,t)=U_0(r)+\sum_{n=1}^{\infty}e^{-Ka_n^2 t}R_n(r). \tag{11}$$

Substituting (2), (10), and (11) in (8) gives the following equations for U_0 and R_n:

$$C_{11}\left(U_0''(r)+\frac{U_0'(r)}{r}\right)-\frac{C_{22}}{r^2}U_0(r)$$

$$=\frac{q_1d_2+q_2d_1+(C_{23}-C_{13})c_0}{r}+q_2d_2\frac{\ln(r/r_2)}{r}, \tag{12}$$

$$C_{11}\left(R_n''(r)+\frac{R_n'(r)}{r}\right)-\frac{C_{22}}{r^2}R_n(r)$$

$$=\frac{(C_{23}-C_{13})c_n}{r}+d_{4n}\left[\frac{J_0(ra_n)}{r}-q_1\alpha_nJ_1(ra_n)\right]$$

$$+d_{5n}\left[\frac{Y_0(ra_n)}{r}-q_1a_nY_1(ra_n)\right]\quad n=1,\ldots,\infty. \tag{14}$$

The solution to these equations is the sum of the solution of the homogeneous equation and a particular solution. The solution of the homogeneous equation (8) is

$$U_g(r,t)=G_1(t)r^{\lambda_1}+G_2(t)r^{\lambda_2};\quad \lambda_{1,2}=\pm\sqrt{C_{22}/C_{11}}. \qquad (15a)$$

In a similar fashion to the parameter $C(t)$, set $G_i(t)$ in the form:

$$G_i(t)=G_{i0}+\sum_{n=0}^{\infty}G_{in}e^{-Ka_n^2 t};\quad i=1,2. \qquad (15b)$$

Since the constants c_j and G_{ij} are yet unknown, we shall indicate the places where they enter in the expressions that follow (these constants are found later from the boundary conditions). For $C_{11}\neq C_{22}$ the solution of (12) for $U_0(r)$ is

$$U_0(r)=G_{10}r^{\lambda_1}+G_{20}r^{\lambda_2}+\frac{C_{23}-C_{13}}{C_{11}-C_{22}}c_0 r+U_0^*(r), \qquad (16a)$$

$$U_0^*(r)=\frac{q_2}{C_{11}-C_{22}}d_2 r\ln(r/r_2)+\left[\frac{q_1d_2+q_2d_1}{C_{11}-C_{22}}-\frac{2C_{11}q_2d_2}{(C_{11}-C_{22})^2}\right]r. \qquad (16b)$$

For $C_{11}=C_{22}$ the corresponding solution of (12) is

$$U_0(r)=G_{10}r+\frac{G_{20}}{r}+\frac{(C_{23}-C_{13})}{2C_{11}}c_0 r\ln(r/r_2)+U_0^*(r), \qquad (17a)$$

$$U_0^*(r)=\frac{q_2}{4C_{11}}d_2 r\ln^2(r/r_2)+\frac{(2q_1-q_2)d_2+2q_2d_1}{4C_{11}}r\ln(r/r_2). \qquad (17b)$$

To solve (14), we use the series expansions of the Bessel functions to obtain a series expansion of the right-hand side (see Appendix II). In the following, γ stands for the Euler's constant ($\simeq 0.577215\ldots$).

For $C_{11}\neq C_{22}$, the solution of (14) for R_n, $n=1,\ldots\infty$, is

$$R_n(r)=G_{1n}r^{\lambda_1}+G_{2n}r^{\lambda_2}+\frac{C_{23}-C_{13}}{C_{11}-C_{22}}c_n r+R_n^*(r), \qquad (18a)$$

$$R_n^*(r)=B_{0n}r+\frac{2}{\pi}\frac{d_{5n}}{(C_{11}-C_{22})}r\ln(ra_n/2)+\sum_{k=0}^{\infty}B_{1nk}r^{2k+3}\ln(ra_n/2)+B_{2nk}r^{2k+3}, \qquad (18b)$$

where

$$B_{0n}=\frac{d_{4n}+(2/\pi)(q_1+\gamma)d_{5n}}{C_{11}-C_{22}}-\frac{4C_{11}d_{5n}}{\pi(C_{11}-C_{22})^2}. \qquad (18c)$$

The coefficients in the sum over k are given in terms of

$$f_{kn}=\left[d_{4n}-\frac{2d_{5n}}{\pi}\left(1+\frac{1}{2}+\ldots+\frac{1}{k+1}-\gamma\right)\right][1+2q_1(k+1)]+\frac{2d_{5n}q_1}{\pi}, \qquad (18d)$$

as follows:

$$B_{1nk}=\frac{2d_{5n}(-1)^{k+1}a_n^{2k+2}[1+2q_1(k+1)]}{\pi 2^{2k+2}[(k+1)!]^2[C_{11}(2k+3)^2-C_{22}]}, \qquad (18e)$$

$$B_{2nk}=\frac{(-1)^{k+1}a_n^{2k+2}}{2^{2k+2}[(k+1)!]^2[C_{11}(2k+3)^2-C_{22}]}\times\left\{f_{kn}-\frac{(4C_{11}/\pi)(2k+3)d_{5n}[1+2q_1(k+1)]}{C_{11}(2_k+3)^2-C_{22}}\right\}. \qquad (18f)$$

In the (unlikely) event that for a certain k, $C_{11}(2k+3)^2=C_{22}$, the term in the sum for this k is replaced by the one in Appendix III.

For $C_{11}=C_{22}$ the solution of (14) for R_n is

$$R_n(r)=G_{1n}r+\frac{G_{2n}}{r}+\frac{C_{23}-C_{13}}{2C_{11}}c_n r\ln(r/r_2)+R_n^*(r), \qquad (19a)$$

$$R_n^*(r)=B_{0n}r\ln(ra_n/2)+\frac{d_{5n}}{2\pi C_{11}}r\ln^2(ra_n/2)+\sum_{k=0}^{\infty}B_{1nk}r^{2k+3}\ln(ra_n/2)+B_{2nk}r^{2k+3}, \qquad (19b)$$

where

$$B_{0n}=\frac{\pi d_{4n}+d_{5n}(2q_1+2\gamma-1)}{2\pi C_{11}}. \qquad (19c)$$

It should be noted that although the sum over the roots a_n is extended from $n=1$ to ∞, only the first few terms are dominant and it usually suffices to include a small number of roots. This issue is discussed in detail in the Results section.

Next, turn to the boundary conditions. We assume that no external tractions exist. Then the conditions on the contour bounding the cross-section (at $r=r_1$ and $r=r_2$) can be written in the following form:

$$\sigma_{rr}(r_i,t)=\tau_{r\theta}(r_i,t)=\tau_{rz}(r_i,t)=0,\qquad i=1,2. \qquad (20)$$

Only the condition for the stress σ_{rr} is not satisfied identically and it is written in terms of the displacement field:

$$C_{11}U_{,r}(r_i,t)+C_{12}\frac{U(r_i,t)}{r}+C_{13}C(t)-q_1T(r_i,t)=0;\quad i=1,2. \qquad (21)$$

By substituting (2), (10), and (11) in (21) and the expressions (17) for $U_0(r)$, gives, in turn, the following two linear equations in G_{10}, G_{20}, c_0:

$$(C_{11}\lambda_1+C_{12})r_i^{\lambda_1-1}G_{10}+(C_{11}\lambda_2+C_{12})r_i^{\lambda_2-1}G_{20}+A_0c_0$$
$$=-C_{11}U_0^{*\prime}(r_i)-C_{12}\frac{U_0^*(r_i)}{r_i}+q_1[d_1+d_2\ln(r_i/r_2)]\quad i=1,2, \qquad (22a)$$

where

$$A_0=\frac{C_{11}+C_{12}}{C_{11}-C_{22}}(C_{23}-C_{13})+C_{13}\quad \text{for}\quad C_{11}\neq C_{22}$$
$$=\frac{C_{23}-C_{13}}{2C_{11}}[C_{11}+(C_{11}+C_{12})\ln(r_i/r_2)]+C_{13}\quad \text{for}\quad C_{11}=C_{22}. \qquad (22b)$$

In a similar fashion, by substituting the expressions (18) for $R_n(r)$, there correspond two linear equations for G_{1n}, G_{2n}, c_n for each n, $n=1,\ldots\infty$, as follows,

$$(C_{11}\lambda_1+C_{12})r_i^{\lambda_1-1}G_{1n}+(C_{11}\lambda_2+C_{12})r_i^{\lambda_2-1}G_{2n}+A_0c_n$$
$$=-C_{11}R_n^{*\prime}(r_i)-C_{12}\frac{R_n^*(r_i)}{r_i}+q_1[d_{4n}J_0(r_ia_n)+d_{5n}Y_0(r_ia_n)];\quad i=1,2. \qquad (23)$$

Now, let us consider the conditions of resultant forces and moments. Since the stresses do not depend on z, these conditions exist in any cross-section. It can be proved (e.g., Lekhnitskii, 1963, although thermal effects are not included), that the conditions of zero-resultant forces along the x- and y-

Table 1 Convergence of the series solution. Values of the *n*th term (at $r=r_2$) of the temperature, displacement, and stress quantities.

	$n=1$	$n=2$	$n=3$
$a_n(\mathrm{m}^{-1})$	87.1	291.0	488.8
$\bar{t}=0.25$			
$T(°\mathrm{C})$	-0.743×10^2	0.144×10^0	-0.426×10^{-5}
$U(\mathrm{m})$	-0.910×10^{-5}	0.142×10^{-7}	0.687×10^{-11}
$\sigma_{\theta\theta}(\mathrm{MN/m^2})$	0.297×10^2	-0.870×10^{-1}	0.133×10^{-4}
$\sigma_{zz}(\mathrm{MN/m^2})$	0.818×10^1	-0.104×10^0	0.948×10^{-6}
$\bar{t}=0.5$			
$T(°\mathrm{C})$	-0.457×10^2	0.635×10^{-3}	-0.976×10^{-12}
$U(\mathrm{m})$	-0.560×10^{-5}	0.629×10^{-10}	0.157×10^{-17}
$\sigma_{\theta\theta}(\mathrm{MN/m^2})$	0.183×10^2	-0.385×10^{-3}	0.305×10^{-11}
$\sigma_{zz}(\mathrm{MN/m^2})$	0.503×10^1	-0.460×10^{-3}	0.217×10^{-12}

axes are satisfied identically. The conditions of zero-resultant moment along *x*- and *y*-axes (and that of zero twisting moment) are also satisfied by the symmetry of the problem. Therefore, it remains only a condition of zero resultant-axial force, P_z:

$$\int_{r_1}^{r_2} \sigma_{zz}(r,t)2\pi r dr = P_z(t) = 0. \tag{24}$$

This gives the last set of equations that are needed to determine the constants G_{ij}, c_j. In terms of

$$q_3 = C_{13}\alpha_r + C_{23}\alpha_\theta + C_{33}\alpha_z, \tag{25}$$

(24) gives

$$\left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_1+1}\right)(r_2^{\lambda_1+1}-r_1^{\lambda_1+1})G_{10}+A_1G_{20}+A_2c_0$$
$$= -E_0(r_1,r_2)+\frac{q_3}{2}\left[\frac{r_2^2-r_1^2}{2}(2d_1-d_2)+d_2r_1^2\ln(r_2/r_1)\right], \tag{26}$$

and for $n=1, \ldots, \infty$,

$$\left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_1+1}\right)(r_2^{\lambda_1+1}-r_1^{\lambda_1+1})G_{1n}+A_1G_{2n}+A_2c_n$$
$$= -E_n(r_1,r_2)+(q_3/a_n)\sum_{i=1}^{2}(-1)^i[d_{4n}r_iJ_1(r_ia_n)$$
$$+d_{5n}r_iY_1(r_ia_n)], \tag{27}$$

where $E_0(r_2,r_1)$ and $E_n(r_1,r_2)$ are given in Appendix IV. The coefficients A_1, A_2 are defined as:

$$A_1 = \left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_2+1}\right)(r_2^{\lambda_2+1}-r_1^{\lambda_2+1}) \quad \text{for} \quad C_{11}\neq C_{22} \tag{28a}$$
$$= C_{13}+(C_{23}-C_{13})\ln(r_2/r_1) \quad \text{for} \quad C_{11}=C_{22}$$

$$A_2 = \frac{r_2^2-r_1^2}{2}\left(C_{33}+\frac{C_{23}^2-C_{13}^2}{C_{11}-C_{22}}\right) \quad \text{for} \quad C_{11}\neq C_{22}$$
$$= \frac{(r_2^2-r_1^2)}{8C_{11}}[4C_{33}C_{11}-(C_{23}-C_{13})^2] \tag{28b}$$
$$+\frac{C_{23}^2-C_{13}^2}{4C_{11}}r_1^2\ln(r_2/r_1) \text{ for } C_{11}=C_{22}.$$

Therefore the constants c_j, G_{ij} and, hence, the displacement U, can be found by solving (21), (26) and (22), (27). After obtaining the displacement field, the stresses can be found by substituting in (7) and (4).

Results and Discussion

Before presenting specific results we shall address several issues that were previously raised. First, in the aforementioned formulation, the reference temperature was assumed to be zero. Since, however, thermal stresses are produced by temperature differentials, the analysis remains the same for any initial temperature other than zero, at which the body is assumed to be stress free. In this case, T_0 is the applied temperature above this initial value.

Second, in producing numerical results, the series expansion for the Bessel's functions (see Appendix II) cannot be used for large arguments. This means that there is a limit to the number of roots a_n of the characteristic equation (3), over which the summation in (11) is performed. Except for very small values of the time t, this does not limit the accuracy of the results. This is because only the first few terms of the series over n are dominant and there is rapid convergence as can be seen from Table 1, which shows the *n*th term of some quantities for the example case that was considered (the specifics of the example case are described in detail next), and for time values $\bar{t}=Kt/(r_2-r_1)^2=0.25$ and 0.5. In view of the almost-zero values for the third term, there is no need to consider more than the first three roots. For very small values of time it becomes, however, necessary to include more terms.

As an illustrative example, the distribution of thermal stresses was determined for a glass/epoxy circular cylinder of inner radius $r_1=20$ mm and outer radius $r_2=36$ mm. It is supposed to be made, for example, by filament winding, with the fibers oriented around the circumference. The moduli in GN/m² and Poisson's ratio for this material are listed next, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction:

$$E_1=13.7,\ E_2=55.9,\ E_3=13.7,\ G_{12}=5.6,\ G_{23}=5.6,$$

$$G_{31}=4.9,\ \nu_{12}=0.068,\ \nu_{23}=0.277,\ \nu_{31}=0.4.$$

The thermal expansion coefficients are: $\alpha_r=40\times10^{-6}/°\mathrm{C}$, $\alpha_\theta=10\times10^{-6}/°\mathrm{C}$, $\alpha_z=40\times10^{-6}/°\mathrm{C}$. For this material, the thermal diffusivity in the radial direction is $K=0.112\times10^{-5}\mathrm{m^2/s}$. Let us assume that the ratio of the convective heat-transfer coefficient between the composite tube and the surrounding medium at $r=r_2$ and the thermal conductivity of the tube in the radial direction is $h=0.15\ \mathrm{m}^{-1}$ (which is a typical value for heat convection to the air). A temperature of $T_0=100\ °\mathrm{C}$ above the reference one is applied at $r=r_1$.

To illustrate the results, the nondimensional radial distance (through the thickness) $\bar{r}=(r-r_1)/(r_2-r_1)$ is used. Figure 2 shows the temperature and Fig. 3 the displacement distribution for time values $\bar{t}=0.25$, 0.5, 1.0, and 10 (the last one is a nearly steady, constant temperature state). The corresponding distribution of stresses σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} are shown in Figs. 4, 5, and 6. The biggest of those is the hoop stress $\sigma_{\theta\theta}$ and its value at the outer surface $\bar{r}=1$ is seen to be larger for $\bar{t}=0.25$ than the steady-state value (for $\bar{t}=10$) by a factor of about 1.5. At the inner surface $\bar{r}=0$, the steady-state ($\bar{t}=10$) stress is compressive and it becomes smaller in magnitude (tending to be tensile) for smaller time values. The radial stress σ_{rr}, is initially mostly tensile and becomes compressive at the final steady state. The axial stress σ_{zz} is compressive closer to the inner surface (small values of $\bar{r}$) but tensile closer to the outer surface; its maximum absolute value is about eight times higher at

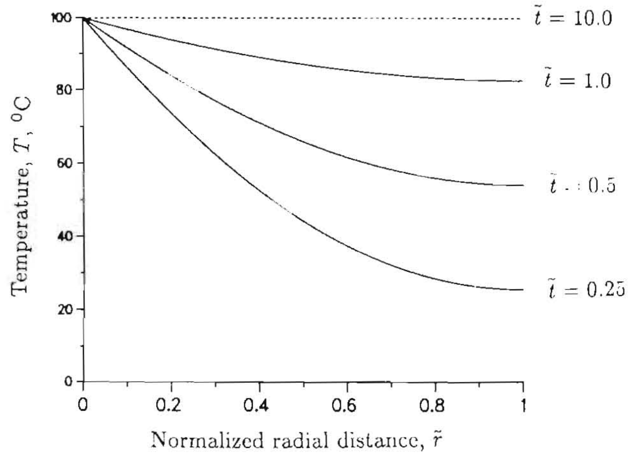

Fig. 2 Radial distribution of the temperature T at different times. The nondimensional time is defined by $\tilde{t} = Kt/(r_2 - r_1)^2$. The dashed line is the nearly steady, constant temperature state.

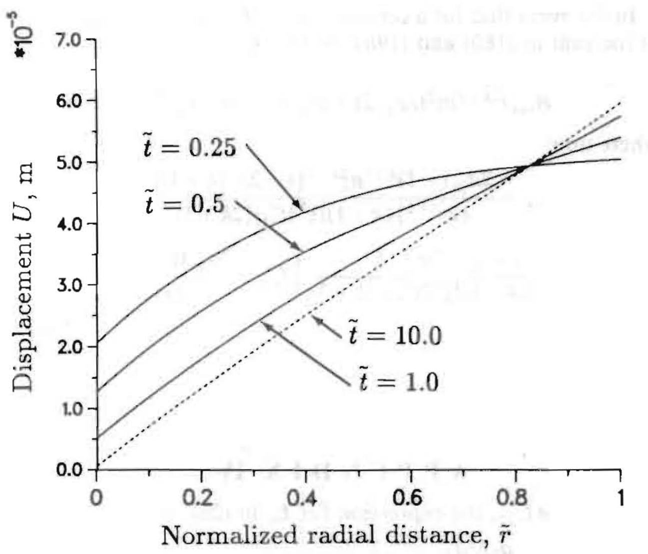

Fig. 3 Radial distribution of the displacement U

$\tilde{t} = 0.25$ than at $\tilde{t} = 10$. It should be pointed out that, although the axial and radial stresses are smaller than the hoop, they may be more critical because the material is weaker in the directions normal to the fibers (typically the ultimate strength of glass/epoxy in the directions normal to the fibers may be less than that in the direction of the fibers by a factor ranging from seven to ten). These results are specific for the example we consider and trends may be different, depending on the mechanical and thermal constants of the material. They show, however, that transient thermal stresses may be of considerable magnitude, the level of which can be determined from the above solution.

Summary

In summary, we have presented a solution for the thermal stresses of a homogeneous, orthotropic hollow cylinder subjected to a constant temperature on the one surface and heat convection into a medium of a different constant temperature at the other surface. Temperature-independent material properties were assumed and a series solution for the displacement was found. Numerical examples were presented for the distribution of the transient thermal stresses, which turned out to be of significant magnitude.

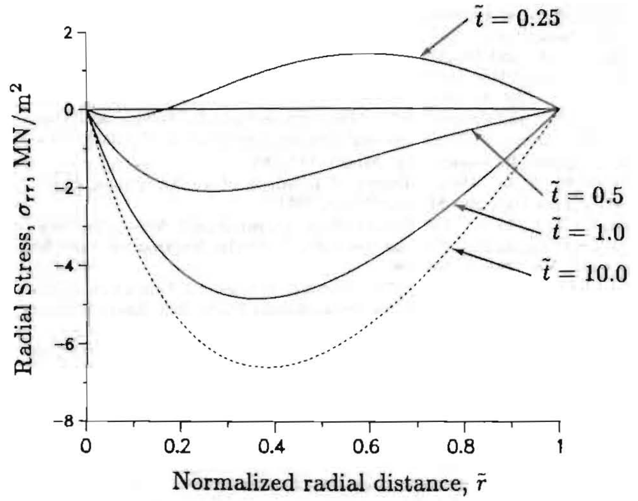

Fig. 4 Distribution of the radial stress σ_{rr}

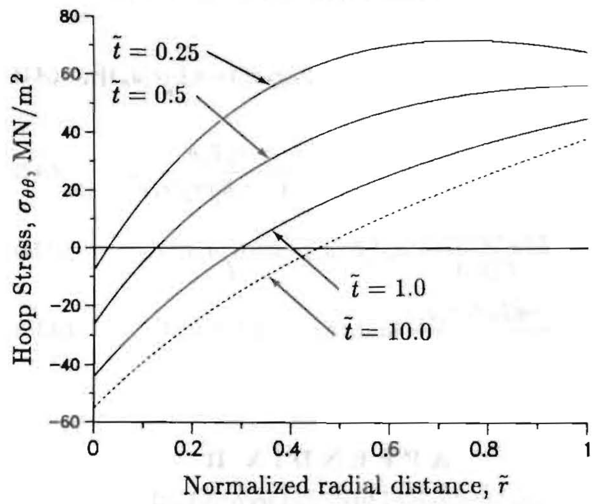

Fig. 5 Distribution of the hoop stress $\sigma_{\theta\theta}$

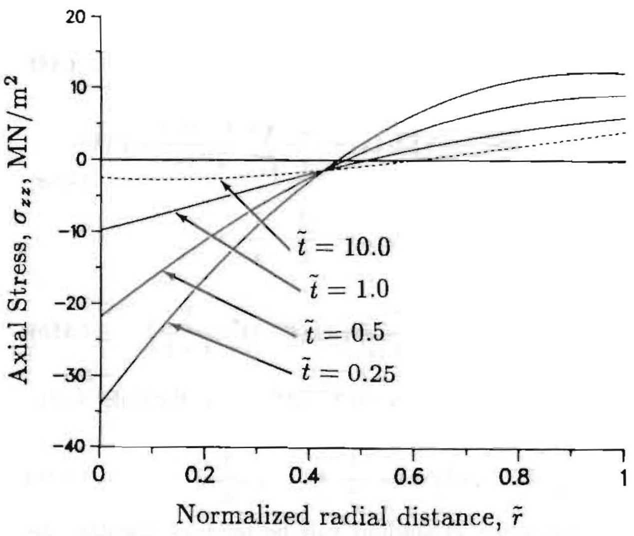

Fig. 6 Distribution of the axial stress σ_{zz}

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APPENDIX I

The constants d_i in the expression for the temperature (3) are given in terms of

$$F(a_n)=(a_n^2+h^2)J_0^2(r_1a_n)-[a_nJ_1(r_2a_n)-hJ_0(r_2a_n)]^2, \quad (A1)$$

as follows:

$$d_1=\frac{T_0}{1+r_2h\ln(r_2/r_1)};\qquad d_2=-\frac{r_2T_0h}{1+r_2h\ln r_2/r_1}, \quad (A2)$$

$$d_{4n}=\frac{\pi T_0Y_0(r_1a_n)}{F(a_n)}[a_nJ_1(r_2a_n)-hJ_0(r_2a_n)]^2, \quad (A3a)$$

$$d_{5n}=-\frac{\pi T_0J_0(r_1a_n)}{F(a_n)}[a_nJ_1(r_2a_n)-hJ_0(r_2a_n)]^2, \quad (A3b)$$

APPENDIX II

The Bessel functions of first- and second-kind of order zero and one have a series of expansion of the form (see e.g., Wylie, 1975)

$$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{2^{2k}(k!)^2};\quad J_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{2^{2k+1}k!(k+1)}, \quad (A4)$$

$$Y_0(x)=\frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_0(x)-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^kx^{2k}}{2^{2k}(k!)^2}\psi(k), \quad (A5a)$$

$$Y_1(x)=\frac{2}{\pi}\left(\ln\frac{x}{2}+\gamma\right)J_1(x)-\frac{2}{\pi}\frac{1}{x}$$
$$-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{2^{2k+1}(k!)(k+1)!}\left(2\psi(k+1)-\frac{1}{k+1}\right). \quad (A5b)$$

In the above expressions $\gamma=0.577215\ldots$ is the Euler's constant and $\psi(k)$ is defined as

$$\psi(k)=1+\frac{1}{2}+\ldots+\frac{1}{k}. \quad (A6)$$

The above series expansions can be used to calculate the Bessel's functions up to a value of the argument of about $x=18$. They are rapidly convergent, especially for small values of the argument (adopting a smallest number limit of $10\times^{-71}$ would require, at most, 72 terms at $x=18$).

Using the series expansion, we obtain the following equation in place of (14):

$$C_{11}\left(R_n''(r)+\frac{R_n'(r)}{r}\right)-\frac{C_{22}}{r^2}R_n(r)$$
$$=\frac{d_{4n}+(2/\pi)(q_1+\gamma)d_{5n}+(C_{23}-C_{13})c_n}{r}$$
$$+\frac{2d_5}{\pi}\frac{\ln(ra_n/2)}{r}+\frac{2d_5}{\pi}\sum_{k=0}^{\infty}$$
$$\frac{(-1)^{k+1}a_n^{2k+2}r^{2k+1}\ln(ra_n/2)}{2^{2k+2}[(k+1)!]^2}[1+2q_1(k+1)]$$
$$+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}a_n^{2k+2}r^{2k+1}}{2^{2k+2}[(k+1)!]^2}f_{kn}, \quad (A7)$$

where f_{kn} is defined in (18*d*).

APPENDIX III

In the event that for a certain k, $C_{11}(2k+3)^2=C_{22}$, the term in the sum in (18*b*) and (19*b*) for this k is

$$B_{1nk}r^{2k+3}\ln^2(ra_n/2)+B_{2nk}r^{2k+3}\ln(ra_n/2), \quad (A8)$$

where now

$$B_{1nk}=\frac{2d_{5n}(-1)^{k+1}a_n^{2k+2}[1+2q_1(k+1)]}{\pi 2^{2k+2}[(k+1)!]^24C_{11}(2k+3)}, \quad (A9a)$$

$$B_{2nk}=\frac{(-1)^{k+1}a_n^{2k+2}}{2^{2k+2}[(k+1)!]^22C_{11}(2k+3)}\left\{f_{kn}-\frac{2d_{5n}[1+2q_1(k+1)]}{2\pi(2k+3)}\right\}. \quad (A9b)$$

APPENDIX IV

For $C_{11}\neq C_{22}$, the expression for E_0 in (26), is:

$$E_0(r_1,r_2)=\frac{q_2r_1^2d_2}{2(C_{11}-C_{22})}(C_{23}+C_{13})\ln(r_2/r_1)+\frac{(r_2^2-r_1^2)}{4(C_{11}-C_{22})}$$
$$\times\left\{(C_{13}-C_{23})q_2d_2+2(C_{13}+C_{23})\left[q_1d_2+q_2d_1-\frac{2C_{11}q_2d_2}{C_{11}-C_{22}}\right]\right\}, \quad (A10a)$$

and the expressions for E_n, $n=1,\ldots\infty$, in (27), are

$$E_n(r_1,r_2)=\frac{(r_2^2-r_1^2)}{2}\left[(C_{23}+C_{13})B_{0n}\right.$$
$$\left.+\frac{1}{\pi}\frac{d_{5n}}{(C_{11}-C_{22})}(C_{13}-C_{23})\right]$$
$$+\frac{1}{\pi}\frac{d_{5n}(C_{13}+C_{23})}{(C_{11}-C_{22})}\sum_{i=1}^{2}(-1)^ir_i^2\ln(r_ia_n/2)+S_n, \quad (A10b)$$

where

$$S_n=\sum_{k=0}^{\infty}\sum_{i=1}^{2}(-1)^i\frac{r_i^{2k+4}\ln(r_ia_n/2)}{2k+4}B_{1nk}[(C_{23}+(2k+3)C_{13}]$$
$$+\sum_{k=0}^{\infty}\sum_{i=1}^{2}(-1)^i\frac{r_i^{2k+4}}{2k+4}\left\{B_{2nk}[C_{23}+(2k+3)C_{13}]\right.$$
$$\left.+B_{1nk}\frac{C_{13}-C_{23}}{2k+4}\right\}. \quad (A11)$$

For $C_{11}=C_{22}$, the expression for E_0 is

$$E_0(r_1,r_2)=\frac{-q_2r_1^2d_2}{8C_{11}}(C_{23}+C_{13})\ln^2(r_2/r_1)$$

$$+\frac{\ln(r_2/r_1)}{8C_{11}}\{r_1^2d_2(C_{13}-C_{23})q_2+r_1^2(C_{13}+C_{23})[(2q_1-q_2)d_2$$

$$+2q_2d_1]\}+\frac{(r_2^2-r_1^2)}{8C_{11}}(C_{23}-C_{13})[(q_2-q_1)d_2-q_2d_1], \quad (A12a)$$

and the expressions for E_n, $n=1, \ldots \infty$, are

$$E_n(r_1,r_2)=\frac{(r_2^2-r_1^2)}{4}(C_{23}-C_{13})\left(\frac{d_{5n}}{2\pi C_{11}}-B_{0n}\right)$$

$$+\sum_{i=1}^{2}\frac{d_{5n}}{4\pi C_{11}}(C_{13}+C_{23})(-1)^i r_i^2\ln^2(r_ia_n/2)$$

$$+\frac{1}{2}\sum_{i=1}^{2}\left[\frac{d_{5n}(C_{13}-C_{23})}{2\pi C_{11}}+(C_{13}+C_{23})B_{0n}\right]$$

$$\times(-1)^i r_i^2\ln(r_ia_n/2)+S_n. \quad (A12b)$$

In the event that $C_{11}(2k+3)^2=C_{22}$, the corresponding k term in the sum S_n in equation $(A11)$ is

$$\sum_{i=1}^{2}(-1)^i\frac{r_i^{2k+4}\ln^2(r_ia_n/2)}{2k+4}[C_{23}+(2k+3)C_{13}]B_{1nk}$$

$$+\sum_{i=1}^{2}(-1)^i\frac{r_i^{2k+4}\ln(r_ia_n/2)}{2k+4}\left\{[C_{23}\right.$$

$$\left.+(2k+3)C_{13}]B_{2nk}+(C_{13}-C_{23})\frac{2B_{1nk}}{2k+4}\right\}$$

$$+\sum_{i=1}^{2}(-1)^i\frac{r_i^{2k+4}}{(2k+4)^2}(C_{23}-C_{13})\left(\frac{2B_{1nk}}{2k+4}-B_{2nk}\right). \quad (A13)$$