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End Fixity Effects on the Buckling and Post-buckling of Delaminated Composites

G. A. Kardomateas

General Motors Research Laboratories, Engineering Mechanics RMB 256, Warren, Michigan 48090-9055, USA

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ABSTRACT

A study of the effects of end fixity (clamped–clamped vs simply-supported) on the buckling and post-buckling of delaminated composites is performed. The study includes the case of a delaminated composite beam-plate on an elastic foundation. First, the analytical solution for the post-buckling behavior of delaminated composites with simply-supported ends, derived via the perturbation technique, is presented. For high values of the foundation modulus the end fixity has little effect on the instability point. For the low values the simply-supported case requires less load for instability. Similarly, the end fixity has little effect on the slope of the energy release rate vs applied load curves for high values of the foundation modulus. For the low values the simply-supported case results in less steep curves, which suggests that the configuration can withstand more load beyond the critical point in the initial post-buckling stage before delamination growth takes place.

1 INTRODUCTION

Problems of delamination growth in composites have received considerable attention in recent years. Analytical models have been developed that deal with the basic premises of the problem.[1–3] An ability to understand the mechanics of this damage mechanism is useful, not only because delaminations can lead to an undesirable strength or stiffness degradation but also because they may have a desirable energy absorption capacity as a result of the relatively large deflections involved.[3]

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In a previous article,[3] an explicit quantitative description of the post-buckling behavior was obtained for a one-dimensional delamination in a compressively loaded laminate. In another related study,[4] the buckling and post-buckling behavior was investigated for the case of a laminate on an elastic foundation. This would simulate the compressive face of a composite boxed beam filled with a soft elastic medium such as foam or a sandwich beam consisting of two thin fiber-reinforced sheets separated by a low stiffness core, which are subjected to a bending load. In those studies the case of a clamped–clamped beam-plate was treated. End fixity can influence both the critical and the post-critical characteristics. In this article we shall first present the analytical post-buckling solutions for the case of a simply-supported beam-plate. The results will then be contrasted to those for a clamped–clamped beam. Both the usual configuration and the case of a beam-plate on an elastic foundation will be treated. Our method of solution is based on the asymptotic analysis of post-buckling behavior as a perturbation series with respect to the slope of the section at the delamination interface.

2 ANALYTICAL FORMULATION

2.1 Instability modes and governing equations

The geometry of the problem is defined in Fig. 1. A homogeneous, orthotropic beam-plate of thickness T, length L, and unit width, containing a strip delamination at a depth H ($H \leq T/2$) from the top surface of the plate, is subjected to an axial compressive force, P, at the ends. The plate is simply supported. In the general case there may be a Winkler-type elastic foundation attached. The delamination extends over the interval $0 \leq x \leq l = 2a$. Over this region the laminate consists of two parts, the part above the delamination ('upper' part, of thickness H) and the part below the delamination ('lower' part, of thickness $T-H$). The remaining laminate outside the delamination interval and of thickness T is referred to as the 'base' laminate. The coordinate systems for the separate parts are shown in Fig. 2. For each part we denote by D_i its bending stiffness:

$$D_i = \frac{Et_i^3}{12(1 - \nu_{13}\nu_{31})}$$

E being the modulus of elasticity along the x axis and t_i being the thickness of the corresponding part; $i =$ u (upper), l (lower) or b (base) parts. Although we assume one value for the modulus throughout the composite, different values for the different parts could be considered.

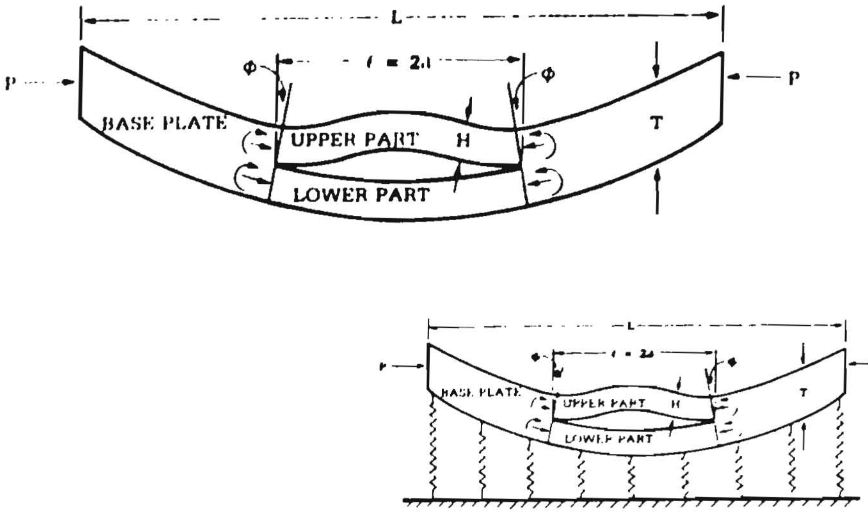

Fig. 1. Definition of the delamination buckling geometry. The system may include an elastic foundation.

Three possible modes of instability can take place,[3] depending on the geometric configuration. Global buckling of the whole beam may occur before any other deflection pattern takes place, typically for small delaminations. The critical load for this limiting mode can be found from an energy formulation.[5] In terms of an integer, m, that is specified so as to

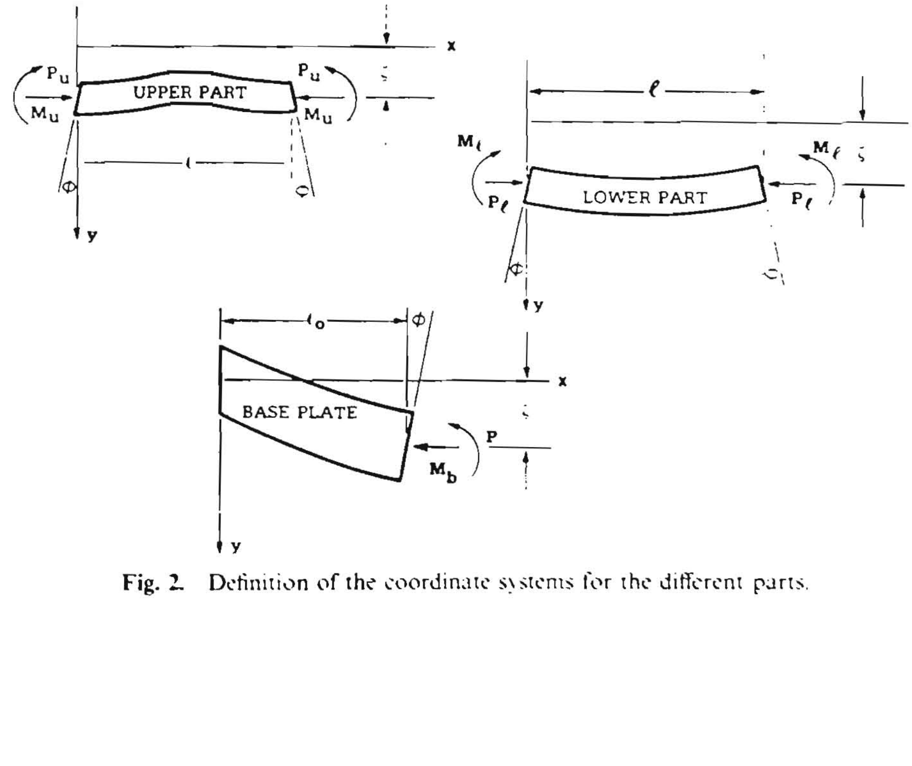

Fig. 2. Definition of the coordinate systems for the different parts.

render the load a minimum, the global instability load for the simply-supported case that assumes a mode shape, $y = a_m \sin(m\pi x/L)$, is

$$P_{\mathrm{glo}} = \frac{\pi^2 D_{\mathrm{b}}}{L^2}\left(m^2 + \frac{\beta L^4}{m^2\pi^4 D_{\mathrm{b}}}\right) \tag{1}$$

where β is the modulus of the foundation. For example, for $\beta L^4/(\pi^4 D_{\mathrm{b}}) < 4$ then $m = 1$, for $4 < \beta L^4/(\pi^4 D_{\mathrm{b}}) < 36$ then $m = 2$, etc. The second limiting case occurs typically for relatively large and thin delaminations and involves only local buckling of the delaminated upper layer, the lower part and the base plate remaining flat. Thus an upper bound to the critical load is expressed from the critical Euler load for the upper segment by

$$P_{\mathrm{loc}} = (T/H)\pi^2 D_{\mathrm{u}}/l^2 \tag{2}$$

In the general, mixed case (to be considered next), transverse deflections for both the upper and lower parts as well as the base plate may occur.

The differential equations for the deflections of the lower ($i = \mathrm{l}$) and base ($i = \mathrm{b}$) parts of the delaminated plate can be written[5]

$$D_i\frac{\mathrm{d}^4 y_i}{\mathrm{d}x^4} + P_i\frac{\mathrm{d}^2 y_i}{\mathrm{d}x^2} = -\beta y_i \tag{3}$$

For the upper part ($i = \mathrm{u}$) we should set $\beta = 0$. This equation is solved in conjunction with the following conditions:

(1) a condition of common deflection, ζ, and of equal slope at the interface section (where the delamination starts) (Fig. 2):

$$y_{\mathrm{u}}|_{x=0,l} = y_{\mathrm{l}}|_{x=0,l} = y_{\mathrm{b}}|_{x=l_0} = \zeta \tag{4a}$$

$$y'_{\mathrm{u}}|_{x=0} = y'_{\mathrm{l}}|_{x=0} = y'_{\mathrm{b}}|_{x=l_0} \tag{4b}$$

(2) the end fixity condition for the base plate:

$$y_{\mathrm{b}}|_{x=0} = y''_{\mathrm{b}}|_{x=0} = 0 \tag{5}$$

(3) axial and shearing force and moment equilibrium at this section:

$$P_{\mathrm{u}} + P_{\mathrm{l}} = P_{\mathrm{b}} = P \qquad V_{\mathrm{u}} + V_{\mathrm{l}} = V_{\mathrm{b}} \tag{6a}$$

$$M_{\mathrm{u}} + M_{\mathrm{l}} + P_{\mathrm{u}}(T - H)/2 - P_{\mathrm{l}}H/2 = M_{\mathrm{b}} \tag{6b}$$

(4) a condition of compatible shortening of the upper and lower parts:

$$(1 - v_{13}v_{31})\frac{P_{\mathrm{u}}l}{EH} - (1 - v_{13}v_{31})\frac{P_{\mathrm{l}}l}{E(T-H)} + \frac{1}{2}\int_0^l y_{\mathrm{u}}'^2\,\mathrm{d}x - \frac{1}{2}\int_0^l y_{\mathrm{l}}'^2\,\mathrm{d}x = Ty'_{\mathrm{u}}\Big|_{x=0} \tag{7}$$

2.2 Solution procedure and buckling load

The perturbation method is used to solve the problem defined above. The pre-buckling state of stress is a state of pure compression:

$$y_{i,0} = 0 \qquad M_{i,0} = 0 \qquad V_{i,0} = 0$$
$$P_{u,0} = P_0 H/T \qquad P_{1,0} = P_0(T - H)/T \qquad P_{b,0} = P_0$$

Let the angle at the interface of the delaminated and base plate be denoted by ϕ (Fig. 1). The deflection and load quantities at each part, $y_i(x)$, P_i, V_i, M_i, are developed into ascending perturbation series with respect to ϕ:

$$y_i(x) = \phi y_{i,1}(x) + \phi^2 y_{i,2}(x) + \cdots \qquad P_i = P_{i,0} + \phi P_{i,1} + \phi^2 P_{i,2} + \cdots \quad \text{(8a)}$$
$$M_i = \phi M_{i,1} + \phi^2 M_{i,2} + \cdots \qquad V_i = \phi V_{i,1} + \phi^2 V_{i,2} + \cdots \quad \text{(8b)}$$

By the definition of the series, at this interface

$$y'_{i,1} = 1 \qquad y'_{i,2} = y'_{i,3} = \cdots = 0 \quad (9)$$

Substituting eqns (8) into the differential equation (3) and equating like powers of ϕ leads to a set of linear differential equations and boundary conditions for each part.

In the first approximation the terms in the first power of ϕ are equated. The solution of the first-order equation for the upper and lower part can be found in Refs 3 and 4. In the following we give the solution for the base plate. The corresponding equation is

$$D_b \frac{d^4 y_{b,1}}{dx^4} + P_0 \frac{d^2 y_{b,1}}{dx^2} + \beta y_{b,1} = 0 \quad \text{(10a)}$$

Define

$$k_{b,0}^2 = P_{b,0}/D_b \qquad \lambda_b = \beta/D_b \quad \text{(10b)}$$

The solution can be expressed in terms of trigonometric functions only or a combination of trigonometric and hyperbolic functions depending on the magnitude of the modulus of the foundation.

For $k_{b,0}^4 > 4\lambda_b$ the solution to the first-order problem equation (10a) satisfying the end fixity conditions of eqn (5) is given by

$$y_{b,1} = \sum_{j=1,2} C_{1j} \sin \delta_j x \quad (11)$$

where δ_j are defined by

$$\delta_{1,2} = \sqrt{[(-k_{b,0}^2 \pm \sqrt{k_{b,0}^4 - 4\lambda_b})/2]} \quad (12)$$

In terms of

$$Q=\sum_{j=1,2}(-1)^j\delta_j\sin\delta_{3-j}l_0\cos\delta_j l_0 \tag{13a}$$

the constants C_{11}, C_{12} are found from eqns (4a) and (9) to be

$$C_{1j}=(-1)^j(\sin\delta_{3-j}l_0-\zeta_1\delta_{3-j}\cos\delta_{3-j}l_0)/Q \tag{13b}$$

The first-order end shear can be expressed in the form

$$V_{b,1}=-D_b(y'''_{b,1}+k^2_{b,0}y'_{b,1})|_{x=l_0}=V^c_{b,1}+\zeta_1 V^f_{b,1} \tag{14}$$

where

$$V^c_{b,1}=(D_b/Q)\delta_1\delta_2\sum_{j=1,2}(-1)^j\delta_j\sin\delta_j l_0\cos\delta_{3-j}l_0 \tag{15a}$$

$$V^f_{b,1}=(D_b/Q)\delta_1\delta_2(\delta_1^2-\delta_2^2)\cos\delta_1 l_0\cos\delta_2 l_0 \tag{15b}$$

The first-order moment at the delamination section can be similarly expressed as

$$M_{b,1}=-D_b y''_{b,1}|_{x=l_0}=M^c_{b,1}+\zeta_1 M^f_{b,1} \tag{16}$$

where

$$M^c_{b,1}=(D_b/Q)(\delta_2^2-\delta_1^2)\sin\delta_1 l_0\sin\delta_2 l_0 \tag{17a}$$

$$M^f_{b,1}=(D_b/Q)\delta_1\delta_2\sum_{j=1,2}(-1)^{3-j}\delta_j\sin\delta_j l_0\cos\delta_{3-j}l_0 \tag{17b}$$

For $k^4_{b,0}<4\lambda_b$ the solution for the first-order equation (10a) is found in terms of δ_1 and δ_2 defined as

$$r=\sqrt{\lambda_b} \qquad \theta=\arccos[-k^2_{b,0}/(2r)] \tag{18a}$$

$$\delta_1=\sqrt{r}\cos(\theta/2) \qquad \delta_2=\sqrt{r}\sin(\theta/2) \tag{18b}$$

to be

$$y_{b,1}=C_{11}\sinh\delta_1 x\cos\delta_2 x+C_{12}\cosh\delta_1 x\sin\delta_2 x \tag{19}$$

By setting

$$Q=(\delta_2\sinh 2\delta_1 l_0-\delta_1\sin 2\delta_2 l_0)/2 \tag{20a}$$

the constants C_{1j} are obtained from eqns (4a) and (9) as follows:

$$C_{11}=[\zeta_1(\delta_2\cosh\delta_1 l_0\cos\delta_2 l_0+\delta_1\sinh\delta_1 l_0\sin\delta_2 l_0)-\cosh\delta_1 l_0\sin\delta_2 l_0]/Q \tag{20b}$$

$$C_{12}=[\zeta_1(\delta_2\sinh\delta_1 l_0\sin\delta_2 l_0-\delta_1\cosh\delta_1 l_0\cos\delta_2 l_0)+\sinh\delta_1 l_0\cos\delta_2 l_0]/Q \tag{20c}$$

The terms defining the first-order shear in eqn (14) are given in this case by

$$V^{c}_{b,1} = -D_b(\delta_1^2 + \delta_2^2)(\delta_1 \sin 2\delta_2 l_0 + \delta_2 \sinh 2\delta_1 l_0)/(2Q) \tag{21a}$$
$$V^{f}_{b,1} = (D_b/Q)(\delta_1^2 + \delta_2^2)2\delta_1\delta_2(\sinh^2 \delta_1 l_0 + \cos^2 \delta_2 l_0) \tag{21b}$$

and the terms for the first-order end moment in eqn (16) are given by

$$M^{c}_{b,1} = -(D_b/Q)2\delta_1\delta_2(\sinh^2 \delta_1 l_0 + \sin^2 \delta_2 l_0) \tag{22a}$$
$$M^{f}_{b,1} = D_b(\delta_1^2 + \delta_2^2)(\delta_2 \sinh 2\delta_1 l_0 + \delta_1 \sin 2\delta_2 l_0)/(2Q) \tag{22b}$$

2.2.1 Characteristic equation

The condition of shear equilibrium (6a), written for the first-order terms, produces an expression for ζ_1 (note that $V_{u,1} = 0$):

$$\zeta_1 = (V^{c}_{1,1} - V^{c}_{b,1})/(V^{f}_{b,1} - V^{f}_{1,1}) \tag{23}$$

The associated equilibrium and compatibility equations (6b) and (7) up to the order ϕ are given in terms of the moments at the interface:

$$M_{u,1} + M_{1,1} - M_{b,1} = P_{1,1}H/2 - P_{u,1}(T-H)/2 \tag{24}$$
$$T\frac{EH(T-H)}{4a(1-\nu_{13}\nu_{31})} = P_{u,1}(T-H)/2 - P_{1,1}H/2 \tag{25}$$

The end moments, $M_{i,1} = -D_i y''_{i,1}$, are given from eqns (16) and (22) in terms of the zero subscript quantities and the quantity ζ_1 which was found in eqn (23). Substituting these expressions into the above two equations, and eliminating the quantity $P_{1,1}H/2 - P_{u,1}(T-H)/2$, gives an equation for the critical buckling load, P_0, as follows:

$$D_u k_{u,0} \cot k_{u,0}a + M^{c}_{1,1} - M^{c}_{b,1} + (V^{c}_{1,1} - V^{c}_{b,1})(M^{f}_{1,1} - M^{f}_{b,1})/(V^{f}_{b,1} - V^{f}_{1,1})$$
$$= -TEH(T-H)/[4a(1-\nu_{13}\nu_{31})] \tag{26}$$

where the quantities $V^{c}_{i,1}$, $V^{f}_{i,1}$, $M^{c}_{i,1}$ and $M^{f}_{i,1}$, $i = 1, b$, in the above equation are given in eqns (15), (21), (17) and (22). This is the characteristic equation for the critical instability load.

2.2.2 Case of no elastic foundation

The solution to the first-order equation (10a) for $\beta = 0$ is

$$y_{b,1} = \frac{\sin k_{b,0}x}{k_{b,0}\cos k_{b,0}l_0} \tag{27a}$$
$$k_{b,0}^2 = P_0/D_b \qquad l_0 = (L-l)/2 \tag{27b}$$

The characteristic equation is again derived by eliminating the quantity

$P_{l,1}H/2 - P_{u,1}(T-H)/2$ in eqns (24) and (25), and is found to be (see also Ref. 3)

$$H^3 k_{u,0} \cot(k_{u,0} l/2) + (T-H)^3 k_{l,0} \cot(k_{l,0} l/2) - T^3 k_{b,0} \tan k_{b,0} l_0 + 6TH(T-H)/l = 0 \tag{28}$$

2.3 Post-buckling solution and delamination growth characteristics

When the terms in ϕ^2 are equated, the second-order differential equations are obtained. The solution for the upper and lower parts can be found in Refs 3 and 4. The solution for the base plate, which is simply supported, will be developed in the following. The differential equation is expressed as

$$D_b y_{b,2}^{(4)} + P_0 y''_{b,2} + \beta y_{b,2} = -P_{b,1} y''_{b,1} \tag{29}$$

The solution to the above equation is a superposition of the general solution and a particular solution.

For $k_{b,0}^4 > 4\lambda_b$ the solution to the second-order equation (29) satisfying the end fixity conditions (5), with δ_j defined in eqn (12), is found to be

$$y_{b,2} = \sum_{j=1,2} C_{2j} \frac{P_1}{D_b} \sin \delta_j x + B_{2j} \frac{P_1}{D_b} x \cos \delta_j x \tag{30a}$$

$$B_{2j} = \frac{(-1)^j C_{1j} \delta_j}{2(\delta_2^2 - \delta_1^2)} \tag{30b}$$

The constants C_{2j} are determined from the conditions (4a) and (9). In terms of Q defined by eqn (13a) and

$$A_1 = \sum_{j=1,2} B_{2j} l_0 \cos \delta_j l_0 \tag{31a}$$

$$A_2 = \sum_{j=1,2} B_{2j}(l_0 \delta_j \sin \delta_j l_0 - \cos \delta_j l_0) \tag{31b}$$

these constants are given by

$$C_{2j} = Q^{-1}[\zeta_2 (D_b/P_1) \delta_{3-j} \cos \delta_{3-j} l_0 - A_1 \delta_{3-j} \cos \delta_{3-j} l_0 + A_2 \sin \delta_{3-j} l_0] \tag{31c}$$

The second-order end shear can be written in the form

$$V_{b,2} = -D_b(y'''_{b,2} + k_{b,0}^2 y'_{b,2})|_{x=l_0} = P_1 V_{b,2}^g + \zeta_2 V_{b,2}^f \tag{32}$$

where

$$V^{g}_{b.2} = -(A_1/B)\delta_1\delta_2(\delta_1^2+\delta_2^2)\cos\delta_1 l_0\cos\delta_2 l_0$$
$$+\sum_{j=1,2}(A_2/Q)\delta_j^3\sin\delta_{3-j}l_0\cos\delta_j l_0$$
$$+B_{2j}\delta_j^2(3\cos\delta_j l_0 - l_0\delta_j\sin\delta_j l_0) \quad (33a)$$

$$V^{f}_{b.2} = (D_b/Q)\delta_1\delta_2(\delta_1^2+\delta_2^2)\cos\delta_1 l_0\cos\delta_2 l_0 \quad (33b)$$

The second-order end moment can also be written in the form

$$M_{b.2} = -D_b y''_{b.2}|_{x=l_0} = P_1 M^{g}_{b.2} + \zeta_2 M^{f}_{b.2} \quad (34)$$

where

$$M^{g}_{b.2} = (A_2/Q)(\delta_1^2+\delta_2^2)\sin\delta_1 l_0\sin\delta_2 l_0$$
$$+\sum_{j=1,2}B_{2j}\delta_j(2\sin\delta_j l_0 + l_0\delta_j\cos\delta_j l_0)$$
$$-(A_1/Q)\delta_j^2\delta_{3-j}\sin\delta_j l_0\cos\delta_{3-j}l_0 \quad (35a)$$

$$M^{f}_{b.2} = \sum_{j=1,2}(D_b/B)\delta_j^2\delta_{3-j}\sin\delta_j l_0\cos\delta_{3-j}l_0 \quad (35b)$$

For $k^4_{b.0} < 4\lambda_b$ the second-order solution satisfying eqns (29) and (5) is found to be

$$y_{b.2} = C_{21}\frac{P_1}{D_b}\sinh\delta_1 x\cos\delta_2 x + C_{22}\frac{P_1}{D_b}\cosh\delta_1 x\sin\delta_2 x$$
$$+B_{21}\frac{P_1}{D_b}x\cosh\delta_1 x\cos\delta_2 x + B_{22}\frac{P_1}{D_b}x\sinh\delta_1 x\sin\delta_2 x \quad (36a)$$

$$B_{2j} = -[C_{11}\delta_{3-j} + (-1)^j C_{12}\delta_j]/(8\delta_1\delta_2) \quad (36b)$$

where δ_j are defined in eqns (18). The constants C_{21} and C_{22} are found from eqns (4a) and (9). In terms of Q defined by eqn (20a) and

$$A_1 = B_{21}l_0\cosh\delta_1 l_0\cos\delta_2 l_0 + B_{22}l_0\sinh\delta_1 l_0\sin\delta_2 l_0 \quad (37a)$$

$$A_2 = B_{21}\cosh\delta_1 l_0\cos\delta_2 l_0 + (B_{21}\delta_1 + B_{22}\delta_2)l_0\sinh\delta_1 l_0\cos\delta_2 l_0$$
$$+(B_{22}\delta_1 - B_{21}\delta_2)l_0\cosh\delta_1 l_0\sin\delta_2 l_0 + B_{22}\sinh\delta_1 l_0\sin\delta_2 l_0 \quad (37b)$$

these constants are given as follows:

$$C_{21} = Q^{-1}\{A_2\cosh\delta_1 l_0\sin\delta_2 l_0 + [\zeta_2(D_b/P_1) - A_1]$$
$$\times(\delta_1\sinh\delta_1 l_0\sin\delta_2 l_0 + \delta_2\cosh\delta_1 l_0\cos\delta_2 l_0)\} \quad (37c)$$

$$C_{22} = Q^{-1}\{[\zeta_2(D_{\mathrm{b}}/P_1) - A_1](\delta_2 \sinh \delta_1 l_0 \sin \delta_2 l_0 - \delta_1 \cosh \delta_1 l_0 \cos \delta_2 l_0) \\ - A_2 \sinh \delta_1 l_0 \cos \delta_2 l_0\} \tag{37d}$$

The second-order end shear is expressed by eqn (32) with the definitions

$$\begin{aligned} V^{\mathrm{c}}_{\mathrm{b}.2} = Q^{-1}[& A_2(\delta_1^2 + \delta_2^2)(\delta_1 \sin 2\delta_2 l_0 + \delta_2 \sinh 2\delta_1 l_0)/2 \\ & - A_1(\delta_1^2 + \delta_2^2)2\delta_1\delta_2(\sinh^2 \delta_1 l_0 + \cos^2 \delta_2 l_0)] \\ & + [B_{21}(\delta_2^2 - \delta_1^2) - 6B_{22}\delta_1\delta_2] \cosh \delta_1 l_0 \cos \delta_2 l_0 \\ & + [B_{22}(\delta_2^2 - \delta_1^2) + 6B_{21}\delta_1\delta_2] \sinh \delta_1 l_0 \sin \delta_2 l_0 \\ & - (\delta_1^2 + \delta_2^2) l_0[(B_{22}\delta_2 - B_{21}\delta_1) \sinh \delta_1 l_0 \cos \delta_2 l_0 \\ & - (B_{21}\delta_2 + B_{22}\delta_1) \cosh \delta_1 l_0 \sin \delta_2 l_0] \end{aligned} \tag{38a}$$

$$V^{\mathrm{f}}_{\mathrm{b}.2} = D_{\mathrm{b}}(\delta_1^2 + \delta_2^2)2\delta_1\delta_2(\sinh^2 \delta_1 l_0 + \cos^2 \delta_2 l_0)/Q \tag{38b}$$

Likewise, the second-order end moment is given by eqn (34) with the definitions

$$\begin{aligned} M^{\mathrm{c}}_{\mathrm{b}.2} = Q^{-1}[& A_2 2\delta_1\delta_2(\sinh^2 \delta_1 l_0 + \sin^2 \delta_2 l_0) \\ & - A_1(\delta_1^2 + \delta_2^2)(\delta_1 \sin 2\delta_2 l_0 + \delta_2 \sinh 2\delta_1 l_0)/2] \\ & + [B_{21}(\delta_2^2 - \delta_1^2) - 2B_{22}\delta_1\delta_2] l_0 \cosh \delta_1 l_0 \cos \delta_2 l_0 \\ & + [B_{22}(\delta_2^2 - \delta_1^2) + 2B_{21}\delta_1\delta_2] l_0 \sinh \delta_1 l_0 \sin \delta_2 l_0 \\ & - 2(B_{21}\delta_1 + B_{22}\delta_2) \sinh \delta_1 l_0 \cos \delta_2 l_0 \\ & + 2(B_{21}\delta_2 - B_{22}\delta_1) \cosh \delta_1 l_0 \sin \delta_2 l_0 \end{aligned} \tag{39a}$$

$$M^{\mathrm{f}}_{\mathrm{b}.2} = D_{\mathrm{b}}(\delta_1^2 + \delta_2^2)(\delta_1 \sin 2\delta_2 l_0 + \delta_2 \sinh 2\delta_1 l_0)/(2Q) \tag{39b}$$

2.3.1 Solution

The shear condition (6a) allows determining ζ_2 in terms of the first-order (yet unknown) forces (note that $V_{\mathrm{u}.2} = P_{\mathrm{u}.1}$) as follows:

$$\zeta_2 = \frac{V^{\mathrm{c}}_{\mathrm{b}.2} - 1}{V^{\mathrm{f}}_{\mathrm{l}.2} - V^{\mathrm{f}}_{\mathrm{b}.2}} P_{\mathrm{u}.1} + \frac{V^{\mathrm{c}}_{\mathrm{b}.2} - V^{\mathrm{c}}_{\mathrm{l}.2}}{V^{\mathrm{f}}_{\mathrm{l}.2} - V^{\mathrm{f}}_{\mathrm{b}.2}} P_{\mathrm{l}.1} \tag{40}$$

The moment equilibrium equation (6b) for the second-order terms is

$$M_{\mathrm{u}.2} + M_{\mathrm{l}.2} - M_{\mathrm{b}.2} = P_{\mathrm{l}.2}H/2 - P_{\mathrm{u}.2}(T - H)/2 \tag{41}$$

while the geometric compatibility equation (7) for the second-order terms is given by

$$\frac{1}{2}\int_0^l y'^2_{\mathrm{u}.1}\,\mathrm{d}x - \frac{1}{2}\int_0^l y'^2_{\mathrm{l}.1}\,\mathrm{d}x = \frac{2(1 - \nu_{13}\nu_{31})l}{EH(T - H)}[P_{\mathrm{l}.2}H/2 - P_{\mathrm{u}.2}(T - H)/2] \tag{42}$$

Notice that the second-order moments, $M_{i.2} = -D_i y''_{i.2}$, at the interface are given from eqns (34), (35) and (39) in terms of the (yet undetermined) first-order end forces. Thus eliminating the quantity $P_{\mathrm{l}.2}H/2 - P_{\mathrm{u}.2}(T - H)/2$ from eqns (41) and (42), and taking into account eqn (40), gives the following equation for the first-order forces $P_{u.1}$ and $P_{l.1}$:

$$
\begin{aligned}
&[M^{s}_{u.2} - M^{s}_{b.2} + (V^{s}_{b.2} - 1)(M^{f}_{l.2} - M^{f}_{b.2})/(V^{f}_{l.2} - V^{f}_{b.2})]P_{u.1} \\
&\quad + [M^{s}_{l.2} - M^{s}_{b.2} + (V^{s}_{b.2} - V^{s}_{l.2})(M^{f}_{l.2} - M^{f}_{b.2})/(V^{f}_{l.2} - V^{f}_{b.2})]P_{l.1} \\
&\qquad = \left(\frac{2k_{u.0}a - \sin 2k_{u.0}a}{4k_{u.0}\sin^2 k_{u.0}a} - S_1\right)\frac{EH(T-H)}{4a(1-\nu_{13}\nu_{31})}
\end{aligned}
\tag{43}
$$

The second equation needed for finding $P_{u.1}$ and $P_{l.1}$ is the first-order equilibrium equation (24) at the interface, namely

$$
P_{l.1}H/2 - P_{u.1}(T-H)/2 = F_0(P_{u.0}, P_{l.0}) \tag{44}
$$

where F_0 is the left-hand side of eqn (26) which depends only on the zero-order quantities. In the above equations $V^{s}_{i.2}$, $V^{f}_{i.2}$, $M^{s}_{i.2}$ and $M^{f}_{i.2}$, $i = l, b$, take the values given in eqns (33), (38), (35) and (39) depending on the relative magnitude of the foundation modulus; S_1 is the shortening of the lower part due to first-order deflections and is given in Refs 3 and 4. The above system of linear equations allows finding $P_{u.1}$ and $P_{l.1}$, and hence the first-order applied end force $P_1 = P_{u.1} + P_{l.1}$.

2.3.2 Case of no elastic foundation

The case of $\beta = 0$ admits as a solution to the second-order equation:

$$
y_{b.2} = c_{b.2}\sin k_{b.0}x + \frac{P_{b.1}}{2D_b k^2_{b.0}\cos k_{b.0}l_0}\, x\cos k_{b.0}x \tag{45}
$$

and, from eqns (9),

$$
c_{b.2} = \frac{P_{b.1}}{2D_b k^3_{b.0}\cos^2(k_{b.0}l_0)}(l_0 k_{b.0}\sin k_{b.0}l_0 - \cos k_{b.0}l_0) \tag{46}
$$

Again, eliminating the quantity $P_{l.2}H/2 - P_{u.2}(T-H)/2$ from eqns (41) and (42) gives the following equation for the first-order forces $P_{u.1}$ and $P_{l.1}$:

$$
\begin{aligned}
&P_{u.1}\left[\frac{\cos(k_{u.0}l/2)}{2k_{u.0}\sin(k_{u.0}l/2)} - \frac{l\cos^2(k_{u.0}l/2)}{4\sin^2(k_{u.0}l/2)} - \frac{l}{4}\right. \\
&\qquad\left. - \frac{\sin k_{b.0}l_0}{2k_{b.0}\cos k_{b.0}l_0} - \frac{l_0\sin^2 k_{b.0}l_0}{2\cos^2 k_{b.0}l_0} - \frac{l_0}{2}\right] \\
&+ P_{l.1}\left[\frac{\cos(k_{l.0}l/2)}{2k_{l.0}\sin(k_{l.0}l/2)} - \frac{l\cos^2(k_{l.0}l/2)}{4\sin^2(k_{l.0}l/2)} - \frac{l}{4}\right. \\
&\qquad\left. - \frac{\sin k_{b.0}l_0}{2k_{b.0}\cos k_{b.0}l_0} - \frac{l_0\sin^2 k_{b.0}l_0}{2\cos^2 k_{b.0}l_0} - \frac{l_0}{2}\right] \\
&\quad = \left[\frac{\sin(k_{l.0}l) - k_{l.0}l}{k_{l.0}\sin^2(k_{l.0}l/2)} - \frac{\sin(k_{u.0}l) - k_{u.0}l}{k_{u.0}\sin^2(k_{u.0}l/2)}\right]\frac{EH(T-H)}{8l(1-\nu_{13}\nu_{31})}
\end{aligned}
\tag{47}
$$

The second equation needed for finding $P_{u,1}$ and $P_{l,1}$ is the first-order equilibrium equation (24) at the interface, namely

$$P_{l,1}H/2 - P_{u,1}(T-H)/2 = \frac{P_{u,0}}{k_{u,0}\tan(k_{u,0}l/2)} + \frac{P_{l,0}}{k_{l,0}\tan(k_{l,0}l/2)} - \frac{P_0\tan k_{b,0}l_0}{k_{b,0}} \tag{48}$$

The above system of linear equations allows the finding of $P_{u,1}$ and $P_{l,1}$, and hence of the first-order applied force $P_1 = P_{u,1} + P_{l,1}$.

2.3.3 Growth characteristics

The post-buckling solution that has been obtained can be used to study the post-critical state of deformation. In addition, the initiation of delamination growth can now be analyzed on the basis of a Griffith-type fracture criterion. Predicting whether the delamination will grow requires an evaluation of the energy release rate. This quantity, which is the differential of the total potential energy with respect to the delamination length, can be easily calculated by using the path-independent J-integral expression in terms of the axial forces and bending moments acting across the various cross-sections adjacent to the tip of the delamination.[6] In terms of the quantities

$$P^* = P(H/T) - P_u \qquad M^* = M_u \qquad M^{**} = P^*T/2 - M^* \tag{49}$$

the energy release rate per unit width is expressed as

$$G = \frac{12(1-\nu_{13}\nu_{31})}{2E}\left\{\frac{P^{*2} + 12(M^*/H)^2}{H} + \frac{P^{*2} + 12[M^{**}/(T-H)]^2}{T-H}\right\} \tag{50}$$

The J-integral method that is used here does not distinguish between the Mode I and II components. It should be noted that future work should consider this question, since there may be a dependence of the fracture toughness on the ratio of those two components.

3 DISCUSSION OF NUMERICAL RESULTS

Numerical examples are presented for the case of $T/H = 6$, $L/H = 200$. The characteristic equation (26) is solved for the critical load P_{0c}. If $P_{0c} < (P_{glo}, P_{loc})$, with P_{glo} and P_{loc} given by eqns (1) and (2), then combined 'mixed' buckling involving out-of-plane deflections for both the delaminated layer and the base plate takes place. Otherwise the critical load is the minimum of the above, corresponding to global buckling (typical of very short delaminations) or local buckling (typical of very large delaminations). As the above analysis shows, the conditions of end attachment, by affecting the

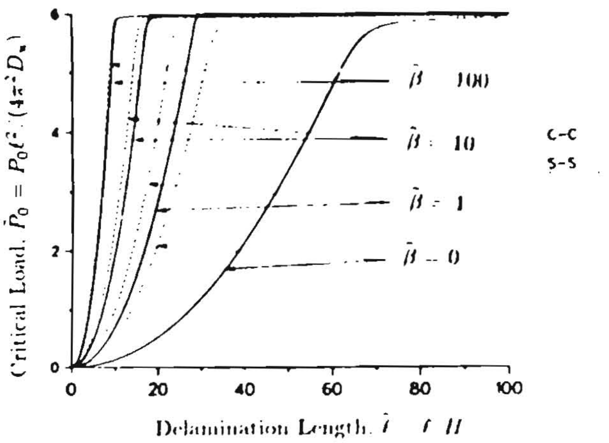

Fig. 3. Buckling load vs delamination length for $T/H = 6$, $L/H = 200$ and a set of foundation moduli, for both the simply-supported (solid line) and clamped–clamped (dashed line) cases. The usual case without an elastic foundation corresponds to $\bar{\beta} = 0$.

equations that describe the deformation of the base plate, influence the performance of the whole system. To illustrate the effect of end fixity, the variation of the critical load, normalized with respect to the Euler load for the delaminated layer, $\bar{P}_0 = P_0 l^2/(4\pi^2 D_u)$, vs delamination length, $\bar{l} = l/H$, for both the simply-supported and clamped–clamped cases, is given in Fig. 3 for a set of values for $\bar{\beta}$. The foundation modulus is normalized as $\bar{\beta} = 3\beta l^4/(16\pi^4 D_b)$. It can be concluded that for high values of the foundation modulus $\bar{\beta}$ the end fixity has a small effect on the instability point. This is because in these cases the instability is mostly controlled by the local buckling mode as opposed to the global one that is more pronounced for low values of $\bar{\beta}$. Notice that in both cases, as $\bar{\beta}$ increases, the curves are shifted to the left, indicating the attainment of loads similar in magnitude to the local buckling ones for smaller delamination lengths.

The combined effect of the location of the delamination through the thickness for different foundation moduli and the end fixity is indicated in Table 1, which gives the values of the delamination length, l/L, for which the characteristic equation has no roots less than the local buckling load given by eqn (2) (the 'local buckling threshold delamination lengths'). As the delaminated layer becomes thinner, local buckling is reached at smaller lengths, and an increase in the foundation modulus reduces this 'threshold length' more effectively. For example, for $H/T = 1/15$ and $\bar{\beta} > 10^4$, local buckling occurs for practically all delamination lengths. In addition, local buckling occurs in general at smaller delamination lengths in the clamped–clamped case than in the simply-supported one. It should also be noted that, for even smaller lengths than the ones given in Table 1, the solution of the

TABLE 1

Values of Delamination Length, l/L, beyond which the Characteristic Equation has no Roots less than the Local Buckling Load ('Local Buckling Threshold')

H/T*	β: 0	1	10	10^2	10^3	10^4
1/6 (C–C)	0·966	0·963	0·961	0·951	0·816	0·511
(S-s)	1·000	1·000	1·000	1·000	0·844	0·512
1/9 (C–C)	0·943	0·936	0·921	0·759	0·516	0·291
(S-s)	1·000	1·000	1·000	0·784	0·516	0·291
1/12 (C–C)	0·603	0·471	0·279	0·161	0·089	0·051
(S-s)	1·000	0·475	0·288	0·161	0·089	0·051
1/15 (C–C)	0·451	0·277	0·166	0·096	0·054	0·031
(S-s)	1·000	0·279	0·169	0·096	0·054	0·031

*C–C: Clamped–Clamped; S-s: Simply-supported.

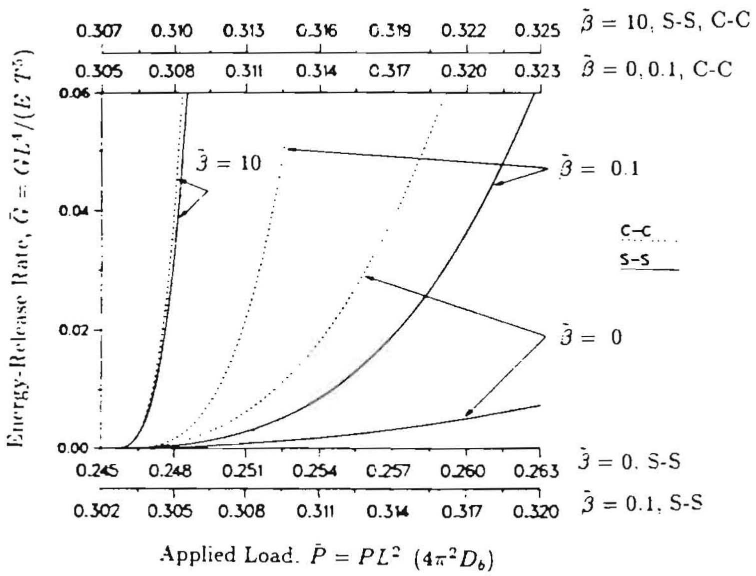

Fig. 4. Strain energy release rate vs applied compressive force during the initial post-buckling stage for both the simply-supported (solid line) and clamped–clamped (dashed line) cases ($T/H = 6$, $L/H = 200$, and delamination length $l/H = 60$). The different load scales are of the same length and correspond to the different initial buckling loads.

characteristic equation is very close to (by less than 1% smaller than) the local buckling load, as also is evident from Fig. 3.

To illustrate the effect of end fixity on the post-critical characteristics, the variation during the initial post-buckling stage of the normalized strain energy release rate, $\bar{G} = G/(ET^5/L^4)$, vs the applied load normalized with respect to the Euler load for the entire beam with no elastic foundation, $\bar{P} = PL^2/(4\pi^2 D_b)$, is plotted in Fig. 4 for the example case of $T/H = 6$, $L/H = 200$, and delamination length $l/H = 60$, for a set of values of the foundation modulus. Both the simply-supported and clamped–clamped cases are considered. Since the critical load changes with $\tilde{\beta}$, different scales (of the same length) are used on the load axis. As for the critical load, the end fixity does not much affect the slope of the G–P curves for the high values of the foundation modulus, $\tilde{\beta}$. For the low values of $\tilde{\beta}$ the simply-supported case results in less steep curves. This decreased slope means that (1) the laminate can withstand more load beyond the critical point in the initial post-buckling stage before delamination growth takes place, and (2) there could be less energy absorbed since growth would not be promoted as much. Notice also that for both cases the curves are steeper for a larger $\tilde{\beta}$.

4 CONCLUSIONS

The effects of end fixity (clamped–clamped vs simply-supported) on the buckling and post-buckling of delaminated composites under compressive loads are investigated. First, the perturbation technique is used to derive the analytical solution for the post-buckling behavior of delaminated composites with simply-supported ends. The results are compared with the characteristics of the clamped–clamped case. Both the usual case and that of a delaminated composite beam-plate on an elastic foundation are treated. End fixity effects are found to be more pronounced for lower values of the foundation modulus.

REFERENCES

1. Chai, H., Babcock, C. D. & Knauss, W. G., One-dimensional modelling of failure in laminated plates by delamination buckling. *Int. J. Solids Structures*, **17** (1981) 1069.
2. Yin, W.-L., Sallam, S. N. & Simitses, G. J., Ultimate axial load capacity of a delaminated beam-plate. *AIAA Journal*, **24** (1986) 123.
3. Kardomateas, G. A. & Schmueser, D. W., Effect of transverse shearing forces on buckling and postbuckling of delaminated composites under compressive loads.