

TORSION OF A FILAMENT WOUND ANISOTROPIC ELLIPTIC CYLINDER WITH VARIABLE MODULI OF ELASTICITY

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Abstract—The problem of torsion of rods produced by filament winding on an elliptical mandrel and possessing variable moduli of elasticity through the thickness (for example, because of a varying lay-up angle) is formulated. The material possesses curvilinear anisotropy, referred to the coordinate system that is inherent to the geometry of the filament wound body. It is further assumed that there are planes of elastic symmetry perpendicular to the shaft axis. Results on the stress and displacement distribution are provided and discussed. Moreover, a study of the influence of a compound bar construction, i.e. a composite bar with an external or internal reinforcement on the maximum interlaminar shear stress, is performed.

NOTATION

$\pm c$	foci of the elliptical section of the mandrel
e_1	major semiaxis of the elliptical section of the mandrel ($= c \cosh a$)
e_2	minor semiaxis of the elliptical section of the mandrel ($= c \sinh a$)
ξ	thicknesswise direction (normal to the layers)
η	direction tangent to the periphery
z	direction along the body axis
u_i	displacement components ($i = \xi, \eta, z$)
v_{0i}	rigid body translation ($i = x, y, z$)
ω_i	rigid body rotation ($i = x, y, z$)
θ	a constant (scaling the twisting moment)
T	thickness of the body
a_{ij}	anisotropic elastic constants
$\Psi(\xi, \eta)$	stress function
M_t	twisting moment
W	warping displacement (equation 7c)
h	$c \sqrt{\sinh^2 a + \sin^2 \eta}$ (equation 4)
q	$h + \xi \frac{c^2 \sinh 2a}{2h^2}$ (equation 5)

INTRODUCTION

Torsion in anisotropic elasticity has been considered for the cases of prismatic or cylindrical bars possessing either rectilinear or cylindrical anisotropy [1]. For the first case, all directions parallel to an orthogonal Cartesian coordinate system are equivalent with respect to the elastic properties. For the second case the same is true for the directions of a polar coordinate system (r, θ, z).

Consider a body produced by filament winding on a mandrel of elliptical cross-section. Such filament wound bars could be used in automotive structural components. For this geometry, the equivalent directions with respect to the elastic properties are the (ξ, η, z) (Fig. 1), where ξ is the direction normal to the layers (thicknesswise direction), η is the tangent to the periphery and z is along the body axis. Hence, this body possesses curvilinear anisotropy referred to the (ξ, η, z) system. For this system of curvilinear coordinates, the infinitesimal elements isolated by the three pairs of coordinate planes have the same elastic properties.

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Note that in Fig. 1, $\xi = 0$ is the inner boundary, $\xi = T$ is the outer boundary, $\eta = 0$ or 2π is the positive major semiaxis and $\eta = \pi$ is the negative major semiaxis.

A formulation dealing with the theory of elasticity of such bodies that are produced by filament winding on an elliptical mandrel was developed in Ref. [2]. The type of curvilinear anisotropy inherent to the geometry of the filament wound body, as described above, was named 'elliptical anisotropy'. Moreover, the solution was presented for the special case of torsion of an orthotropic body. This case corresponds in practice, for example, to filament winding of an orthotropic composite sheet at a constant 90° angle. Therefore, that analysis cannot be applied to the case of laying-up this sheet at a different angle, since orthotropy is not preserved under a coordinate axis rotation. In addition, there may be a case of hybrid construction (layers of different material) that we may wish to analyze.

In this paper, the theory of torsion of bodies produced by filament winding on a cylindrical mandrel of elliptical cross-section is generalized to the case of rods with variable moduli of elasticity through the thickness. This work will also be an extension to the solution for the case of a shaft of circular cross-section [1]. In the following, we assume that the stresses do not vary along the generator and that there are no body forces.

THEORY

Consider a mandrel of elliptical cross-section with semiaxes $c \cosh a$ and $c \sinh a$ (foci at $\pm c$) (Fig. 1). Since the body produced by filament winding on this mandrel has constant thickness, the outer bounding surface is not a true ellipse, unlike the internal one (in the limit of infinite thickness the outer contour is a circle). This feature, as well as the fact that the anisotropy is defined in a curvilinear coordinate system, makes the present problem unique. Now, for the body produced by filament winding on that mandrel, the moduli of elasticity are assumed to depend on the coordinate ξ and, consequently, the coefficients of deformation a_{ij} , where $i, j = 1, 6$ are functions of ξ . Furthermore, it is assumed that at each point there is one plane of elastic symmetry perpendicular to the z axis (axis of the body). Hence, the constants $a_{14}, a_{24}, a_{34}, a_{46}, a_{15}, a_{25}, a_{35}, a_{56}$ are equal to zero. Therefore, the equations of the generalized Hooke's law (stress-strain relations) have the form:

$$\varepsilon_{\xi\xi} = a_{11}\sigma_{\xi\xi} + a_{12}\sigma_{\eta\eta} + a_{13}\sigma_{zz} + a_{16}\tau_{\xi\eta}, \quad (1a)$$

$$\varepsilon_{\eta\eta} = a_{12}\sigma_{\xi\xi} + a_{22}\sigma_{\eta\eta} + a_{23}\sigma_{zz} + a_{26}\tau_{\xi\eta}, \quad (1b)$$

$$\varepsilon_{zz} = a_{13}\sigma_{\xi\xi} + a_{23}\sigma_{\eta\eta} + a_{33}\sigma_{zz} + a_{36}\tau_{\xi\eta}, \quad (1c)$$

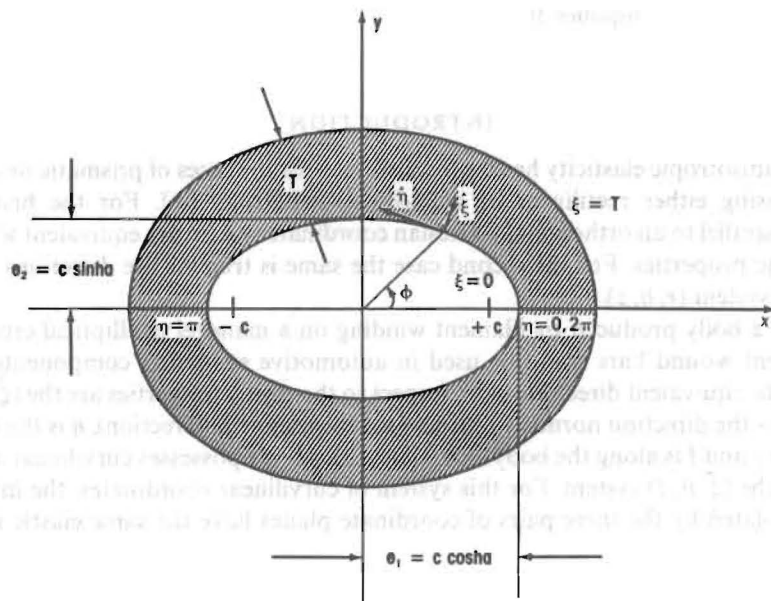


FIG. 1. Cross-section of the filament wound elliptic cylinder.

$$\gamma_{\eta z} = a_{44}\tau_{\eta z} + a_{45}\tau_{\xi z}, \quad (1d)$$

$$\gamma_{\xi z} = a_{45}\tau_{\eta z} + a_{55}\tau_{\xi z}, \quad (1e)$$

$$\gamma_{\xi\eta} = a_{16}\sigma_{\xi\xi} + a_{26}\sigma_{\eta\eta} + a_{36}\sigma_{zz} + a_{66}\tau_{\xi\eta}. \quad (1f)$$

The equations of equilibrium in the system of curvilinear coordinates (ξ, η, z) and the strain-displacement equations are given in Ref. [2]. We seek a solution for pure torsion (the forces being distributed over the ends and reducing at either of them to a twisting moment M_t) by setting

$$\sigma_{\xi\xi} = \sigma_{\eta\eta} = \sigma_{zz} = \tau_{\xi\eta} = 0, \quad (2)$$

which leaves only the equation

$$(q\tau_{\xi z})_{,\xi} + \tau_{\eta z, \eta} = 0, \quad (3)$$

with the definitions [2]:

$$h = h(\eta) = c\sqrt{\sinh^2 a + \sin^2 \eta}, \quad (4)$$

$$q = \left[\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right]^{1/2} = h + \xi \frac{c^2 \sinh 2a}{2h^2}. \quad (5)$$

These definitions arise in accordance with the curvilinear coordinate system being used; for example, for a polar coordinate system (r, θ) , we would have r in place of q in equation (3).

Combining equations (2) and (1) gives the strain-displacement equations as follows:

$$u_{\xi, \xi} = 0, \quad u_{\eta, \eta} + \frac{\partial q}{\partial \xi} u_{\xi} = 0, \quad u_{z, z} = 0, \quad (6a)$$

$$u_{\xi, \eta} + qu_{\eta, \xi} - \frac{\partial q}{\partial \xi} u_{\eta} = 0, \quad (6b)$$

$$u_{\eta, z} + \frac{1}{q} u_{z, \eta} = a_{44}\tau_{\eta z} + a_{45}\tau_{\xi z}, \quad (6c)$$

$$u_{\xi, z} + u_{z, \xi} = a_{45}\tau_{\eta z} + a_{55}\tau_{\xi z}. \quad (6d)$$

The first four equations, (6a) and (6b), are satisfied identically if we set

$$u_{\xi} = z\bar{\theta} \frac{\partial h}{\partial \eta} + \frac{c}{h} (v_{0x} \sinh a \cos \eta + v_{0y} \cosh a \sin \eta) + \omega_z \frac{\partial h}{\partial \eta}, \quad (7a)$$

$$u_{\eta} = z\bar{\theta} \left(\xi + h \frac{\partial q}{\partial \xi} \right) + \frac{c}{h} (-v_{0x} \cosh a \sin \eta + v_{0y} \sinh a \cos \eta) + \omega_z \left(\xi + h \frac{\partial q}{\partial \xi} \right), \quad (7b)$$

$$u_z = W(\xi, \eta) + c\omega_x \sin \eta \left(\sinh a + \frac{\xi}{h} \cosh a \right) - c\omega_y \cos \eta \left(\cosh a + \frac{\xi}{h} \sinh a \right) + v_{0z}, \quad (7c)$$

where $\bar{\theta}$ is a constant that turns out to depend on the magnitude of the twisting moment, and $W(\xi, \eta)$ is the warping displacement. In the above expressions, the displacements have been defined within a rigid body translation and rotation expressed in the Cartesian coordinate system by (v_{0x}, v_{0y}, v_{0z}) and $(\omega_x, \omega_y, \omega_z)$ respectively.

By substituting the above expressions for u_{ξ}, u_{η} into equations (6c), (6d), the following equations are produced for W :

$$a_{45}\tau_{\eta z} + a_{55}\tau_{\xi z} - \frac{\partial W}{\partial \xi} = \bar{\theta} \frac{\partial h}{\partial \eta}, \quad (8a)$$

$$a_{44}\tau_{\eta z} + a_{45}\tau_{\xi z} - \frac{1}{q} \frac{\partial W}{\partial \eta} = \bar{\theta} \left(\xi + h \frac{\partial q}{\partial \xi} \right). \quad (8b)$$

Notice that the elastic constants a_{ij} , $i, j = 4, 5$ depend on ξ , i.e. $a_{ij} = a_{ij}(\xi)$.

We can satisfy the equation of equilibrium (3) by introducing the stress function $\Psi(\xi, \eta)$:

$$\tau_{\xi z} = \frac{1}{q} \Psi_{,\eta}; \quad \tau_{\eta z} = -\Psi_{,\xi}. \quad (9)$$

Using equation (9) and eliminating W from (8a), (8b) yields

$$qa_{44} \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{1}{q} a_{55} \frac{\partial^2 \Psi}{\partial \eta^2} + \left(\frac{\partial q}{\partial \xi} a_{44} + qa'_{44} \right) \frac{\partial \Psi}{\partial \xi} - \left(\frac{1}{q^2} \frac{\partial q}{\partial \eta} a_{55} + a'_{45} \right) \frac{\partial \Psi}{\partial \eta} - 2a_{45} \frac{\partial^2 \Psi}{\partial \xi \partial \eta} = -2\bar{\theta}q, \quad (10)$$

where the prime on the elastic constants denotes the derivative with respect to ξ .

This equation is solved in conjunction with the boundary conditions of $\tau_{\xi z} = 0$ on the inner and outer bounding surface, which translates into the conditions of constant values, c_i , of the stress function on the boundary:

$$\Psi = c_i \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = T.$$

One of those two constants, for example on the outer boundary, can be chosen arbitrarily to be zero, hence we let $c_2 = 0$.

The other constant, c_1 , is found from the condition of single valued displacement u_z , expressed by the vanishing of the contour integral around the periphery of the inner boundary; furthermore, by using equation (7c),

$$\int_0^{2\pi} \left(\frac{\partial u_z}{\partial \eta} \right)_{\xi=0} d\eta = \int_0^{2\pi} \left(\frac{\partial W}{\partial \eta} \right)_{\xi=0} d\eta = 0. \quad (11)$$

Substituting equation (8b) for $\partial W/\partial \eta$, we obtain

$$\int_0^{2\pi} \left[h(a_{44}\tau_{\eta z})_{\xi=0} - \bar{\theta} \frac{c^2 \sinh 2a}{2} \right] d\eta = 0. \quad (12)$$

Now let us consider the conditions at the ends of a body of finite length. Since $\sigma_{zz} = 0$, the resultant axial force P_z and bending moments about the x, y axes are zero and the only loading is the twisting moment M_t , given by [2]

$$\iint \left[\left(\xi + h \frac{\partial q}{\partial \xi} \right) \tau_{\eta z} + \frac{\partial h}{\partial \eta} \tau_{\xi z} \right] q d\xi d\eta = M_t. \quad (13)$$

This equation will serve to determine the arbitrary constant $\bar{\theta}$, which is a scaling factor for the end loading. To eliminate this constant from equation (10), we set

$$\Psi(\xi, \eta) = \bar{\theta}\psi(\xi, \eta), \quad (14)$$

and substitute in equations (10) and (12). Then we have to solve the following system for the stress function ψ :

—differential equation

$$qa_{44}\psi_{,\xi\xi} + \frac{1}{q} a_{55}\psi_{,\eta\eta} + \left(\frac{\partial q}{\partial \xi} a_{44} + qa'_{44} \right) \psi_{,\xi} - \left(\frac{1}{q^2} \frac{\partial q}{\partial \eta} a_{55} + a'_{45} \right) \psi_{,\eta} - 2a_{45}\psi_{,\xi\eta} = -2q, \quad (15)$$

—boundary conditions

$$\psi = 0 \quad \text{at} \quad \xi = T, \quad (16)$$

$$\int_0^{2\pi} h(a_{44}\psi_{,\xi})_{\xi=0} d\eta = -c^2\pi \sinh 2a. \quad (17)$$

Subsequently, the constant $\bar{\theta}$ is found from the end condition, equation (13), to be

$$\bar{\theta} = M_t / C_R, \quad (18)$$

where

$$C_R = \int_{\eta=0}^{\eta=2\pi} \int_{\xi=0}^{\xi=T} \left[- \left(\xi + h \frac{\partial q}{\partial \xi} \right) q \psi_{,\xi} + \frac{\partial h}{\partial \eta} \psi_{,\eta} \right] d\xi d\eta. \quad (19)$$

Once the stress function and hence the stresses have been determined, the warping (out-of-plane) displacement W can be found by numerically integrating equation (8a). Notice that the in-plane displacements have been found in equations (7a, b).

A limiting case of the problem being considered is the case of a filament wound circular cylinder. The solution for the stresses for a circular section has been obtained in Ref. [1]. The limiting case of the curvilinear anisotropy referred to the (ξ, η, z) system considered here is cylindrical anisotropy (i.e. anisotropy referred to the r, θ, z system). For N layers, with the k layer interfaces at radii r_k, r_{k+1} (so that the inner radius of the cylinder is $r_1 = a$ and the outer $r_{N+1} = b$), the solution for the stresses in a circular cylinder under torsion is given by

$$\tau_{\theta z}^k = \bar{\theta} \frac{r}{a_{44}^k} = \frac{M_t}{C_1} \frac{r}{a_{44}^k}, \quad \tau_{rz} = 0. \quad (20a)$$

In the above equation, C_1 is the torsional rigidity, given by

$$C_1 = 2\pi \int_a^b \frac{r^3}{a_{44}(r)} dr = \frac{\pi}{2} \sum_{k=1}^N \frac{r_{k+1}^4 - r_k^4}{a_{44}^k}. \quad (20b)$$

So, for a circular cylinder the stress $\tau_{\theta z}$ within a layer is proportional to the shear modulus of the material $G_{\theta z}^k = 1/a_{44}^k$ and varies linearly with respect to the distance r .

For this limiting case, the displacements are in turn (neglecting any rigid body translation and assuming rotation only about the z axis)

$$u_r = 0, \quad u_\theta = \bar{\theta} r z + \omega_z r, \quad (21a)$$

$$W = \bar{\theta} \int \frac{a_{45}(r)}{a_{44}(r)} r dr. \quad (21b)$$

For a cylinder of N layers, with k being the layer number, the displacements at each layer are:

$$W^1 = \frac{\bar{\theta}}{2} \frac{a_{45}^1}{a_{44}^1} (r^2 - r_1^2), \quad (22a)$$

$$W^k = \frac{\bar{\theta}}{2} \frac{a_{45}^k}{a_{44}^k} (r^2 - r_k^2) + \frac{\bar{\theta}}{2} \sum_{i=1}^{k-1} \frac{a_{45}^i}{a_{44}^i} (r_{i+1}^2 - r_i^2), \quad k > 1. \quad (22b)$$

From equation (21b) it is seen that the cross-sections of a circular cylinder subjected to torsion remain planar only if $a_{45} = 0$. This is not true for the case of the cylindrical body of elliptical cross-section.

RESULTS AND DISCUSSION

An example of a rod with variable moduli of elasticity is a shaft produced by filament winding of several layers at different lay-up angles on a mandrel of elliptical cross-section. Naturally, each of the layers possesses 'elliptical' anisotropy as described in the introduction, with the axis of anisotropy coinciding with the axis of the shaft, so that there are planes of elastic symmetry normal to the axis of the shaft. Moreover, we assume that there are no initial stresses on the rod (the latter can appear in the manufacturing process) and that the layers are perfectly bonded along their contact surfaces (so that the displacements remain continuous across layer interfaces).

To obtain the stress distribution, the finite difference method was used to solve equations (15) and (17), together with the condition $\psi|_{\xi=T} = 0$. In this context, the structure of equation (15) shows that the domain can be reduced by symmetry conditions to one half of the

elliptical cross-section, i.e. for $0 \leq \eta \leq \pi$. Then, the values of the function ψ for $\pi \leq \eta \leq 2\pi$ are given by $\psi(\xi, \eta) = \psi(\xi, \eta - \pi)$. In addition, owing to the local nature of the finite difference operators, the coefficient matrix is highly banded. By storing only those terms of the coefficient matrix that fall within the bandwidth, a simple Gauss elimination scheme can be used to solve large numbers of equations with maximum economy. Let us assume that we have M_h points on the half ellipse. Since, by the symmetry discussed above, for $j=1$, $\psi_{i,j-1} = \psi_{i,M_h-1}$, and for $j=M_h$, $\psi_{i,j+1} = \psi_{i,2}$, central differences [3] for both with respect to ξ and with respect to η for $j=1, 3, \dots, M_h$ can be used. Concerning equation (17), forward difference formulas are used to obtain $\psi_{,\xi}$ at $\xi=0$. The semibandwidth of the resulting coefficient matrix is equal to $2M_h + 3$.

Let us first look at the effect of the stacking sequence on the state of stress. Consider as an example a four-ply construction with ply thickness t_p and mandrel dimensions $e_1/t_p = 35/3$, $e_2/t_p = 20/3$ ($e_{1,2}$ are the major and minor semiaxes respectively). The material used is glass-fiber unidirectional, with the following typical constants [1]: $E_1 = 57 \text{ GN m}^{-2}$, $E_2 = 14 \text{ GN m}^{-2} = E_3$; $G_{12} = 5.7 \text{ GN m}^{-2} = G_{31}$, $G_{23} = 5 \text{ GN m}^{-2}$; $\nu_{12} = 0.277$, $\nu_{23} = 0.400$, $\nu_{31} = 0.068$. Notice that for a 90° lay-up, $1 \equiv \eta$, $2 \equiv z$, $3 \equiv \xi$. The elastic constants for a particular lay-up angle are determined by a transformation of those elastic constants under the corresponding rotation of the coordinate system [1]. Figure 2 shows the through-the-thickness variation of the stress $\tau_{\eta z}$ for two stacking sequences (from inside towards outside surface): $[90^\circ + 45^\circ - 45^\circ 90^\circ]$ and $[-45^\circ 90^\circ 90^\circ + 45^\circ]$, and in comparison with a $[90^\circ 90^\circ 90^\circ 90^\circ]$ lay-up. A transition to a lower lay-up angle results in a drop in the stress $\tau_{\eta z}$, while a transition to a higher lay-up angle is accompanied by an increase. Since $\tau_{\eta z}$ increases through the thickness, it could be desirable to place the lower lay-up angles on the outside.

In the following we will consider as an example case the lay-up sequence $[-45^\circ 90^\circ 90^\circ + 45^\circ]$. The solution for a circular cylinder is known [1] and is given by equation (20). To

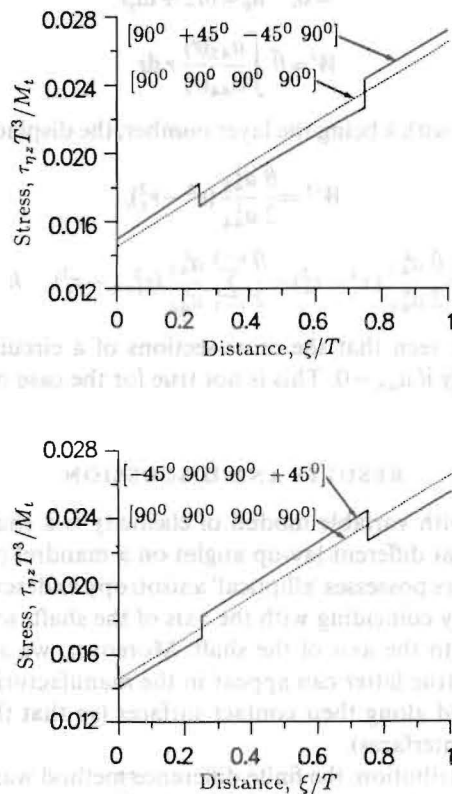


FIG. 2. Thicknesswise distribution (along ξ) of the shear stress $\tau_{\eta z}$ at the minor axis site $\phi = 90^\circ$, illustrating the effects of the different stacking sequences.

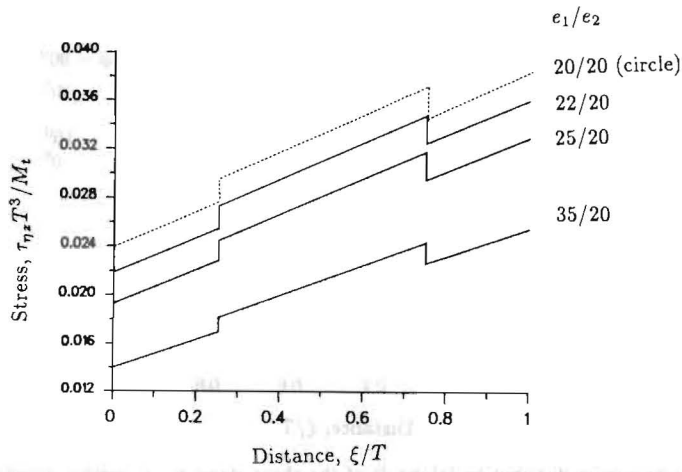


FIG. 3. Shear stress, $\tau_{\eta z}$, at $\phi = 90^\circ$, vs thickness, ξ , curves for a decreasing major to minor mandrel axis ratio. Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$. The dashed curve is the solution for a circular cross-section.

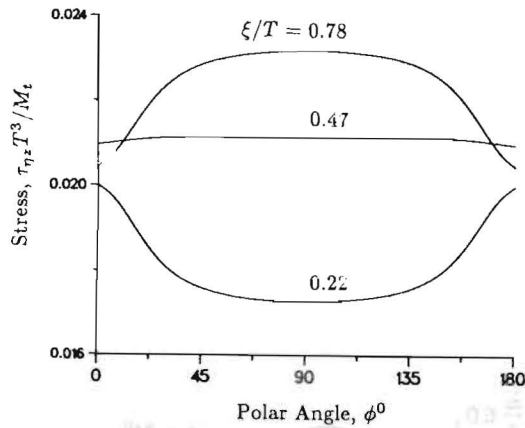


FIG. 4. Angular distribution of the shear stress $\tau_{\eta z}$ at certain distances ξ through the thickness. Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$.

compare with the present one for the elliptical cross-section, Fig. 3 gives the $\tau_{\eta z}$ versus thickness ξ curves for diminishing major/minor axis ratios (the minor axis dimension is kept constant). The solution is seen to approach the one for the circular section as the major/minor axis ratio tends to become unity.

Figure 4 shows the angular distribution of the shearing stress $\tau_{\eta z}$ at certain distances ξ through the thickness. The distribution is of the same type as for the orthotropic case [2]. For a distance ξ below a particular value, the shear stress has its maximum at the major axis and its minimum at the minor one. The exact opposite happens for layers that are above this threshold value. The variation of $\tau_{\eta z}$ with the thicknesswise distance ξ is shown in Fig. 5. The stress $\tau_{\eta z}$ undergoes little change through the thickness near the major axis, whereas near the minor axis the stress increases markedly with distance. Furthermore, the curves have a common intersection point at about $\xi/T = 0.47$, where there is practically no angular variation (see also Fig. 4).

The distribution of the stress $\tau_{\xi z}$ with respect to both the polar angle ϕ and the thickness ξ is shown in Fig. 6. It is seen that this component of stress, which is two orders of magnitude less than the stress $\tau_{\eta z}$, follows the same pattern as for the orthotropic case. The maximum value is shifted outwards for points closer to the minor axis. Notice that now $\tau_{\xi z}$ is not zero on the minor or major axis, unlike in the orthotropic case.

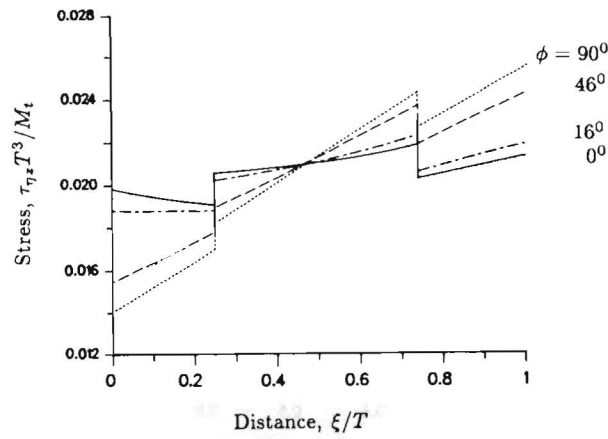


FIG. 5. Thicknesswise distribution (along ξ) of the shear stress τ_{rz} at certain angular locations. Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$.

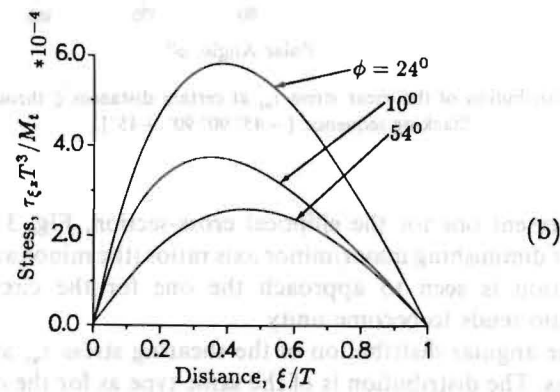
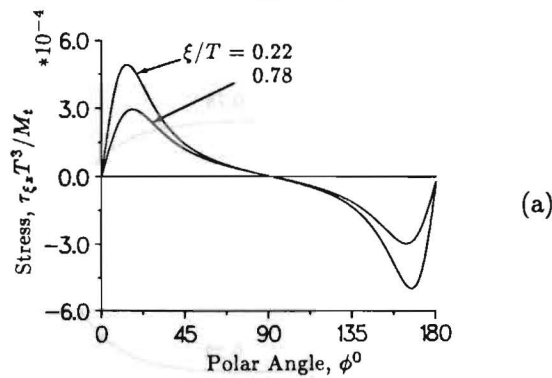


FIG. 6. Angular (a) and thicknesswise (b) distribution of the shear stress τ_{xz} . Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$.

It is interesting to examine the displacements and in particular the warping of the cross-section. The displacement W can be obtained by numerically integrating equation (17a). Assume zero displacement at the inner edge: $W|_{\xi=0}=0$. Figure 7 illustrates the angular variation of W . Figure 8 shows the thicknesswise distribution of the warping displacement. Near the major and minor axis the displacement is controlled by the elastic constant a_{45} and thus it increases or decreases depending on the sign of a_{45} , or is constant if $a_{45}=0$, as in the center (90° lay-up) section. However, away from the major or minor axes, it varies monotonically since it is controlled by the other elastic constants (which have the same sign

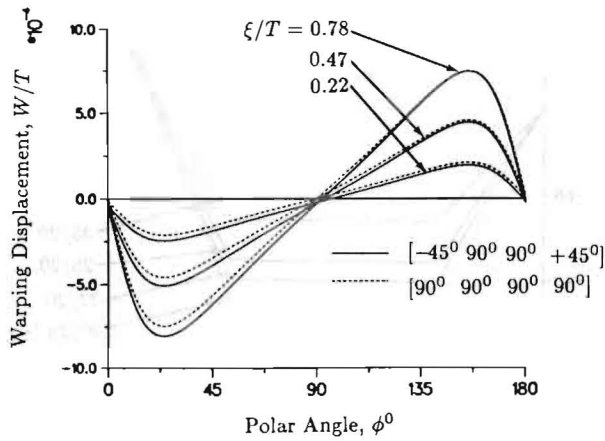


FIG. 7. Angular variation of the warping displacement W at certain distances through the thickness.

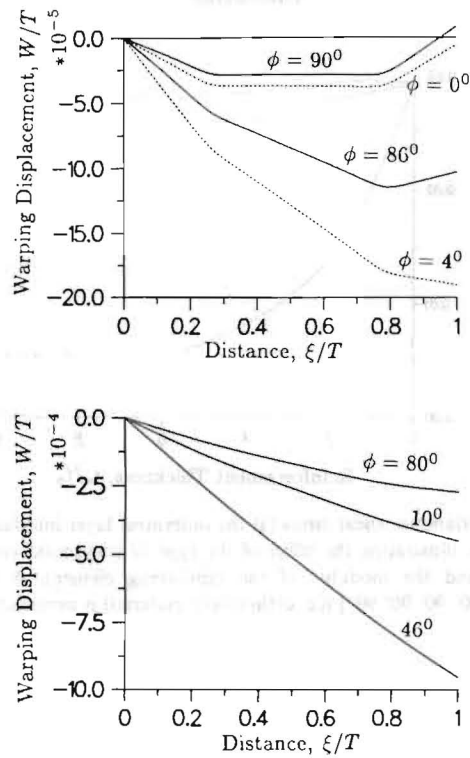


FIG. 8. Thicknesswise distribution (along ξ) of the warping displacement W at certain angular positions. Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$.

for either the $\pm 45^\circ$ or 90° lay-up angles). For an orthotropic material, the displacement W is zero at the major and minor axes (see also Fig. 7). As the major/minor axis ratio tends to unity, the displacement W tends to the limiting case of a circular cylinder (Fig. 9). For this last limiting case, the cross-sections remain planar if $a_{45} = 0$, and non-zero warping displacements occur only if $a_{45} \neq 0$. For the elliptical cross-section, however, warping occurs even in the orthotropic case (Fig. 7).

Compound bars

In certain applications it is desirable that a section of the bar has a compound construction, in the form of an external sleeve, made of different material, which is placed over the filament wound bar or an internal filler (which could also be hollow) placed inside the composite bar. The analysis developed in the previous sections can be used to study such

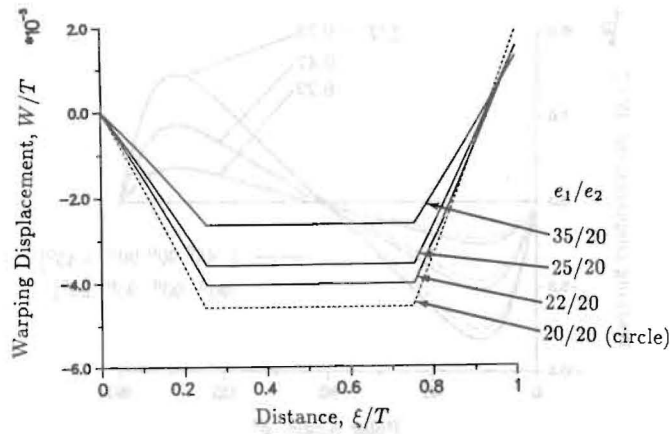


FIG. 9. Effect of a change in the major to minor axis ratio of the mandrel on the thicknesswise distribution of the warping displacement W at the minor axis location $\phi = 90^\circ$. Stacking sequence: $[-45^\circ 90^\circ 90^\circ +45^\circ]$. The dashed curve corresponds to the solution for the limiting case of a circular cross-section.

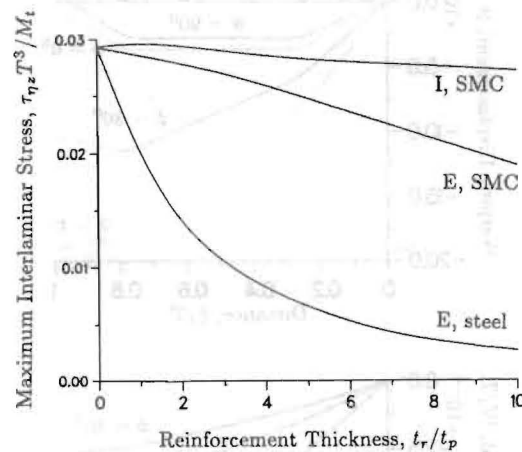


FIG. 10. Maximum interlaminar shear stress (at the outermost layer interface) as a function of the reinforcement thickness, illustrating the effect of the type of compound construction system (i.e. internal or external) and the modulus of the reinforcing element. A stacking sequence of $[90^\circ 90^\circ 90^\circ 90^\circ]$ (i.e. orthotropic material) is assumed.

a compound bar since the whole construction can be considered as a shaft composed of different materials through the thickness. This is, however, a two-dimensional analysis and, therefore, when studying the effect of a compound bar construction on the interlaminar shear stresses, it should be kept in mind that the results could be influenced locally by end constraints.

In the following we shall refer to the external sleeve or internal filler as the 'reinforcement', although the specific application may not be a reinforcing element. As an example we consider 13 layers of glass-fiber (of thickness t_p) that are wound at 90° lay-up angle on an elliptical mandrel with major semi-axis $e_1/t_p = 35$ and minor semi-axis $e_2/t_p = 20$. Let us consider as materials for the reinforcement a low modulus SMC R-50 with $G_{\eta z} = G_{\xi z} = 2.5 \text{ GN m}^{-2}$ and a high modulus steel with $G_{\eta z} = G_{\xi z} = 79 \text{ GN m}^{-2}$. Figure 10 shows the maximum interlaminar shear stress (which occurs at the interface between the 12th and 13th layers) as a function of the reinforcement thickness (SMC R-50 reinforcement) for an internal and an external type construction. It is seen that placing the reinforcing bar externally reduces the maximum interlaminar shear stress by a larger amount. Figure 10 also compares the performance of the low versus high modulus reinforcing materials in reducing the interlaminar stress in an external type system. The high modulus is seen to be superior (of course, in this particular case the weight penalty should also be considered).

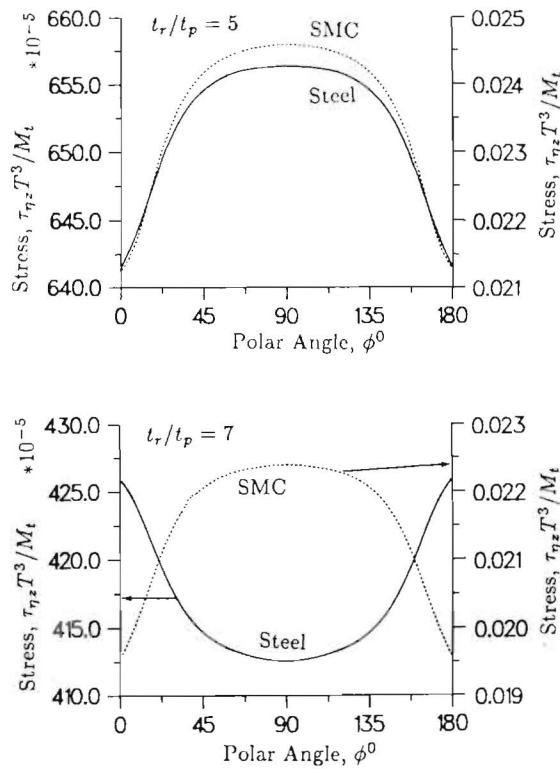


FIG. 11. Angular distribution of the shear stress τ_{rz} (at the outermost layer interface) in an external construction system, illustrating the combined effect of the modulus of the reinforcement material and thickness on the angular stress distribution. Stacking sequence of the bar: $[90^\circ 90^\circ 90^\circ 90^\circ]$ (orthotropic).

The reinforcing bar can also change the angular distribution of stress. Figure 11 shows the angular distribution of τ_{rz} at the outmost layer interface (between the 12th and 13th layers) for a set of reinforcement thicknesses and for the two materials considered in an external construction system. Although for the SMC reinforcement the maximum stress always occurs at the minor axis, for the steel one the maximum stress is found to be at the major axis for anchor thicknesses beyond about $t_r/t_p = 6$.

CONCLUSIONS

An elastic solution for the torsion of filament wound cylinders of elliptical cross-section with variable moduli of elasticity through the thickness (for example, owing to a varying lay-up angle) was presented. The material possesses curvilinear anisotropy, referred to the coordinate system that is defined by the geometry of the filament wound body, and it is assumed that there are planes of elastic symmetry perpendicular to the axis of the body. The differential equation and boundary conditions that govern the stress function were derived and solved via a finite-difference technique. Specific results on the thicknesswise and angular distribution of stresses and displacements, and the effects of the stacking sequence, were presented. The analysis was also applied to studying the effects of a compound bar construction (composite bar with an external or internal reinforcement), and it was found that an external reinforcement reduces the maximum interlaminar shear stress by a larger amount.

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