BENDING OF A CYLINDRICALLY ORTHOTROPIC CURVED BEAM WITH LINEARLY DISTRIBUTED ELASTIC CONSTANTS

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SUMMARY

We develop the analytical solution for the state of stress in pure bending of a cylindrically orthotropic curved beam with the elastic constants being linear functions of the radial distance *r*. The solution is found in a series form. An illustrative example shows the distortion of the stress distribution from the usual constant-modulus case. As a related problem, we also consider the case of a ring with linearly varying elastic constants through the thickness under internal and/or external pressure.

1. Introduction

THE problem of bending of anisotropic curved beams has been considered in the literature, for example by Lekhnitskii (**1**), and solutions are known for the case of a homogeneous beam (constant moduli and Poisson's ratios throughout). In engineering applications of composite parts in the form of curved beams there is often a gradient in the elastic constants through the thickness. The stress distribution in this case of variable elastic constants is more complicated and a solution is known only when the constants change along the radius according to a power law, and can be found in (**2**). Such a case does not, however, reflect the actual distribution of the moduli in practice; instead, practical applications are closely represented by a linear variation (with two non-zero coefficients) of the elastic constants through the thickness. In this paper we present the solution to the latter problem. We consider a plane curved rod under the action of bending moments. For simplicity we assume that the body is orthotropic and that there are no body forces. After deriving the governing equations, the solution is produced by a series expansion. Results of the analysis for an illustrated example are discussed. We also discuss the results for the related problem of a ring under internal and/or external pressure in which the elastic constants vary linearly through the thickness.

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2. Formulation

Let there be a plane rod of uniform thickness h and of unit width having cylindrical orthotropy and bounded by two concentric circles of radii a and b and two radial segments forming an arbitrary angle less than 2π. The beam is loaded at both ends by opposite moments M. Figure 1 shows a cross-section of the beam at the middle plane. The generalized Hooke's law is written

$$\left.\begin{aligned}\varepsilon_{rr} &= a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta},\\ \varepsilon_{\theta\theta} &= a_{12}\sigma_{rr} + a_{22}\sigma_{\theta\theta},\\ \gamma_{r\theta} &= a_{66}\tau_{r\theta},\end{aligned}\right\} \tag{1}$$

where now $a_{ij} = a_{ij}(r)$. As in the case of constant a_{ij}, the stresses and strains depend only on the radial distance r and $\tau_{r\theta} = 0$. Assuming that

$$\sigma_{rr} = \frac{f_0(r)}{r}, \qquad \sigma_{\theta\theta} = f_0'(r), \tag{2}$$

the only remaining equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{3}$$

is satisfied. The compatibility equation needs to be satisfied and is written

$$\frac{\partial^2(r\varepsilon_{\theta\theta})}{\partial r^2} - \frac{\partial\varepsilon_{rr}}{\partial r} = 0. \tag{4}$$

Now the elastic constants are assumed to be linear functions of r:

$$a_{ij} = a_{ijc} + a_{ijg}r. \tag{5}$$

Using (1), (2), (3), the compatibility equation integrates to

$$(a_{22c} + a_{22g}r)r^2 f_0''(r) + (a_{22c} + 2a_{22g}r)rf_0'(r) - [a_{11c} + (a_{11g} - a_{12g})r]f_0(r) = Cr, \tag{6}$$

C being the constant of integration to be determined from the boundary conditions.

Next, we shall solve equation (6). First we shall obtain the solution to the homogeneous equation. Let us represent it in a series expansion as

$$F_0(r) = \sum_{k=0}^{\infty} c_k r^{s+k}. \tag{7}$$

The equation becomes

$$\sum_{k=0}^{\infty} c_k\{[a_{22c}(s+k)^2 - a_{11c}] + r[a_{22g}(s+k)(s+k+1) - (a_{11g} - a_{12g})]\}r^{s+k} = 0. \tag{8}$$

We now equate the coefficient of each power of r to zero. The coefficient of the lowest power or r, which is r^s, gives the indicial equation

$$a_{22c}s^2 - a_{11c} = 0. \tag{9}$$

Thus $s_{1,2} = \pm(a_{11c}/a_{22c})^{\frac{1}{2}}$.

From the coefficient of r^{s+k} we obtain

$$c_k[a_{22c}(s+k)^2 - a_{11c}] + c_{k-1}[a_{22g}(s+k-1)(s+k) - (a_{11g} - a_{12g})] = 0. \tag{10}$$

This recurrence relation determines successively the coefficients $c_1, c_2, \ldots$ in terms of c_0. Since we have two solutions corresponding to s_1, s_2 we shall denote the corresponding coefficients by $c_{i,1}$, $c_{i,2}$. For the root $s_1 = (a_{11c}/a_{22c})^{\frac{1}{2}}$ of the indicial equation we find for example that

$$c_{1,1} = c_{0,1}\frac{(a_{11g} - a_{12g}) - a_{22g}s_1(s_1+1)}{a_{22c}(s_1+1)^2 - a_{11c}}, \tag{11a}$$

and in general

$$c_{k,1} = c_{0,1}\prod_{m=1}^{k}\frac{[(a_{11g} - a_{12g}) - a_{22g}(s_1+m-1)(s_1+m)]}{[a_{22c}(s_1+m)^2 - a_{11c}]}. \tag{11b}$$

We can arbitrarily set $c_{0,1} = c_{0,2} = 1$.

Let us denote the two series solutions corresponding to $s_{1,2}$ by $F_{1,2}(r, a_{ij})$. Then the general solution of the homogeneous equation is any linear combination of those two independent solutions, namely

$$F_0 = A_1F_1(r, a_{ij}) + A_2F_2(r, a_{ij}). \tag{12}$$

Before proceeding to the particular solution, let us discuss the convergence of the series (7). We shall use the Gauss test that requires taking the ratio of two consecutive terms

$$\left|\frac{c_k r^{s+k}}{c_{k+1}r^{s+k+1}}\right| = \frac{[a_{22c}(s+k+1)^2 - a_{11c}]}{[a_{22g}(s+k)(s+k+1) - (a_{11g} - a_{12g})]}\frac{1}{r}$$
$$= \frac{1}{|r|}\left\{\frac{a_{22c}}{a_{22g}} + \frac{a_{22c}}{a_{22g}}\frac{1}{k} + O\left(\frac{1}{k^2}\right)\right\}. \tag{13}$$

From the Gauss test we conclude that the series (11) is absolutely convergent if $|a_{22c}| > |a_{22g}r|$; if this condition is not satisfied, a series expansion in descending powers of r is needed.

Now we shall find a particular solution. We use the relevant theory and obtain the general solution of (6) using both solutions F_1 and F_2 of the corresponding homogeneous equation by the method of variation of parameters (3). A convenient expression can be obtained as follows. The differential equation can be written in the form

$$f''(r) + P(r)f'(r) + Q(r)f(r) = R(r), \tag{14a}$$

where

$$P(r)=\frac{1}{r}+\frac{a_{22g}}{a_{22c}+a_{22g}r}, \qquad R(r)=\frac{C}{(a_{22c}+a_{22g}r)r}. \tag{14b}$$

The particular solution is

$$F_p(r)=F_1(r)\int\frac{F_2(r)R(r)}{F_1'(r)F_2(r)-F_1(r)F_2'(r)}\,dr-F_2(r)\int\frac{F_1(r)R(r)}{F_1'(r)F_2(r)-F_1(r)F_2'(r)}\,dr. \tag{15}$$

It can be proved by successively substituting $F_{1,2}(r)$ in (14a) and eliminating the $Q(r)f(r)$ term that

$$F_1'(r)F_2(r)-F_1(r)F_2'(r)=C^*\exp\left\{-\int P(\rho)\,d\rho\right\}=\frac{C^*}{r(a_{22c}+a_{22g}r)}. \tag{16}$$

The constant C^* can be found from the definition of the power series and the above equation for $r=1$:

$$C^*=(a_{22c}+a_{22g})\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\{c_{k,1}(s_1+k)c_{l,2}-c_{k,1}c_{l,2}(s_2+l)\}. \tag{17}$$

Therefore, the particular solution is

$$F_p(r)=\left[F_1(r)\int F_2(\rho)R(\rho)\exp\left(\int P(x)\,dx\right)d\rho\right.$$
$$\left.-F_2(r)\int F_1(\rho)R(\rho)\exp\left(\int P(x)\,dx\right)d\rho\right]\Big/C^*, \tag{18a}$$

which becomes

$$F_p(r)=\frac{C}{C^*}\left\{F_1(r)\int F_2(\rho)\,d\rho-F_2(r)\int F_1(\rho)\,d\rho\right\}. \tag{18b}$$

Integrating gives

$$F_p(r)=\frac{C}{C^*}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{c_{k,1}c_{l,2}}{s_2+l+1}-\frac{c_{k,2}c_{l,1}}{s_1+l+1}\right\}r^{s_1+s_2+k+l+1}=CF_p^*(r). \tag{18c}$$

So the general solution of (6) is obtained; namely

$$f_0(r)=CF_p^*(r)+A_1F_1(r)+A_2F_2(r), \tag{19}$$

and the stresses can now be found from (2).

The constants C, A_1 and A_2 are found from the traction-free boundary conditions of

$$\sigma_{rr}(r)=0 \quad \text{at} \quad r=a,\, b, \tag{20a}$$

and the condition that the stresses reduce to the pure moment M per unit width:

$$\int_a^b \sigma_{\theta\theta}(r)r\,dr = -M, \qquad \int_a^b \sigma_{\theta\theta}(r)\,dr = 0. \tag{20b}$$

Equation $(20b)_2$ is automatically satisfied once the traction-free conditions (20a) are. Therefore, we end up with the following system of equations:

$$\left.\begin{aligned} &CF_p^*(a) + A_1F_1(a) + A_2F_2(a) = 0, \qquad CF_p^*(b) + A_1F_1(b) + A_2F_2(b) = 0,\\ &C[G_p^*(b) - G_p^*(a)] + A_1[G_1(b) - G_1(a)] + A_2[G_2(b) - G_2(a)] = M, \end{aligned}\right\} \tag{21}$$

where we define

$$\left.\begin{aligned} G_{1,2}(r) &= \sum_{k=0}^{\infty} \frac{c_{k,(1,2)}(s_{1,2}+k)}{(s_{1,2}+k+1)} r^{s_{1,2}+k+1},\\ G_p^*(r) &= \frac{1}{C^*}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{c_{k,1}c_{l,2}}{s_2+l+1} - \frac{c_{k,2}c_{l,1}}{s_1+k+1}\right\}\\ &\quad\times\frac{s_1+s_2+k+l+1}{s_1+s_2+k+l+2} r^{s_1+s_2+k+l+2}. \end{aligned}\right\} \tag{22}$$

The above system of linear equations provides the constants C, A_1, A_2 and finally we obtain expressions for the stresses as follows:

$$\left.\begin{aligned} \sigma_{rr} &= \frac{C}{C^*}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{c_{k,1}c_{l,2}}{s_2+l+1} - \frac{c_{k,2}c_{l,1}}{s_1+k+1}\right\} r^{s_1+s_2+k+l}\\ &\quad + A_1\sum_{k=0}^{\infty} c_{k,1}r^{s_1+k-1} + A_2\sum_{k=0}^{\infty} c_{k,2}r^{s_2+k-1},\\ \sigma_{\theta\theta} &= \frac{C}{C^*}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{c_{k,1}c_{l,2}}{s_2+l+1} - \frac{c_{k,2}c_{l,1}}{s_1+k+1}\right\}\\ &\quad\times(s_1+s_2+k+l+1)r^{s_1+s_2+k+l}\\ &\quad + A_1\sum_{k=0}^{\infty} c_{k,1}(s_1+k)r^{s_1+k-1} + A_2\sum_{k=0}^{\infty} c_{k,2}(s_2+k)r^{s_2+k-1},\\ \tau_{r\theta} &= 0. \end{aligned}\right\} \tag{23}$$

Concerning the displacement field, we use the strain-displacement relations and equations (1):

$$\frac{\partial u_r}{\partial r} = \varepsilon_{rr}(r), \qquad \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \varepsilon_{\theta\theta}(r), \qquad \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \gamma_{r\theta} = 0. \tag{24}$$

Integrating (4) we obtain

$$\frac{\partial(r\varepsilon_{\theta\theta})}{\partial r} = \varepsilon_{rr} = C, \tag{25}$$

C being the constant that has been determined already in the above solution.

Now integrate $(24)_1$ and (25) by taking into account that both $\varepsilon_{\theta\theta}$ and ε_{rr} are functions of r only, and then substitute the resulting expressions into $(24)_2$ to obtain

$$u_r = \int \varepsilon_{rr} r \, dr + f_1(\theta), \qquad u_\theta = Cr\theta - \int f_1(\theta) \, d\theta + f_2(r). \tag{26}$$

Substituting in $(24)_3$ gives differential equations for $f_1(\theta)$ and $f_2(r)$ which result finally in the following expressions for the displacements:

$$u_r = \int \varepsilon_{rr}(r) \, dr + d_1 \cos \theta + d_2 \sin \theta, \qquad u_\theta = Cr\theta - d_1 \sin \theta + d_2 \cos \theta + d_3 r. \tag{27}$$

The constants d_1, d_2, d_3 correspond to a rigid-body displacement which is undetermined in this stress boundary-value problem; they can be found once the conditions of constraint are specified. Considering, for instance, the centroid of the cross-section from which θ is measured (Fig. 1), and also an element of the radius at this point, as rigidly fixed, the conditions of constraint

$$u_r = u_\theta = \partial u_\theta / \partial r = 0 \quad \text{at } \theta = 0 \quad \text{and} \quad r = r_0 = \tfrac{1}{2}(a + b)$$

give $d_2 = d_3 = 0$ and $d_1 = -\int \varepsilon_{rr}(r) \, dr|_{r=r_0}$. Thus the displacement field can be determined. From $(27)_2$ we see that the displacement u_θ consists of a translatory part $-d_1 \sin \theta$, and a rotation of the cross-section by the angle $C\theta$ about the centre of curvature; that is, cross-sections remain planar.

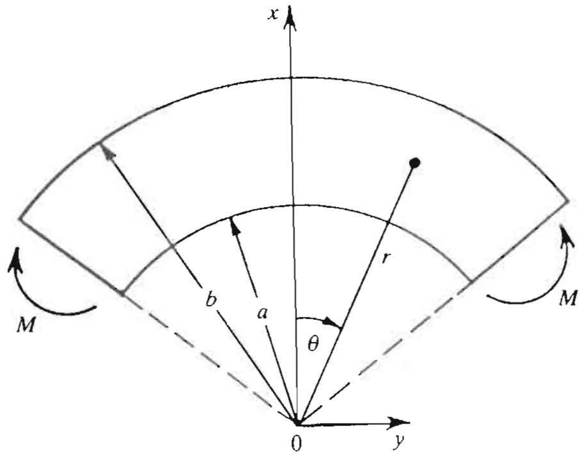

FIG. 1. Definition of the geometry and the problem

3. Results and discussion

As an illustrative example consider a curved beam made of graphite–epoxy with variable elastic constants through the thickness of inside radius $a = 1$ m and $b/a = 1{\cdot}5$. Assume that the moduli at the outer radius $r = b$ are as follows in giga-pascals (notice that 1 corresponds to the radial (r) direction and 2 corresponds to the tangential (θ) direction);

$$E_{1b} = 8{\cdot}0, \qquad E_{2b} = 293{\cdot}0, \qquad G_{12b} = 3{\cdot}1, \qquad \nu_{12b} = 0{\cdot}247.$$

The corresponding compliance constants are

$$a_{11b} = 1/E_{1b}, \qquad a_{12b} = -\nu_{12b}/E_{1b}, \qquad a_{22b} = 1/E_{2b}, \qquad a_{66b} = 1/G_{12b}.$$

Let us assume that the elastic constants vary according to (5) and the steepness in variations is expressed by the parameter m as follows:

$$a_{ijg} = a_{ijc} m. \tag{28}$$

As an example we assume that $m = -0{\cdot}50$ which would give a ratio of compliance constants at the outside/inside edges equal to $\frac{1}{2}$.

First the convergence of the series solution is illustrated in Table 1, which shows the kth term of the series expansion of the functions $F_1(r)$ and $F_2(r)$ at $r = b$. The series seems to be rapidly converging in a satisfactory manner.

Figures 2, 3 show the distribution of the normalized stresses $\sigma_{\theta\theta}h/M$ and $\sigma_{rr}h/M$ through the thickness. The curves are compared with the stresses for constant moduli (dashed curve), given by Lekhnitskii (1):

$$\left.\begin{aligned}
\sigma_{rr} &= -\frac{M}{2b^2hg}\left[1 - \frac{1-c^{k+1}}{1-c^{2k}}\left(\frac{r}{b}\right)^{k-1} - \frac{1-c^{k-1}}{1-c^{2k}}c^{k+1}\left(\frac{b}{a}\right)^{k+1}\right],\\
\sigma_{\theta\theta} &= -\frac{M}{2b^2hg}\left[1 - \frac{1-c^{k+1}}{1-c^{2k}}k\left(\frac{r}{b}\right)^{k-1} + \frac{1-c^{k-1}}{1-c^{2k}}c^{k+1}k\left(\frac{b}{a}\right)^{k+1}\right],
\end{aligned}\right\} \tag{29a}$$

where

$$\left.\begin{aligned}
&c = a/b, \qquad k = (a_{11}/a_{22})^{\frac{1}{2}},\\
&g = \frac{1-c^2}{2} - \frac{k}{k+1}\frac{(1-c^{k+1})^2}{1-c^{2k}} + \frac{kc^2}{k-1}\frac{(1-c^{k-1})^2}{1-c^{2k}}.
\end{aligned}\right\} \tag{29b}$$

The most interesting one is $\sigma_{\theta\theta}(r)$. Due to the gradient in the elastic constants the stress is by absolute value 27 per cent higher at the outer surface ($r = b$) and 23 per cent lower at the inner surface ($r = a$) relative to the homogeneous case. The distribution does not follow a linear (as in the

TABLE 1. *Convergence of the series solution. Values of the kth term (at $r = b$) of the functions $F_1(r)$ and $F_2(r)$ (in N/m)*

	$k = 0$	$k = 10$	$k = 25$	$k = 50$	$k = 100$
F_1	$0{\cdot}117 \times 10^{2}$	-0.854×10^{-2}	$-0{\cdot}413 \times 10^{-4}$	$-0{\cdot}148 \times 10^{-7}$	$-0{\cdot}409 \times 10^{-14}$
F_2	$0{\cdot}858 \times 10^{-1}$	$0{\cdot}735 \times 10^{-1}$	-0.350×10^{-3}	$-0{\cdot}813 \times 10^{-7}$	$-0{\cdot}191 \times 10^{-13}$

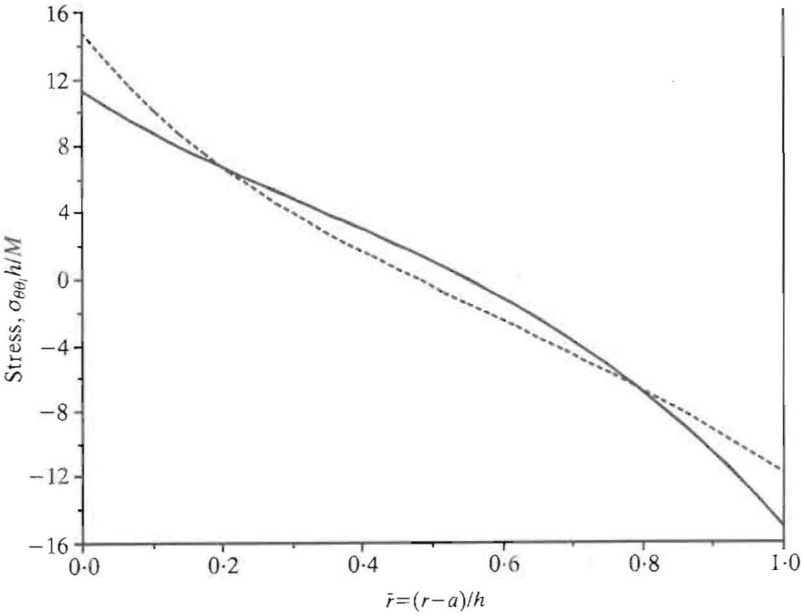

FIG. 2. Distribution of the stress $\sigma_{\theta\theta}$ at the cross-section in a curved beam of thickness h for the example case considered. The broken line represents the case of a homogeneous beam with non-varying elastic constants throughout

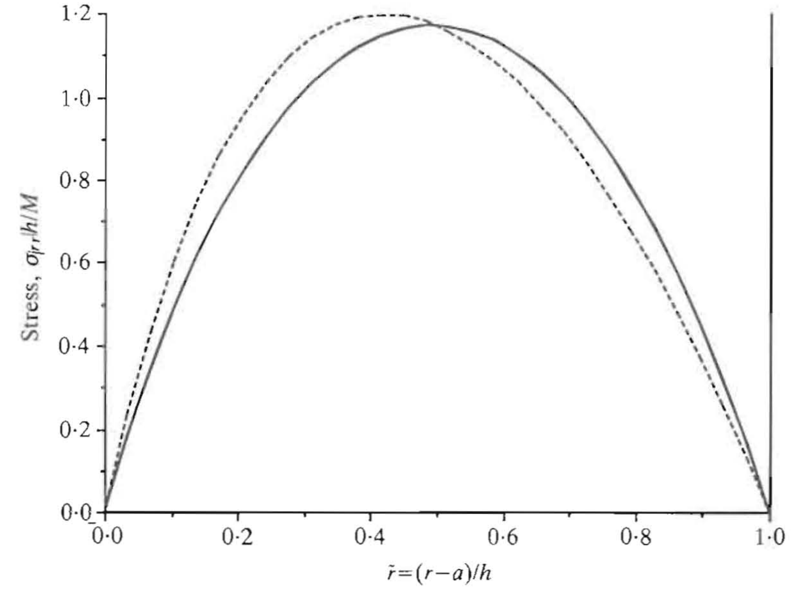

FIG. 3. Stress distribution σ_{rr} through the thickness in a curved beam of thickness h for the example case considered. The broken line represents the case of a homogeneous beam with non-varying elastic constants throughout

straight-bar case) or hyperbolic (as in the isotropic elementary strength-of-materials case) law. The other component of stress σ_{rr} is of much smaller magnitude and follows the same pattern; relative to the homogeneous case the curve is shifted so that the stress is increased at points closer to the outside edge ($r = b$) and reduced towards the inside edge ($r = a$). This component of stress is neglected in the elementary strength-of-materials theory. Figure 4 shows the variation of the normalized hoop stress $\sigma_{\theta\theta}h/M$ at the outside (solid line) and inside (dashed line) fibres as a function of the steepness parameter m in (28). The resulting curve is nonlinear with a higher slope at the larger values of m, which means that there is more stress reduction or increase per unit change in moduli at the large differential between elastic constants on the inside and outside edges.

Table 2 shows the values for the normalized hoop stress $\sigma_{\theta\theta}h/M$ at the fibres on the outside and inside edge $r = b$ and $r = a$, respectively. The range of the steepness parameter m in (28) is from 0 to $-0{\cdot}5$, which in turn makes the ratio of the moduli at the inner and outer fibres between 1·0 and 0·5. The values of $\sigma_{\theta\theta}h/M$ for the outer/inner fibres range from $-10{\cdot}56/+13{\cdot}84$ for $m = 0{\cdot}0$, that is, constant moduli throughout, to $-13.64/+11.25$ for $m = -0.5$; that is, moduli on the inside edge being half that on the

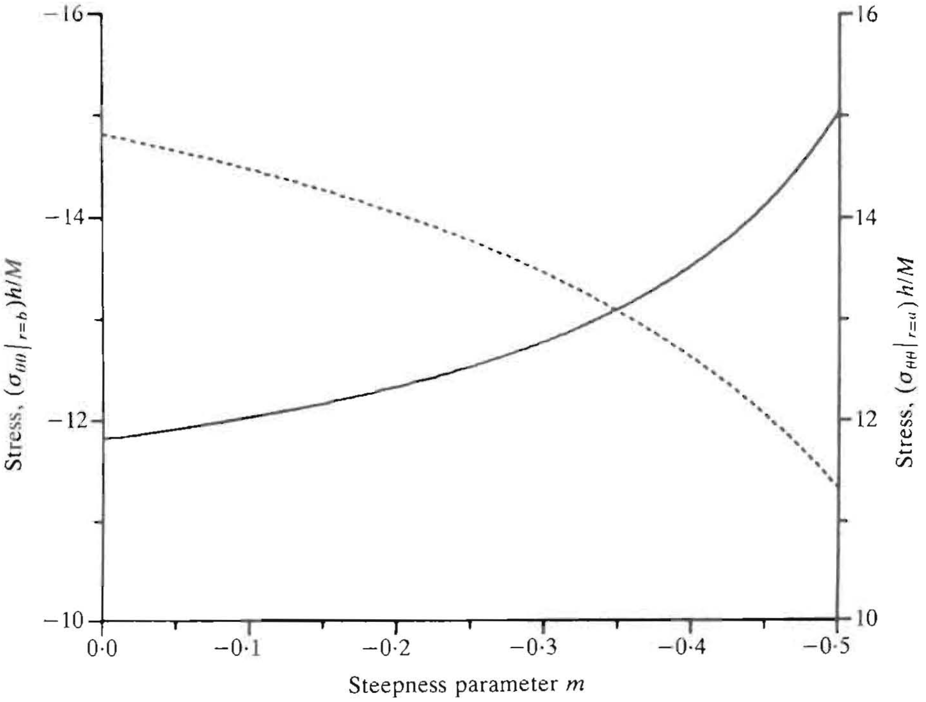

FIG. 4. Stress at the outside edge $\sigma_{\theta\theta}|_{r=b}$ (solid line) and at the inside edge $\sigma_{\theta\theta}|_{r=a}$ (dashed line) plotted against the steepness parameter m

TABLE 2. *Comparison of outside/inside fibre stresses in a curved beam under pure moment M*

Steepness parameter m, m^{-1}	Inside/outside moduli ratio	$(\sigma_{\theta\theta}\|_{r=b} h/M)/(\sigma_{\theta\theta}\|_{r=a} h/M)$
0·00	1·0	−11·82/ + 14·81 (homogeneous case)
$-\frac{1}{6}$	0·9	−12·22/+14·19
$-\frac{2}{7}$	0·8	−12·70/ + 13·54
$-\frac{3}{8}$	0·7	−13·29/ + 12·85
$-\frac{4}{9}$	0·6	−14·05/ + 12·11
$-\frac{1}{2}$	0·5	−15·05/ + 11·32

outside. The maximum value of the radial stress $\sigma_{rr}h/M$ showed a small variation, ranging from 1·23 for $m = 0·0$ to 1·19 for $m = -0·5$.

A related case is the stress distribution in a ring with linearly varying elastic constants through the thickness under internal and/or external pressure. In this case $u_\theta = 0$ due to symmetry and $C = 0$ in (6). The solution follows the same pattern but now only two constants, A_1 and A_2 in expressions (19), need to be determined from the boundary conditions of internal (p) and external (q) pressures:

$$\sigma_{rr}(a) = -p, \qquad \sigma_{rr}(b) = -q.$$

Results for the case of a ring loaded only on the inner contour ($q = 0$) and for material and radii data as in the previous curved-beam example are shown in Fig. 5 for the distribution of σ_{rr}/p and in Fig. 6 for that of $\sigma_{\theta\theta}/p$. In these figures the results are compared with the homogeneous case (dashed line) which is given by Lekhnitskii (2):

$$\sigma_{rr} = \frac{pc^{k+1}}{1-c^{2k}}\left[\left(\frac{r}{b}\right)^{k-1} - \left(\frac{b}{r}\right)^{k+1}\right], \qquad \sigma_{\theta\theta} = \frac{pc^{k+1}k}{1-c^{2k}}\left[\left(\frac{r}{b}\right)^{k-1} + \left(\frac{b}{r}\right)^{k+1}\right],$$

where c and k are given by $(29\mathrm{b})_{1,2}$. It is seen that the radial (compressive) stress σ_{rr} is smaller in the case of variable moduli at all points throughout the thickness, whereas the hoop stress $\sigma_{\theta\theta}$ at the outer edge ($r = b$) is twice that of the homogeneous case and about 25 per cent smaller at the inside edge ($r = a$). The effect of the steepness parameter m is illustrated in Table 3 which shows the values for the normalized hoop stress $\sigma_{\theta\theta}/p$ at the fibres on the outside and inside edges of the ring $r = b$ and $r = a$, respectively. The values of $\sigma_{\theta\theta}/p$ for the outer/inner fibres range from +0·70/+6·15 for $m = 0·0$, that is, constant moduli throughout, to +1·41/+4·65 for $m = -0·5$, that is, moduli on the inside edge being half that on the outside.

As for the displacements, for a ring under pressure, due to the rotational symmetry we have $C = 0$ and $u_\theta = 0$ and the displacement u_r has radial dependence only. From $(27)_1$, using the series solution (23) for the

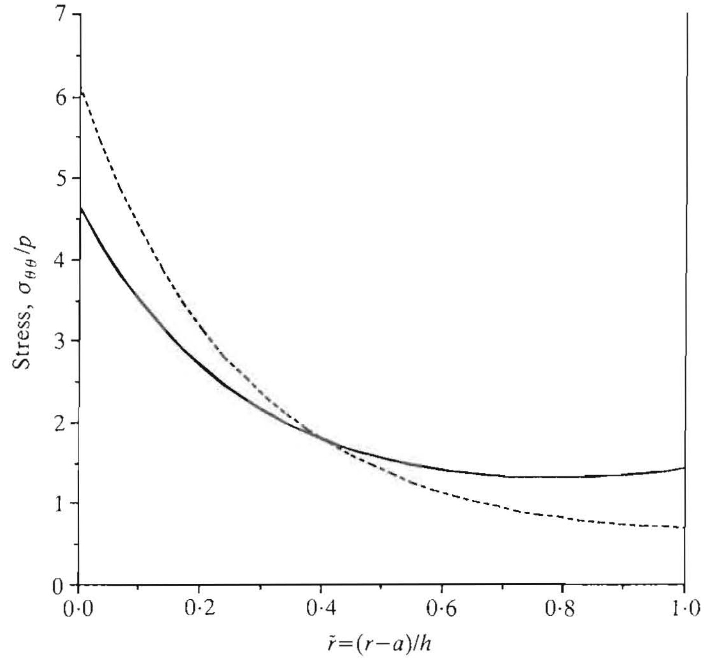

FIG. 5. Variation of the stress $\sigma_{\theta\theta}$ through the thickness for the case of a ring of thickness h under internal pressure p for the example case of linearly varying elastic constants. The dashed line represents the homogeneous case (non-varying elastic constants throughout)

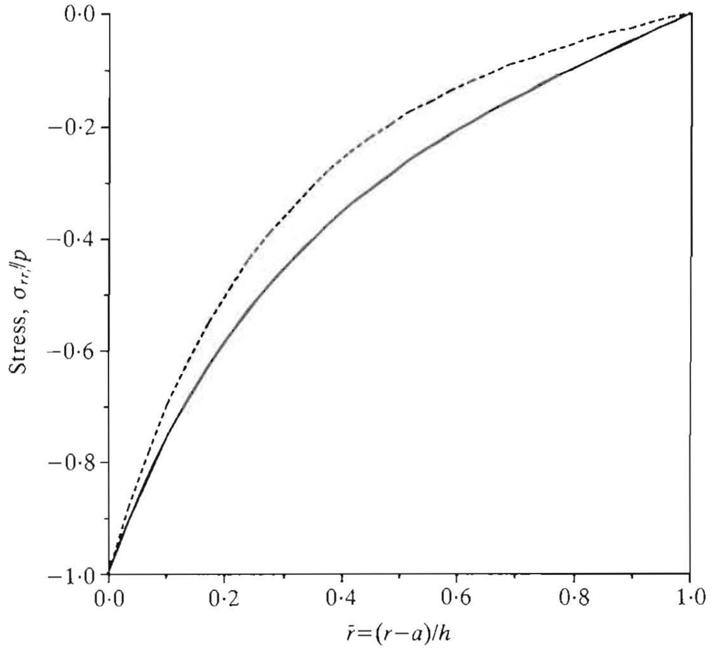

FIG. 6. Variation of the radial stress σ_{rr} at the cross-section for the case of a ring of thickness h under internal pressure p for the example case of linearly varying elastic constants. The dashed line represents the homogeneous case (non-varying elastic constants throughout)

TABLE 3. *Comparison of outside/inside fibre stresses in a ring under internal pressure p*

Steepness parameter m, m^{-1}	Inside/outside moduli ratio	$(\sigma_{\theta\theta}\,|_{r=b}/p)/(\sigma_{\theta\theta}\,|_{r=a}/p)$
0·00	1·0	+0·70/ + 6·15 (homogeneous case)
$-\frac{1}{6}$	0·9	+0·77/ + 5·88
$-\frac{2}{7}$	0·8	+0·87/ + 5·60
$-\frac{3}{8}$	0·7	+0·99/ + 5·31
$-\frac{4}{9}$	0·6	+1·17/ + 4·99
$-\frac{1}{2}$	0·5	+1·41/ + 4·65

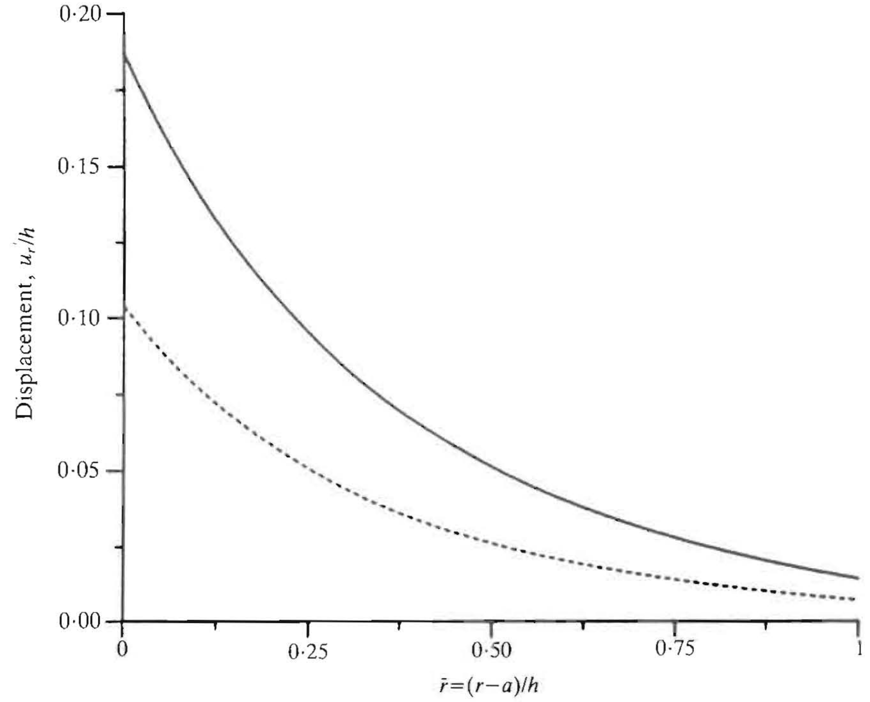

FIG. 7. Distribution of the displacement u_r for a ring of thickness h under internal pressure p for the example case of linearly varying elastic constants. The broken line represents the homogeneous case (non-varying elastic constants throughout)

stresses we obtain the displacement

$$u_r = \sum_{i=1,2} \sum_{k=0}^{\infty} c_{ki} A_i \left\{ \frac{a_{11c}}{s_i + k} + a_{12c} + \frac{a_{11g} + a_{12g}(s_i + k)}{s_i + k + 1} r \right\} r^{s_i + k}.$$

Figure 7 shows the distribution of the radial displacement u_r through the thickness for the example case considered (solid line), as compared with the homogeneous, non-varying elastic-constant case (dashed line), given by

$$u_r = \frac{pc^{k+1}}{1 - c^{2k}} \left[(a_{12} + a_{22}k)\left(\frac{r}{b}\right)^k - (a_{12} - a_{22}k)\left(\frac{b}{r}\right)^k \right].$$

The displacement u_r is seen to be higher for the non-homogeneous case, especially near the inside edge.

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