G. A. Kardomateas
Assistant Professor,
School of Aerospace Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0150
Mem. ASME

The Initial Phase of Transient Thermal Stresses due to General Boundary Thermal Loads in Orthotropic Hollow Cylinders

The stresses and displacements in the initial phase of applying a thermal load on the bounding surfaces of an orthotropic hollow circular cylinder are obtained using the Hankel asymptotic expansions for the Bessel functions of the first and second kind. Such a load may be constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface. The material properties are assumed to be independent of temperature. A constant applied temperature at the one surface and convection into a medium at a different temperature at the other surface is used to illustrate the variation of stresses with time and through the thickness in the initial transient phase.

Introduction

Meeting the need for materials which can function usefully at different temperature levels is one of the most challenging problems facing our technology. The difficulty is compounded by the fact that operating conditions involve not only elevated temperature levels but frequently also severe temperature gradients. Such temperature differentials may produce thermal stresses significant enough to limit the material life. Some of the more important applications of fiber-reinforced composites involve the configuration of a hollow cylinder, typically produced by filament winding on a cylindrical mandrel. For example, composite tubes of considerable thickness could be used in such parts as automotive suspension components. The manufacturing stage includes a thermal treatment phase, during which thermal stresses of considerable magnitude may be generated. Other applications may be for launch tubes or landing gears.

Relevant work on understanding the thermally-induced stresses in such configurations was done by Kalam and Tauchert (1978), Hyer and Cooper (1986), and Avery and Herakovich (1986). These studies dealt with the steady-state problem. A solution to the transient thermal stress problem was derived by Kardomateas (1989). This solution involved a series expansion of Bessel functions of the first and second kind and it is valid only beyond a certain time value. This is because the series expansions cannot be used for large arguments; small time values require including an increasing number of terms and therefore large arguments. This paper gives the solution for this case of the very beginning of applying the thermal loads (the initial phase of transient thermal stresses). The analysis is limited to orthotropic tubes, (for example, composite tubes with the fibers aligned circumferentially). The material properties are assumed to be independent of temperature, the stresses not to vary along the generators and that there are no body forces.

In this paper the body is assumed to be subjected to any kind of thermal load such as constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface; in this respect this paper also extends the work by Kardomateas (1989), which was done for the specific case of a constant temperature on the one surface of the cylinder and heat convection to a different medium at the other. We shall present an example for the distribution of stresses and displacements in the initial phase of transient stresses, where it is assumed that one surface of the cylinder is at a constant temperature and at the other there is heat convection into a medium at the reference temperature.

Derivation of the Governing Equations

Consider a circular cylindrical shell $r_1 \leq r \leq r_2$ whose axis coincides with the axis of z. The initial (reference) temperature is assumed to be zero. The thermal problem consists of the heat conduction equation

$$K\left(\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t)}{\partial r}\right) = \frac{\partial T(r,t)}{\partial t} \quad (r_1<r<r_2, t>0), \qquad (1a)$$

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for presentation at the Winter Annual Meeting, Dallas, Texas, Nov. 25-30, 1990.

Discussion on this paper should be addressed to the Technical Editor, Leon M. Keer, The Technological Institute, Northwestern University, Evanston, IL 60208, and will be accepted until two months after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by the ASME Applied Mechanics Division, Feb. 2, 1989; final revision, June 21, 1989.

Paper No. 90-WA/APM-16.

where K is the thermal diffusivity of the composite in the r direction, and the initial and boundary conditions

$$T(r,t=0)=0 \text{ at } r_1 \le r \le r_2, \tag{1b}$$

$$h_1 \frac{\partial T(r,t)}{\partial r}\Big|_{r=r_1} - h_2 T(r_1,t) = h_3 \ (t>0), \tag{1c}$$

$$h_1^* \frac{\partial T(r,t)}{\partial r}\Big|_{r=r_2} + h_2^* T(r_2,t) = h_3^* \ (t>0), \tag{1d}$$

where h_1, h_2, h_1^*, h_2^* are constants which may be positive or zero (provided h_1 and h_2, or h_1^* and h_2^* do not both vanish) and h_3 and h_3^* are constants. By choice of these constants the general results include all combinations of constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface. A solution for the temperature field is given by Carslaw and Jaeger (1959) in terms of the Bessel functions of the first and second kind J_n and Y_n, as follows

$$T(r,t) = d_1 + d_2 \ln(r/r_1) + d_3 \ln(r/r_2) + \sum_{n=1}^{\infty} e^{-K a_n^2 t}[d_{4n} J_0(ra_n) + d_{5n} Y_0(ra_n)], \tag{2}$$

where $\pm a_n$ are the roots (all real and simple) of:

$$[h_1 x J_1(r_1 x) + h_2 J_0(r_1 x)][h_1^* x Y_1(r_2 x) - h_2^* Y_0(r_2 x)] - [h_1 x Y_1(r_1 x) + h_2 Y_0(r_1 x)][h_1^* x J_1(r_2 x) - h_2^* J_0(r_2 x)]. \tag{3}$$

The constants d_i are given in Appendix A.

In cylindrical coordinates (r, θ, z), the constitutive equations for the orthotropic body are given in terms of the elastic constants C_{ij} and the thermal expansion coefficients, α_i:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} - \alpha_r \Delta T \\ \epsilon_{\theta\theta} - \alpha_\theta \Delta T \\ \epsilon_{zz} - \alpha_z \Delta T \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}. \tag{4}$$

Since the temperature field is independent of the angular and axial coordinates, the stresses and strains will be a function of r and t only.

Following the analysis by Kardomateas (1989), the strain field for this class of problems is expressed in terms of the radial displacement $U(r, t)$:

$$\epsilon_{rr} = \frac{\partial U(r,t)}{\partial r}; \ \epsilon_{\theta\theta} = \frac{U(r,t)}{r}; \ \epsilon_{zz} = C(t), \tag{5a}$$

$$\gamma_{\theta z} = \gamma_{rz} = \gamma_{r\theta} = 0. \tag{5b}$$

There remain one equilibrium equation for the nonvanishing stresses:

$$\frac{\partial \sigma_{rr}(r,t)}{\partial r} + \frac{\sigma_{rr}(r,t) - \sigma_{\theta\theta}(r,t)}{r} = 0. \tag{6}$$

Define

$$q_1 = C_{11}\alpha_r + C_{12}\alpha_\theta + C_{13}\alpha_z, \tag{7a}$$

$$q_2 = (C_{11} - C_{12})\alpha_r + (C_{12} - C_{22})\alpha_\theta + (C_{13} - C_{23})\alpha_z, \tag{7b}$$

$$q_3 = C_{13}\alpha_r + C_{23}\alpha_\theta + C_{33}\alpha_z. \tag{7c}$$

Set

$$U(r,t) = U_0(r) + \sum_{n=1}^{\infty} e^{-K a_n^2 t} R_n(r), \tag{8}$$

and express the parameter $C(t)$ in the form

$$C(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-K a_n^2 t}. \tag{9}$$

Substituting equations (4), (5) into (6) and using the definitions (7), (8), and (9) results in the following differential equations:

$$C_{11}\left(U_0''(r) + \frac{U_0'(r)}{r}\right) - \frac{C_{22}}{r^2} U_0(r) = \frac{q_1(d_2+d_3) + q_2 d_1 + (C_{23} - C_{13})c_0}{r} + q_2 d_2 \frac{\ln(r/r_1)}{r} + q_2 d_3 \frac{\ln(r/r_2)}{r}, \tag{10}$$

$$C_{11}\left(R_n''(r) + \frac{R_n'(r)}{r}\right) - \frac{C_{22}}{r^2} R_n(r) = \frac{(C_{23} - C_{13})c_n}{r} + d_{4n}\left[\frac{J_0(ra_n)}{r} - q_1 a_n J_1(ra_n)\right] + d_{5n}\left[\frac{Y_0(ra_n)}{r} - q_1 a_n Y_1(ra_n)\right] \quad n = 1, \ldots \infty. \tag{11}$$

Solution

Define

$$\lambda_{1,2} = \pm \sqrt{C_{22}/C_{11}}. \tag{12}$$

For $C_{11} \ne C_{22}$, the solution of (10) for $U_0(r)$ is

$$U_0(r) = G_{10} r^{\lambda_1} + G_{20} r^{\lambda_2} + \frac{C_{23} - C_{13}}{C_{11} - C_{22}} c_0 r + U_0^*(r), \tag{13a}$$

$$U_0^*(r) = \frac{q_2}{C_{11} - C_{22}} [d_2 r \ln(r/r_1) + d_3 r \ln(r/r_2)] + \left[\frac{q_2(d_2+d_3) + q_2 d_1}{C_{11} - C_{22}} - \frac{2C_{11} q_2 (d_2 + d_3)}{(C_{11} - C_{22})^2}\right] r. \tag{13b}$$

For $C_{11} = C_{22}$, the corresponding solution is

$$U_0(r) = G_{10} r + \frac{G_{20}}{r} + \frac{(C_{23} - C_{13})}{2C_{11}} c_0 r \ln(r/r_2) + U_0^*(r), \tag{14a}$$

$$U_0^*(r) = \frac{q_2}{4C_{11}} [d_2 r \ln^2(r/r_1) + d_3 r \ln^2(r/r_2)] + \frac{(2q_1 - q_2)(d_2 + d_3) + 2q_2 d_1}{4C_{11}} r \ln(r/r_2), \tag{14b}$$

where G_i are constants, yet unknown, which, together with the constants c_i, are found later from the boundary conditions.

To solve equation (11) for large arguments, we use the Hankel asymptotic expansions of the Bessel functions of the first and second kind. For $C_{11} \ne C_{22}$, the solution for R_n is expressed

$$R_n(r)=G_{1n}r^{\lambda_1}+G_{2n}r^{\lambda_2}+\frac{C_{23}-C_{13}}{C_{11}-C_{22}}c_nr+R_n^*(r), \quad (15a)$$

and for $C_{11} = C_{22}$, the solution for R_n is

$$R_n(r)=G_{1n}r+\frac{G_{2n}}{r}+\frac{C_{23}-C_{13}}{2C_{11}}c_nr\ln(r/r_2)+R_n^*(r). \quad (15b)$$

The function $R_n^*(r)$ is the particular solution of the equation that results from equation (11) if the right-hand side includes only the Bessel functions terms. For small values of the argument the function $R_n^*(r)$ was given in Kardomateas (1989) (although a specific thermal load was considered in that work, the expression holds true for the general thermal load considered here).

For large arguments, we use the Hankel asymptotic expansions of the Bessel functions of the first and second kind (see Appendix B). Employing the substitution

$$\rho=ra_n;\ R_n^*(r)=R_n^{**}(\rho), \quad (16)$$

gives the following equation for $R_n^{**}(\rho)$

$$C_{11}a_n^2\left(R_n^{**''}(\rho)+\frac{R_n^{**'}(\rho)}{\rho}\right)-C_{22}a_n^2\frac{R_n^{**}(\rho)}{\rho^2}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^ka_n\psi_1(k)}{(2k)!(8\rho)^{2k}\rho\sqrt{\pi\rho}}$$

$$\times\{(d_{4n}+d_{5n})(\sin\rho-a_{1,k}\rho\cos\rho+a_{2,k}\rho^2\sin\rho)+$$

$$+(d_{4n}-d_{5n})(\cos\rho+a_{1,k}\rho\sin\rho+a_{2,k}\rho^2\cos\rho)\}, \quad (17)$$

where

$$a_{1,k}=q_1\frac{4k+1}{4k-1}-\frac{16k}{(4k-1)^2};\ a_{2,k}=\frac{16kq_1}{(4k-1)(4k-3)}. \quad (18)$$

and $\psi_1(k)$ is defined in Appendix B.

The solution of equation (17) for the function $R_n^{**}(\rho)$ is found to be

$$R_n^{**}(\rho)=\sum_{n=0}^{\infty}p_{k,1}^n\rho^{-2k-1/2}\cos\rho+s_{k,1}^n\rho^{-2k-1/2}\sin\rho+$$

$$+p_{k,2}^n\rho^{-2k-3/2}\cos\rho+s_{k,2}^n\rho^{-2k-3/2}\sin\rho. \quad (19)$$

The coefficients $p_{k,1}^n$, $s_{k,1}^n$, $p_{k,2}^n$, $s_{k,2}^n$ are determined by considering the terms in the sum (18) that contribute to the terms $\rho^{-2k-1/2}\cos\rho$, $\rho^{-2k-1/2}\sin\rho$, $\rho^{-2k-3/2}\cos\rho$, $\rho^{-2k-3/2}\sin\rho$ in the right-hand side of equation (17a). We obtain the following recursive formulas for $p_{k,1}^n$, $s_{k,1}^n$,

$$p_{k,1}^nC_{11}=p_{k-1,1}^n[C_{11}(-2k+3/2)^2-C_{22}]+s_{k-1,2}^nC_{11}2(-2k+1)$$

$$+\frac{(-1)^k\psi_1(k)a_{1,k}}{(2k)!8^{2k}a_n\sqrt{\pi}}(d_{4n}+d_{5n}), \quad (20a)$$

$$s_{k,1}^nC_{11}=s_{k-1,1}^n[C_{11}(-2k+3/2)^2-C_{22}]-p_{k-1,2}^nC_{11}2(-2k+1)$$

$$+\frac{(-1)^{k+1}\psi_1(k)a_{1,k}}{(2k)!8^{2k}a_n\sqrt{\pi}}(d_{4n}-d_{5n}), \quad (20b)$$

and for $p_{k,2}^n$, $s_{k,2}^n$,

$$p_{k,2}^nC_{11}=p_{k-1,2}^n[C_{11}(-2k+1/2)^2-C_{22}]-s_{k,1}^nC_{11}4k$$

$$+\left\{\frac{(-1)^{k+1}\psi_1(k)}{(2k)!8^{2k}a_n\sqrt{\pi}}+\frac{(-1)^k\psi_1(k+1)a_{2,k+1}}{[2(k+1)]!8^{2(k+1)}a_n\sqrt{\pi}}\right\}(d_{4n}-d_{5n}), \quad (20c)$$

$$s_{k,2}^nC_{11}=s_{k-1,2}^n[C_{11}(-2k+1/2)^2-C_{22}]+p_{k,1}^nC_{11}4k$$

$$+\left\{\frac{(-1)^{k+1}\psi_1(k)}{(2k)!8^{2k}a_n\sqrt{\pi}}+\frac{(-1)^k\psi_1(k+1)a_{2,k+1}}{[2(k+1)]!8^{2(k+1)}a_n\sqrt{\pi}}\right\}(d_{4n}+d_{5n}). \quad (20d)$$

The process starts from $k = 1$ and the starting values for $k = 0$ are from equations (17) and (18) as follows:

$$p_{0,1}=-\frac{q_1(d_{4n}+d_{5n})}{a_n\sqrt{\pi}}\cdot s_{0,1}=\frac{q_1(d_{4n}-d_{5n})}{a_n\sqrt{\pi}}, \quad (21a)$$

$$p_{0,2}=(-8+3q_1)\frac{(d_{4n}-d_{5n})}{8a_n\sqrt{\pi}};\ s_{0,2} \quad (21b)$$

$$=(-8+3q_1)\frac{(d_{4n}+d_{5n})}{8a_n\sqrt{\pi}}. \quad (21b)$$

A fine point of the analysis will be addressed now. The solution (19) is a particular solution of equation (17). It was derived based on the Hankel asymptotic expansions of the Bessel functions for values of the argument $\rho = ra_n \geq \rho_{tr} = 18.0$ (see Appendix B). A corresponding solution, $R_{nS}^*(r)$, for values of the argument $\rho \leq \rho_{tr}$ had been derived by Kardomateas (1989), based on a series expansion for the Bessel functions. Since, for a given root a_n the argument ρ ranges from r_1a_n to r_2a_n, there may be a transition point from one solution to the other for $R_n^*(r)$ in the expression (15). Both solutions are particular ones and may be different. Therefore, at that transition point a homogeneous solution term should be added to (19) so that

$$R_{nL}^{**}(\rho)=d_{6n}\rho^{\lambda_1}+d_{7n}\rho^{\lambda_2}+R_n^{**}(\rho);\ R_n^*(r)=R_{nL}^{**}(\rho), \quad (22a)$$

where d_{6n} and d_{7n} are determined from the condition of equal value and slope at the transition point

$$R_{nL}^{**}(\rho_{tr})=R_{nS}^*(\rho_{tr}/a_n);\ R_{nL}^{**'}(\rho_{tr})a_n=R_{nS}^{*'}(\rho_{tr}/a_n). \quad (22b)$$

The unknown constants c_j, G_{ij} are found from the conditions of zero external traction and zero resultant force. These are modified from Kardomateas (1989) to include the general thermal load case. The traction-free condition, $\sigma_{rr}(r_i, t) = 0$, gives the following two linear equations in G_{10}, G_{20}, c_0.

$$(C_{11}\lambda_1+C_{12})r_i^{\lambda_1-1}G_{10}+(C_{11}\lambda_2+C_{12})r_i^{\lambda_2-1}G_{20}+A_0c_0=$$

$$=-C_{11}U_0^{*'}(r_i)-C_{12}\frac{U_0^*(r_i)}{r_i}+q_1[d_1+d_2\ln(r_i/r_1)$$

$$+d_3\ln(r_i/r_2)]\ i=1,2, \quad (23a)$$

where

$$A_0=\frac{C_{11}+C_{12}}{C_{11}-C_{22}}(C_{23}-C_{13})+C_{13}\ \text{for}\ C_{11}\neq C_{22}$$

$$=\frac{C_{23}-C_{13}}{2C_{11}}[C_{11}+(C_{11}+C_{12})\ln(r_i/r_2)]+C_{13}\ \text{for}\ C_{11}=C_{22} \quad (23b)$$

and two linear equations for G_{1n}, G_{2n}, c_n, $n = 1, \infty$,

$$(C_{11}\lambda_1+C_{12})r_i^{\lambda_1-1}G_{1n}+(C_{11}\lambda_2+C_{12})r_i^{\lambda_2-1}G_{2n}+A_0c_n=$$

$$=-C_{11}a_nR_{nL}^{**'}(r_ia_n)-C_{12}\frac{R_{nL}^{**}(r_ia_n)}{r_i}+q_1[d_{4n}J_0(r_ia_n)$$

$$+d_{5n}Y_0(r_ia_n)]\ i=1,2. \quad (23c)$$

The condition of zero resultant axial force, $P_z = \int_{r_1}^{r_2}\sigma_{zz}(r, t)2\pi r\,dr = 0$, gives the last set of equations that are needed to determine the constants G_i, c_i. For $C_{11} \neq C_{22}$

$$\left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_1+1}\right)(r_2^{\lambda_1+1}-r_1^{\lambda_1+1})G_{10}+A_1G_{20}+A_2c_0=$$

$$= -E_0(r_1,r_2) + \frac{q_3}{2}\left[\frac{(r_2^2-r_1^2)}{2}(2d_1-d_2-d_3) + (d_2r_2^2+d_3r_1^2)\ln(r_2/r_1)\right], \qquad (24a)$$

and for $n = 1, \infty$,

$$\left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_1+1}\right)(r_2^{\lambda_1+1}-r_1^{\lambda_1+1})G_{1n}+A_1G_{2n}+A_2c_n =$$

$$= -E_n(r_1,r_2)+(q_3/a_n)\sum_{i=1}^{2}(-1)^i[d_{4n}r_iJ_1(r_ia_n)+d_{5n}r_iY_1(r_ia_n)], \qquad (24b)$$

where $E_0(r_2, r_1)$ and $E_n(r_1, r_2)$ are defined in Appendix C. The coefficients A_1, A_2 are defined as:

$$A_1 = \left(C_{13}+\frac{C_{23}-C_{13}}{\lambda_2+1}\right)(r_2^{\lambda_2+1}-r_1^{\lambda_2+1}) \text{ for } C_{11}\neq C_{22}$$

$$= C_{13}+(C_{23}-C_{13})\ln(r_2/r_1) \text{ for } C_{11}=C_{22} \qquad (24c)$$

$$A_2 = \frac{r_2^2-r_1^2}{2}\left(C_{33}+\frac{C_{23}^2-C_{13}^2}{C_{11}-C_{22}}\right) \text{ for } C_{11}\neq C_{22}$$

$$= \frac{(r_2^2-r_1^2)}{8C_{11}}[4C_{33}C_{11}-(C_{23}-C_{13})^2]$$

$$+ \frac{C_{23}^2-C_{13}^2}{4C_{11}}r_1^2\ln(r_2/r_1) \text{ for } C_{11}=C_{22}. \qquad (24d)$$

After the displacements have been determined, the stresses are found by substituting in equations (5) and (4).

Discussion

As an example, consider a glass/epoxy circular cylindrical shell of inner radius r_1 = 20 mm and outer radius r_2 = 36 mm, with circumferential fibers. The moduli in GN/m^2 and Poisson's ratios for this material are listed next, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction:

$$E_1 = 13.7,\ E_2 = 55.9,\ E_3 = 13.7,\ G_{12} = 5.6,\ G_{23} = 5.6,$$
$$G_{31} = 4.9,\ \nu_{12} = 0.068,\ \nu_{23} = 0.277,\ \nu_{31} = 0.4.$$

The thermal expansion coefficients are: α_r = 40 × 10^{-6}/°C, α_θ = 10 × 10^{-6}/°C, α_z = 40 × 10^{-6}/° C. For this material,

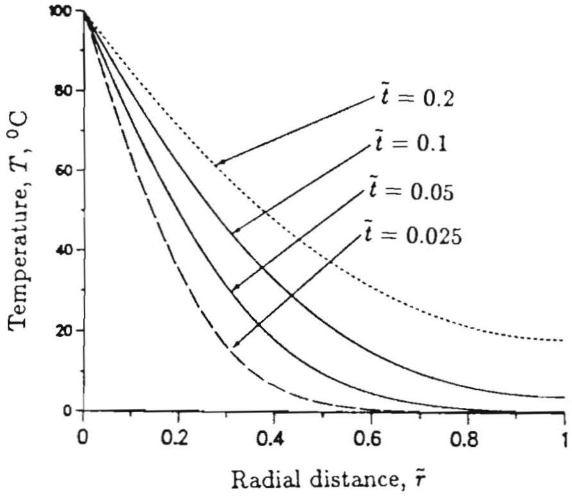

Fig. 1 Temperature T at different times for constant applied temperature T_0 = 100°C at the inner surface and heat convection at the outer surface. The nondimensional time is defined by $\bar{t} = Kt/(r_2 - r_1)^2$.

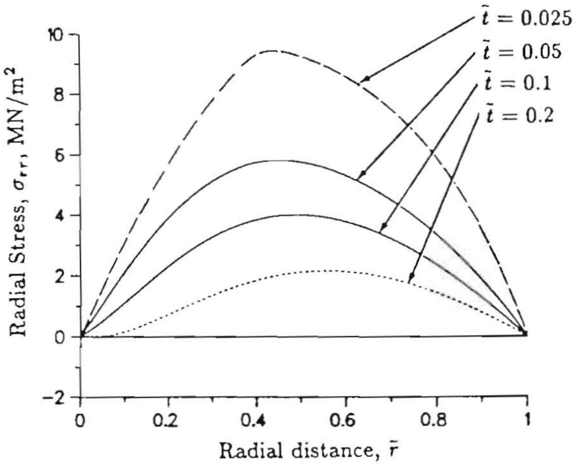

Fig. 3 Radial stress σ_{rr} versus normalized radial distance

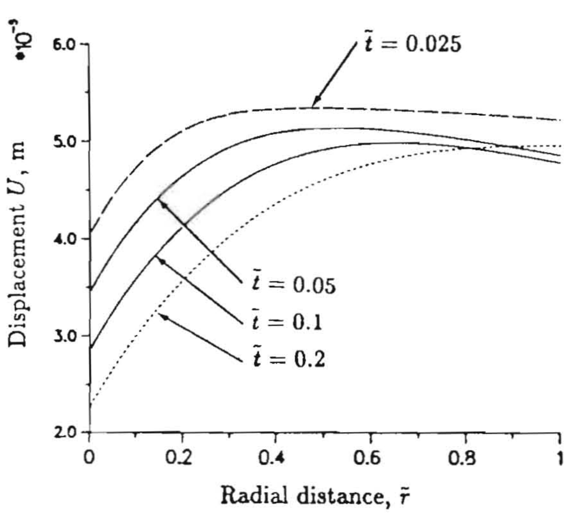

Fig. 2 Through the thickness distribution of the radial displacement U

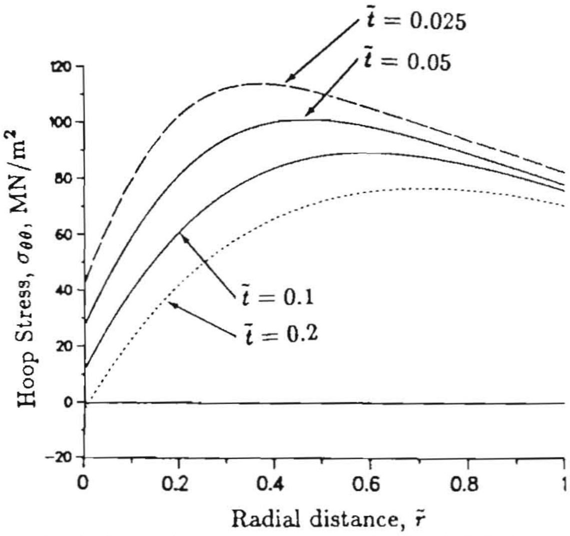

Fig. 4 Hoop stress $\sigma_{\theta\theta}$ versus normalized radial distance

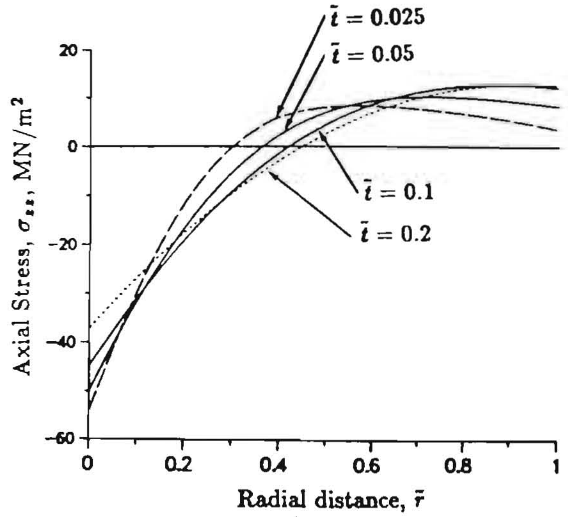

Fig. 5 Axial stress σ_{zz} versus normalized radial distance

the thermal diffusivity in the radial direction is $K = 0.112 \times 10^{-5}$ m^2/sec. Let us assume that a temperature of $T_0 = 100°$C above the reference one is applied at $r = r_1$ while there is heat convection to the surrounding air at $r = r_2$. In this case $h_1 = h_3^* = 0$, $h_1^* = h_2 = 1$, $h_3 = -T_0$, $h_2^* = h$ where h is the ratio of the convective heat-transfer coefficient between the composite tube and the surrounding medium at $r = r_2$ and the thermal conductivity of the tube in the radial direction. A typical value of this parameter for heat convection to the air is $h = 0.15$ m^{-1}. As was mentioned by Kardomateas (1989), although the reference temperature is taken equal to zero, the analysis is the same for any initial temperature other than zero, in which case T_0 is the applied temperature above this reference value.

In presenting the results, the nondimensional radial distance (i.e., through the thickness) is used, defined by

$$\bar{r} = \frac{r - r_1}{r_2 - r_1}.$$

Figure 1 shows the temperature and Fig. 2 the displacement distribution for time values $\bar{t} = 0.20$, 0.10, 0.05, and 0.025. Temperature gradients are steeper for small time values. The corresponding distribution of stresses σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} are shown in Figs. 3, 4, and 5. The biggest of those is the hoop stress $\sigma_{\theta\theta}$ and its maximum value is seen to be larger for $\bar{t} = 0.025$ than for $\bar{t} = 0.20$ by a factor of about 1.8 (Fig. 4). The location of the maximum stress also changes with time. The radial stress σ_{rr} is tensile and its maximum value at $\bar{t} = 0.025$ is 2.75 times that at $\bar{t} = 0.20$ (Fig. 3). For the same example case, Kardomateas (1989) found that under steady-state conditions, the radial stress is compressive throughout and the steady-state hoop stress is compressive if $\bar{r} < 0.45$. This underlines the strong dependence of the stress distribution on the time scale. The axial stress σ_{zz} has a maximum compressive value at the inner surface; at $\bar{t} = 0.025$, its magnitude is 1.4 times that at $\bar{t} = 0.20$ (Fig. 5). The axial component is important because the material is normally weaker in the directions perpendicular to the fibers.

Table 1 shows the number of terms N_{max} for the summation over n in the solution (8) that are required for convergence at different times. Convergence is defined here so as the successive last three terms are decreasing by at least an order of magnitude and the ratio of the last term over the first one for the stresses, temperature, and displacement is less than 10^{-5}. Only three of the roots, a_n, are in the "small argument" region; the rest are in the "large argument" (Hankel asymptotic expansion) region. In fact, in this initial phase of transient thermal stresses the roots in the "large argument" region are dominant and more terms are required for smaller time values.

Table 1 Number of terms N_{max} of the solution (8) required for convergence*. Only three of the roots a_n are in the "small argument" region; the rest are in the "large argument" (Hankel asymptotic expansions) region.

$\bar{t}$	N_{max}
0.200	4
0.100	5
0.050	6
0.025	8
0.010	11

*Convergence is defined here so as the successive last three terms are decreasing by at least an order of magnitude and the ratio of the last term over the first one on the stresses, temperature, and displacement is less than 10^{-5}.

References

Abramowitz, M., and Stegun, I. A., eds., 1970, "Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables," National Bureau of Standards Applied Mathematics Series No. 55, 9th printing, Washington, D.C.

Avery, W. B., and Herakovich, C. T., 1986, "Effect of Fiber Anisotropy on Thermal Stresses in Fibrous Composites," ASME JOURNAL OF APPLIED MECHANICS, Vol. 53, pp. 751–756.

Carslaw, H. S., and Jaeger, J. C., 1959, *Conduction of Heat in Solids*, 2nd ed., Oxford at the Clarendon Press, U.K.

Hyer, M. W., and Cooper, D. E., 1986, "Stresses and Deformations in Composite Tubes Due to a Circumferential Temperature Gradient," ASME JOURNAL OF APPLIED MECHANICS, Vol. 53, pp. 757–764.

Kalam, M.A., and Tauchert, T. R., 1978, "Stresses in an Orthotropic Elastic Cylinder due to a Plane Temperature Distribution $T(r, \theta)$," *Journal of Thermal Stresses*, Vol. 1, pp. 13–24.

Kardomateas, G. A., 1989, "Transient Thermal Stresses in Cylindrically Orthotropic Composite Tubes," ASME JOURNAL OF APPLIED MECHANICS, Vol. 56, pp. 411–417.

APPENDIX A

The constants d_1, d_2, d_3 are expressed in terms of

$$Q = r_1 h_2 h_1^* + r_2 h_1 h_2^* + r_1 r_2 h_2 h_2^* \ln(r_2/r_1), \qquad (A1a)$$

as follows:

$$d_1 = (r_2 h_3^* h_1 - r_1 h_3 h_1^*)/Q, \qquad (A1b)$$

$$d_2 = r_1 r_2 h_3^* h_2/Q;\ d_3 = r_1 r_2 h_3 h_2^*/Q. \qquad (A1c)$$

The constants d_{4n}, d_{5n} are given in terms of

$$F_1(a_n) = [h_1^* a_n J_1(r_2 a_n) - h_2^* J_0(r_2 a_n)]$$
$$\times \{h_3[h_1^* a_n J_1(r_2 a_n) - h_2^* J_0(r_2 a_n)] - h_3^*[h_1 a_n J_1(r_1 a_n) + h_2 J_0(r_1 a_n)]\}, \qquad (A2a)$$

$$F_2(a_n) = (h_1^{*2} a_n^2 + h_2^{*2})[h_1 a_n J_1(r_1 a_n) + h_2 J_0(r_1 a_n)]^2 - (h_1^2 a_n^2 + h_2^2)[h_1^* a_n J_1(r_2 a_n) - h_2^* J_0(r_2 a_n)]^2, \qquad (A2b)$$

as follows:

$$d_{4n} = -\pi \frac{F_1(a_n)}{F_2(a_n)} [h_1 a_n Y_1(r_1 a_n) + h_2 Y_0(r_1 a_n)], \qquad (A3a)$$

$$d_{5n} = \pi \frac{F_1(a_n)}{F_2(a_n)} [h_1 a_n J_1(r_1 a_n) + h_2 J_0(r_1 a_n)]. \qquad (A3b)$$

Finally, it should be noted that the case of $h_2 = h_2^* = 0$ is excluded (although it can be treated along the same lines).

APPENDIX B

For large arguments, we can use the Hankel asymptotic expansions for the Bessel functions (see, e.g., Abramowitz and Stegun, 1970) to obtain the following expressions

$$J_0(x)=A_0(x)\sin x+B_0(x)\cos x;\ J_1(x)=B_1(x)\sin x-A_1(x)\cos x, \tag{B1a}$$

$$Y_0(x)=B_0(x)\sin x-A_0(x)\cos x;\ Y_1(x)=-A_1(x)\sin x-B_1(x)\cos x. \tag{B1b}$$

The functions $A_0(x)$, $A_1(x)$, $B_0(x)$, $B_1(x)$ are given in terms of

$$\psi_1(k)=1^2.3^2.5^2\ldots(4k-1)^2,\ k=1,\infty;\ \psi_1(0)=1, \tag{B2}$$

as follows:

$$(\pi x)^{1/2}A_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\psi_1(k)}{(2k)!(8x)^{2k}}\left[1-\frac{16kx}{(4k-1)^2}\right], \tag{B3a}$$

$$(\pi x)^{1/2}A_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\psi_1(k)(4k+1)}{(2k)!(8x)^{2k}(4k-1)}\left[1-\frac{16kx}{(4k-3)(4k+1)}\right], \tag{B3b}$$

$$(\pi x)^{1/2}B_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k\psi_1(k)}{(2k)!(8x)^{2k}}\left[1+\frac{16kx}{(4k-1)^2}\right], \tag{B3c}$$

$$(\pi x)^{1/2}B_1(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\psi_1(k)(4k+1)}{(2k)!(8x)^{2k}(4k-1)}\left[1+\frac{16kx}{(4k-3)(4k+1)}\right]. \tag{B3d}$$

In the numerical calculations of the foregoing series, the summation process is terminated at the point where the term becomes less than a specified small number, which is taken here to be 10^{-16}. In this way, the above series of the Hankel asymptotic expansion can be used to calculate the Bessel functions for values of the argument $x\ge 18$. The series converges rapidly and the number of terms required in the summation over k is at most 13 at $x=18.0$, being smaller for larger values of the argument.

APPENDIX C

For $C_{11}\ne C_{22}$, the expression for E_0 in equation (24a), is:

$$E_0(r_1,r_2)=\frac{q_2(r_2^2d_2+r_1^2d_3)}{2(C_{11}-C_{22})}(C_{23}+C_{13})\ln(r_2/r_1)+\frac{(r_2^2-r_1^2)}{4(C_{11}-C_{22})}$$
$$\times\left\{(C_{13}-C_{23})q_2(d_2+d_3)+2(C_{13}+C_{23})\left[q_1(d_2+d_3)+q_2d_1-\frac{2C_{11}q_2(d_2+d_3)}{C_{11}-C_{22}}\right]\right\}. \tag{C1a}$$

For $C_{11}=C_{22}$, the expression for E_0 is

$$E_0(r_1,r_2)=\frac{q_2(r_2^2d_2-r_1^2d_3)}{8C_{11}}(C_{23}+C_{13})\ln^2(r_2/r_1)$$
$$+\frac{\ln(r_2/r_1)}{8C_{11}}\{(r_2^2d_2+r_1^2d_3)(C_{13}-C_{23})q_2+r_1^2(C_{13}+C_{23})[(2q_1-q_2)(d_2+d_3)+2q_2d_1]\}$$
$$+\frac{(r_2^2-r_1^2)}{8C_{11}}(C_{23}-C_{13})[(q_2-q_1)(d_2+d_3)-q_2d_1], \tag{C1b}$$

and the expressions for E_n, $n=1,\infty$ in equation (27), (they involve an integration of $R_{nL}^{**}(\rho)$) are for the case of large arguments

$$E_n=C_{13}[r_2R_{nL}^{**}(r_2a_n)-r_1R_{nL}^{**}(r_1a_n)]+(C_{23}-C_{13})a_n^{-1}\sum_{i=1}^{2}(-1)^i\left[\frac{d_{6n}(r_ia_n)^{\lambda_1+1}}{\lambda_1+1}+d_{7n}F_i\right]$$
$$+(C_{23}-C_{13})a_n^{-1}\sum_{k=0}^{\infty}\left\{\left(p_{k,1}^n+\frac{s_{k,2}^n}{2k+1/2}\right)A_k^n+\left(s_{k,1}^n-\frac{p_{k,2}^n}{2k+1/2}\right)B_k^n+\right.$$
$$\left.+\sum_{i=1}^{2}\frac{(-1)^{i+1}}{2k+1/2}\left[p_{k,2}^n(r_ia_n)^{-2k-1/2}\cos(r_ia_n)+s_{k,2}^n(r_ia_n)^{-2k-1/2}\sin(r_ia_n)\right]\right\}. \tag{C2a}$$

The constant A_k^n and B_k^n are defined by the recursive formulas

$$A_k^n=-\frac{1}{(2k-1/2)(2k-3/2)}A_{k-1}^n+\sum_{i=1}^{2}(-1)^i\left\{\frac{(r_ia_n)^{-2k+3/2}\sin(r_ia_n)}{(2k-1/2)(2k-3/2)}-\frac{(r_ia_n)^{-2k+1/2}\cos(r_ia_n)}{2k-1/2}\right\}, \tag{C2b}$$

$$B_k^n=-\frac{1}{(2k-1/2)(2k-3/2)}B_{k-1}^n+\sum_{i=1}^{2}(-1)^{i+1}\left\{\frac{(r_ia_n)^{-2k+3/2}\cos(r_ia_n)}{(2k-1/2)(2k-3/2)}+\frac{(r_ia_n)^{-2k+1/2}\sin(r_ia_n)}{2k-1/2}\right\}. \tag{C2c}$$

The initial values are

$$A_0^n=\int_{r_1a_n}^{r_2a_n}\rho^{-1/2}\cos\rho\,d\rho;\ B_0^n=\int_{r_1a_n}^{r_2a_n}\rho^{-1/2}\sin\rho\,d\rho. \tag{C3}$$

The constants F_i, $i=1, 2$ in equation (C2a) are defined

$$F_i=\frac{(r_ia_n)^{\lambda_2+1}}{\lambda_2+1}\ \text{for}\ C_{11}\ne C_{22}$$
$$=\ln(r_ia_n)\ \text{for}\ C_{11}=C_{22}. \tag{C4}$$

The expressions of E_n for small arguments were given by Kardomateas (1989) and are the same for this case of general thermal loading.