The Initial Phase of Transient Thermal Stresses due to General Boundary Thermal Loads in Orthotropic Hollow Cylinders

The stresses and displacements in the initial phase of applying a thermal load on the bounding surfaces of an orthotropic hollow circular cylinder are obtained using the Hankel asymptotic expansions for the Bessel functions of the first and second kind. Such a load may be constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface. The material properties are assumed to be independent of temperature. A constant applied temperature at the one surface and convection into a medium at a different temperature at the other surface is used to illustrate the variation of stresses with time and through the thickness in the initial transient phase.

Introduction

Meeting the need for materials which can function usefully at different temperature levels is one of the most challenging problems facing our technology. The difficulty is compounded by the fact that operating conditions involve not only elevated temperature levels but frequently also severe temperature gradients. Such temperature differentials may produce thermal stresses significant enough to limit the material life. Some of the more important applications of fiber-reinforced composites involve the configuration of a hollow cylinder, typically produced by filament winding on a cylindrical mandrel. For example, composite tubes of considerable thickness could be used in such parts as automotive suspension components. The manufacturing stage includes a thermal treatment phase, during which thermal stresses of considerable magnitude may be generated. Other applications may be for launch tubes or landing gears.

Relevant work on understanding the thermally-induced stresses in such configurations was done by Kalam and Tauchert (1978), Hyer and Cooper (1986), and Avery and Herakovich (1986). These studies dealt with the steady-state problem. A solution to the transient thermal stress problem was derived by Kardomateas (1989). This solution involved a series expansion of Bessel functions of the first and second kind and it is valid only beyond a certain time value. This is because the series expansions cannot be used for large arguments; small time values require including an increasing number of terms and therefore large arguments. This paper gives the solution for this case of the very beginning of applying the thermal loads (the initial phase of transient thermal stresses). The analysis is limited to orthotropic tubes, (for example, composite tubes with the fibers aligned circumferentially). The material properties are assumed to be independent of temperature, the stresses not to vary along the generators and that there are no body forces.

In this paper the body is assumed to be subjected to any kind of thermal load such as constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface; in this respect this paper also extends the work by Kardomateas (1989), which was done for the specific case of a constant temperature on the one surface of the cylinder and heat convection to a different medium at the other. We shall present an example for the distribution of stresses and displacements in the initial phase of transient stresses, where it is assumed that one surface of the cylinder is at a constant temperature and at the other there is heat convection into a medium at the reference temperature.

Derivation of the Governing Equations

Consider a circular cylindrical shell \( r_1 \leq r \leq r_2 \) whose axis coincides with the axis of \( z \). The initial (reference) temperature is assumed to be zero. The thermal problem consists of the heat conduction equation

\[
K \left( \frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r} \frac{\partial T(r,t)}{\partial r} \right) = \frac{\partial T(r,t)}{\partial t} \quad (r_1 < r < r_2, t > 0),
\]
where \( K \) is the thermal diffusivity of the composite in the \( r \) direction, and the initial and boundary conditions
\[
T(r, t=0) = 0 \quad \text{at} \quad r_1 \leq r \leq r_2, \quad (1b)
\]
\[
h_1 \frac{\partial T(r, t)}{\partial r} \big|_{r=r_1} - h_2 T(r_1, t) = h_1 \psi(t > 0), \quad (1c)
\]
\[
h_1^* \frac{\partial T(r, t)}{\partial r} \big|_{r=r_2} + h_2^* T(r_2, t) = h_1^* \psi(t > 0), \quad (1d)
\]
where \( h_1, h_2, h_1^*, h_2^* \) are constants which may be positive or zero (provided \( h_1 \) and \( h_2 \), or \( h_1^* \) and \( h_2^* \) do not both vanish) and \( h_1 \) and \( h_2 \) are constants. By choice of these constants the general results include all combinations of constant temperature, constant heat flux, zero heat flux, or heat convection to a different medium at either surface. A solution for the temperature field is given by Carslaw law and Jaeger (1959) in terms of the Bessel functions of the first and second kind \( J_n \) and \( Y_n \), as follows
\[
T(r, t) = d_1 + d_2 \ln(r/r_1) + d_3 \ln(r/r_2)
\]
\[
+ \sum_{n=1}^{\infty} e^{-\kappa d_n^2 \psi} [d_{n+} Y_n(a_n) + d_{n-} J_n(a_n)],
\]
(2)
where \( \pm a_n \) are the roots (all real and simple) of:
\[
[h_1 x J_1(x) + h_2 x Y_1(x)] [h_1^* x J_1(x) + h_2^* x Y_1(x)]
\]
\[
- [h_1 x J_1(x) + h_2 x Y_1(x)] [h_1^* x J_1(x) + h_2^* x Y_1(x)].
\]
(3)
The constants \( d_n \) are given in Appendix A.

In cylindrical coordinates \((r, \theta, z)\), the constitutive equations for the orthotropic body are given in terms of the elastic constants \( C_{ij} \) and the thermal expansion coefficients, \( \alpha_{ij} \):
\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta \theta} \\
\sigma_{zz} \\
\tau_{r \theta} \\
\tau_{r z} \\
\tau_{\theta z}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{rr} \\
\epsilon_{\theta \theta} \\
\epsilon_{zz} \\
\gamma_{r \theta} \\
\gamma_{r z} \\
\gamma_{\theta z}
\end{bmatrix} - \alpha_{ij} \Delta T.
\]
Since the temperature field is independent of the angular and axial coordinates, the stresses and strains will be a function of \( r \) and \( t \) only.

Following the analysis by Kardomateas (1989), the strain field for this class of problems is expressed in terms of the radial displacement \( U(r, t) \):
\[
\begin{align}
\epsilon_{rr} &= \frac{3U(r, t)}{r} \quad \text{and} \quad \epsilon_{\theta \theta} = \frac{U(r, t)}{r} \quad \text{at} \quad r > 0. \\
\gamma_{r \theta} &= \gamma_{r z} = \gamma_{\theta z} = 0.
\end{align}
\]
(5a)
(5b)

There remain one equilibrium equation for the nonvanishing stresses:
\[
\frac{\partial \sigma_{rr}(r, t)}{\partial r} + \frac{\sigma_{rr}(r, t) - \sigma_{\theta \theta}(r, t)}{r} = 0.
\]
(6)
Define

\[ q_1 = C_{11} \alpha_x + C_{12} \alpha_y + C_{13} \alpha_z, \]
\[ q_2 = (C_{11} - C_{12}) \alpha_x + (C_{12} - C_{22}) \alpha_y + (C_{13} - C_{23}) \alpha_z, \]
\[ q_3 = C_{13} \alpha_x + C_{23} \alpha_y + C_{33} \alpha_z. \]
(7a)
(7b)
(7c)

Set
\[
U(r, t) = U_0(t) + \sum_{n=1}^{\infty} e^{-\kappa d_n^2 \psi} R_n(r),
\]
(8)
and express the parameter \( C(t) \) in the form
\[
C(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\kappa d_n^2 \psi}.
\]
(9)
Substituting equations (4), (5) into (6) and using the definitions (7), (8), and (9) results in the following differential equations:
\[
C_{11} \left( R_0''(r) + \frac{R_0'(r)}{r} \right) - \frac{C_{22}}{r^2} U_0(r)
\]
\[
= q_1 (d_2 + d_4) + q_2 d_2 + (C_{22} - C_{12}) C_0
\]
\[
+ q_2 d_2 \ln(r/r_1) + q_3 d_1 \ln(r/r_2),
\]
(10)
\[
C_{11} \left( R_n''(r) + \frac{R_n'(r)}{r} \right) - \frac{C_{22}}{r^2} R_n(r)
\]
\[
= \frac{(C_{22} - C_{12}) C_0}{r} + d_4 \left[ J_n(a_n) - d_n \left( Y_n(a_n) \right) \right] - q_1 a_n J_n(a_n) + d_n \left( Y_n(a_n) \right)
\]
(11)

Solution

Define
\[
\lambda_{1,2} = \pm \sqrt{C_{22}/C_{11}}.
\]
(12)
For \( C_{11} \neq C_{22} \), the solution of (10) for \( U_0(r) \) is
\[
U_0(r) = G_1 \psi^{1/2} + G_2 \psi^{3/2} + \frac{C_{22} - C_{12}}{C_{11} - C_{22}} C_0 + U_0'(r),
\]
(13a)
\[
U_0'(r) = \frac{q_2}{C_{11} - C_{22}} \left[ d_2 \ln(r/r_1) + d_3 \ln(r/r_2) \right]
\]
\[
+ \left[ \frac{q_1 (d_2 + d_4) + q_2 d_2}{C_{11} - C_{22}} - \frac{2C_{11} q_2 (d_2 + d_3)}{(C_{11} - C_{22})^2} \right] \right] r.
\]
(13b)
For \( C_{11} = C_{22} \), the corresponding solution is
\[
U_0(r) = G_1 \psi^{1/2} + G_2 \psi^{3/2} + \frac{(C_{22} - C_{12}) C_0}{2C_{11}} \psi \ln(r/r_2) + U_0'(r),
\]
(14a)
\[
U_0'(r) = \frac{q_3}{4C_{11}} \left[ d_2 \ln^2(r/r_1) + d_3 \ln^2(r/r_2) \right]
\]
\[
+ \left[ \frac{2q_1 - q_2}{C_{11}} (d_2 + d_3) + 2q_2 d_2 \ln(r/r_2) \right] r \ln(r/r_2),
\]
(14b)
where \( G_1 \) and \( G_2 \) are constants, yet unknown, which, together with the constants \( c_n \), are found later from the boundary conditions.

To solve equation (11) for large arguments, we use the Han­kel asymptotic expansions of the Bessel functions of the first and second kind. For \( C_{11} \neq C_{22} \), the solution for \( R_n \) is expressed
\[ R_a(r) = G_{10}r^{2} + G_{20}r^{4} + C_{12}r^{2} - C_{12}r^{4} + R_a^*(r). \]  \hspace{1cm} (15a)\]

and for \( C_{11} = C_{22} \), the solution for \( R_a \) is

\[ R_a(r) = G_{10}r^{2} + \frac{G_{20}r^{4}}{r} + C_{12}r^{2} \ln(r/r_2) + R_a^*(r). \]  \hspace{1cm} (15b)\]

The function \( R_a^*(r) \) is the particular solution of the equation that results from equation (11) if the right-hand side includes only the Bessel functions terms. For small values of the argument the function \( R_a^*(r) \) was given in Kardomateas (1989) (although a specific thermal load was considered in that work, the expression holds true for the general thermal load considered here).

For large arguments, we use the Hankel asymptotic expansions of the Bessel functions of the first and second kind (see Appendix B). Employing the substitution

\[ \rho = ra_n, \quad R_a^*(r) = R_a^*(\rho), \]  \hspace{1cm} (16)\]

gives the following equation for \( R_a^*(\rho) \)

\[ C_{11}a_{10}^2 \left( R_a^{**}(\rho) + \frac{R_a^{***}(\rho)}{\rho} \right) - C_{22}a_{10}^2 R_a^{**}(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k a_{10} \psi(k)}{(2k)!} \frac{(d_{an} + d_{am})}{\sqrt{\rho}} \times (d_{an} - d_{am}) \cos \rho + \frac{(d_{an} - d_{am})}{\sqrt{\rho}} \sin \rho, \] \hspace{1cm} (17)\]

where

\[ a_{10} = \frac{q_{10}(\rho) + q_{10}(\rho)}{2}, \quad a_{10} = \frac{d_{4n} + d_{5n}}{2}, \quad \psi(k) = \frac{\sin(4n + 5n)}{2(k - 1)!}, \] \hspace{1cm} (18)\]

and \( \psi(k) \) is defined in Appendix B.

The solution of equation (17) for the function \( R_a^*(\rho) \) is found to be

\[ R_a^{**}(\rho) = \sum_{n=0}^{\infty} p_{n0}^0 \rho^{-2k-1/2} \cos \rho + s_{n0}^0 \rho^{-2k-1/2} \sin \rho + \] \[ + \frac{p_{n0}^0 \rho^{-2k-3/2} \cos \rho + s_{n0}^0 \rho^{-2k-3/2} \sin \rho}{\rho}. \] \hspace{1cm} (19)\]

The coefficients \( p_{n0}^0, s_{n0}^0, p_{n2}^0, s_{n2}^0 \) are determined by considering the terms in the sum (18) that contribute to the terms \( \rho^{-2k-1/2} \cos \rho, \rho^{-2k-1/2} \sin \rho, \rho^{-2k-3/2} \cos \rho, \rho^{-2k-3/2} \sin \rho \) in the right-hand side of equation (17a). We obtain the following recursive formulas for \( p_{n1}^0, s_{n1}^0 \)

\[ p_{n1}^0, C_{11} = p_{n1}^0 - \frac{C_{11}}{C_{11}} - C_{12}(2k + 3/2)^2 - C_{12} + s_{n1}^0, C_{11}(2k + 1) \] \[ + \frac{(-1)^k \psi(k)}{(2k)!} \frac{(d_{an} + d_{am})}{\sqrt{\rho}}, \] \hspace{1cm} (20a)\]

and for \( p_{n2}^0, s_{n2}^0 \)

\[ p_{n2}^0, C_{11} = p_{n2}^0 - \frac{C_{11}}{C_{11}} - C_{12}(2k + 1/2)^2 - C_{12} + s_{n2}^0, C_{11}(2k + 4k) \] \[ + \frac{(-1)^k \psi(k + 1)}{(2k + 1)!} \frac{(d_{an} + d_{am})}{\sqrt{\rho}}, \] \hspace{1cm} (20c)\]

and for \( p_{n1}^2, s_{n1}^2 \)

\[ s_{n1}^2, C_{11} = s_{n1}^2 - \frac{C_{11}}{C_{11}} - C_{12}(2k + 1/2)^2 - C_{12} + p_{n1}^2, C_{12}(4k) \] \[ + \frac{(-1)^k \psi(k + 1)}{(2k)!} \frac{(d_{an} + d_{am})}{\sqrt{\rho}}. \] \hspace{1cm} (20c)\]

The process starts from \( k = 1 \) and the starting values for \( k = 0 \) are from equations (17) and (18) as follows:

\[ \rho_{01} = -q_{10}(d_{an} + d_{am}) a_{10} \sqrt{\pi}, \quad s_{01} = q_{10}(d_{an} - d_{am}) a_{10} \sqrt{\pi}, \] \hspace{1cm} (21a)\]

\[ \rho_{02} = (-8 + 3q_{10}) (d_{an} - d_{am}) a_{10} \sqrt{\pi}, \quad s_{02} = 8a_{10} \sqrt{\pi}. \] \hspace{1cm} (21b)\]

A fine point of the analysis will be addressed now. The solution (19) is a particular solution of equation (17). It was derived based on the Hankel asymptotic expansions of the Bessel functions for values of the argument \( \rho = ra_n \gg \rho_0 \) at 18.0 (see Appendix B). A corresponding solution, \( R_{a1}^*(r) \), for values of the argument \( \rho = \rho_0 \) had been derived by Kardomateas (1989), based on a series expansion for the Bessel functions. Since, for a given root \( a_n \) the argument \( \rho \) ranges from \( ra_n \) to \( \rho_0 a_n \) there may be a transition point from one solution to the other for \( R_a^*(r) \) in the expression (15). Both solutions are particular ones and may be different. Therefore, at that transition point a homogeneous solution term should be added to (19) so that

\[ R_a^*(\rho) = d_{10} \rho^{2k+1} + d_{10} \rho^{2k} + R_a^*(\rho); \quad R_a^*(r) = R_a^*(\rho). \] \hspace{1cm} (22a)\]

where \( d_{10} \) and \( d_{10} \) are determined from the condition of equal value and slope at the transition point

\[ R_a^*(\rho_0) = R_{a2}^*(\rho_0/a_n); \quad R_a^*(\rho_0/a_n) = R_{a2}^*(\rho_0/a_n). \] \hspace{1cm} (22b)\]

The unknown constants \( C_{10}, C_{20} \) are found from the conditions of zero external traction and zero resultant force. These are modified from Kardomateas (1989) to include the general thermal load case. The traction-free condition, \( d_{10}r_{10}, t = 0 \), gives the following two linear equations in \( G_{10}, G_{20}, c_0 \)

\[ (C_{11}^2 + C_{12}^2)^2 - C_{12}^2 G_{10} + (C_{11}^2 + C_{12}^2)^2 G_{20} + A_{0} c_0 = 0, \] \hspace{1cm} (23a)\]

\[ A_0 = C_{11}^{10} + C_{12}^{10} (C_{11}^2 - C_{12}^2) + C_{11} \] \[ = C_{22} - C_{22} \left[ C_{11} + (C_{11} + C_{12}) \text{ln}(r/r_2) \right] + C_{11} \] \[ = C_{22} - C_{22} \] \hspace{1cm} (23b)\]

and two linear equations for \( G_{10}, G_{20}, c_n, n = 1, \infty \)

\[ (C_{11}^2 + C_{12}^2)^2 - C_{12}^2 G_{10} + (C_{11}^2 + C_{12}^2)^2 G_{20} + A_{0} c_n = 0, \] \hspace{1cm} (23c)\]

The condition of zero resultant axial force, \( F_z = \int_{r_1}^{r_2} a_{12}(r) \] \[ 1/2 \times r \, dr = 0, \] gives the last set of equations that are needed to determine the constants \( G, c \). For \( C_{11} \neq C_{22} \)

\[ C_{11} + \frac{C_{22} - C_{11}}{\lambda_1 + 1} (r_1^{2l+1} - r_2^{2l+1}) G_{10} + A_1 G_{20} + A_2 c_0 = 0. \] \hspace{1cm} (23d)\]
\[
E_0(r_1, r_2) + \frac{q_1}{2} \left( \frac{(r_2^2 - r_1^2)}{2} (2d_1 - d_2 - d_3) \right) + (d_1 r_2^2 + d_2 r_1^2) \ln(r_2/r_1)
\]

(24a)

and for \( n = 1, \infty \),

\[
(C_{13} + \frac{C_{23} - C_{13}}{\lambda_1 + 1}) \left( \phi_{11}^{n+1} - \phi_{21}^{n+1} \right) G_{1n} + A_1 G_{2n} + A_2 c_n =
\]

\[
= -E_n(r_1, r_2) + \frac{q_1}{r_n} \sum_{i=1}^{n} (-1)^i \left( d_{e_{r_1}} l_1 (r_1 \rho_n) + d_{e_{r_1}} Y_1 (r_1 \rho_n) \right).
\]

(24b)

where \( E_n(r_2, r_1) \) and \( E_n(r_1, r_2) \) are defined in Appendix C. The coefficients \( A_1, A_2 \) are defined as:

\[
A_1 = \left( C_{13} + \frac{C_{23} - C_{13}}{\lambda_2 + 1} \right) \left( \phi_{22}^{n+1} - \phi_{21}^{n+1} \right) \text{ for } C_{11} \neq C_{22}
\]

\[
= C_{13} + (C_{23} - C_{13}) \ln(r_2/r_1) \text{ for } C_{11} = C_{22}
\]

(24c)

After the displacements have been determined, the stresses are found by substituting in equations (5) and (4).

Discussion

As an example, consider a glass/epoxy circular cylindrical shell of inner radius \( r_1 = 20 \text{ mm} \) and outer radius \( r_2 = 36 \text{ mm} \), with circumferential fibers. The moduli in GN/m² and Poisson’s ratios for this material are listed next, where 1 is the radial (\( r \)), 2 is the circumferential (\( \theta \)), and 3 the axial (\( z \)) direction:

\[
E_1 = 13.7, \quad E_2 = 55.9, \quad E_3 = 13.7, \quad G_{12} = 5.6, \quad G_{23} = 5.6, \quad G_{31} = 4.9, \quad \nu_{12} = 0.068, \quad \nu_{23} = 0.277, \quad \nu_{13} = 0.4.
\]

The thermal expansion coefficients are: \( \alpha_r = 40 \times 10^{-6}/^\circ \text{C} \), \( \alpha_\theta = 10 \times 10^{-6}/^\circ \text{C} \), \( \alpha_z = 40 \times 10^{-6}/^\circ \text{C} \). For this material,
the thermal diffusivity in the radial direction is \( K = 0.112 \times 10^{-6} \text{ m}^2/\text{sec} \). Let us assume that a temperature of \( T_0 = 100^\circ \text{C} \) above the reference one is applied at \( r = r_1 \) while there is heat convection to the surrounding air at \( r = r_2 \). In this case \( h_1 = h_2 = 0; h_3 = h_4 = 1, h_5 = -T_0, h_6^2 = h \) where \( h \) is the ratio of the convective heat-transfer coefficient between the composite tube and the surrounding medium at \( r = r_2 \) and the thermal conductivity of the tube in the radial direction. A typical value of this parameter for heat convection to the air is \( h = 0.15 \text{ m}^2/\text{sec} \). As was mentioned by Kardomateas (1989), although the reference temperature is taken equal to zero, the analysis is the same for any initial temperature other than zero, in which case \( T_0 \) is the applied temperature above this reference value.

In presenting the results, the non-dimensional radial distance (i.e., through the thickness) is used, defined by

\[
F = \frac{r - r_1}{r_2 - r_1}
\]

Figure 1 shows the temperature and Fig. 2 the displacement distribution for time values \( \tilde{t} = 0.20, 0.10, 0.05, \) and 0.025. Temperature gradients are steeper for small time values. The biggest of those is the hoop stress \( \sigma_{\theta\theta} \) and its maximum value is seen to be larger for \( \tilde{t} = 0.025 \) than for \( \tilde{t} = 0.20 \) by a factor of about 1.8 (Fig. 4). The location of the maximum stress also changes with time. The radial stress \( \sigma_r \) is tensile and its maximum value at \( \tilde{t} = 0.025 \) is 2.75 times that at \( \tilde{t} = 0.20 \) (Fig. 3). For the same example case, Kardomateas (1989) found that under steady-state conditions, the radial stress is compressive throughout and the steady-state hoop stress is compressive if \( \tilde{t} < 0.45 \). This underlines the strong dependence of the stress distribution on the time scale.

The axial stress \( \sigma_z \) has a maximum compressive value at the inner surface; at \( \tilde{t} = 0.25 \), its magnitude is 1.4 times that at \( \tilde{t} = 0.20 \) (Fig. 5). The axial component is important because the material is normally weaker in the directions perpendicular to the fibers.

Table 1 shows the number of terms \( N_{\text{max}} \) for the summation over \( n \) in the solution (8) that are required for convergence at different times. Convergence is defined here so as the successive last three terms are decreasing by at least an order of magnitude and the ratio of the last term over the first one on the stresses, temperature, and displacement is less than \( 10^{-6} \). Only three of the roots, \( a_n \), are in the "small argument" region; the rest are in the "large argument" (Hankel asymptotic expansion) region. In fact, in this initial phase of transient thermal stresses the roots in the "large argument" region are dominant and more terms are required for smaller time values.

### References


### APPENDIX A

The constants \( d_1, d_2, d_3 \) are expressed in terms of

\[
\begin{align*}
Q &= r_1 h_2 h_1^2 + r_1 h_1 h_2 + r_1 r_2 h_2^2 \ln(r_2/r_1),
\end{align*}
\]

as follows:

\[
\begin{align*}
d_1 &= (r_1 h_2 h_1 - r_1 h_2^2)/Q ,
\end{align*}
\]

\[
\begin{align*}
d_2 &= r_1 h_1 h_2^2/Q ,
\end{align*}
\]

\[
\begin{align*}
d_3 &= r_1 h_1 h_2^2/Q.
\end{align*}
\]

The constants \( d_{1n}, d_{3n} \) are given in terms of

\[
\begin{align*}
F_1(a_n) &= [h_1 a_n J_1(r_2 a_n) - h_2 J_1'(r_2 a_n)] \\
&\quad \times \{ h_1 a_n J_1(r_2 a_n) - h_2 J_1'(r_2 a_n) \} \\
&\quad - [h_1 a_n J_1(r_2 a_n) + h_2 J_2(r_2 a_n)] ,
\end{align*}
\]

\[
\begin{align*}
F_2(a_n) &= (h_1^2 a_n^2 + h_2^2) [h_1 a_n J_1(r_2 a_n) + h_2 J_2(r_2 a_n)]^2 \\
&\quad - (h_1^2 a_n^2 + h_2^2) [h_1 a_n J_1(r_2 a_n) - h_2 J_2(r_2 a_n)]^2 ,
\end{align*}
\]

as follows:

\[
\begin{align*}
d_{1n} &= -\pi \frac{F_1(a_n)}{F_2(a_n)} [h_1 a_n Y_1(r_2 a_n) + h_2 Y_2(r_2 a_n)] ,
\end{align*}
\]

\[
\begin{align*}
d_{3n} &= \pi \frac{F_1(a_n)}{F_2(a_n)} [h_1 a_n n J_1(r_2 a_n) + h_2 J_2(r_2 a_n)] .
\end{align*}
\]

Finally, it should be noted that the case of \( h_1 = h_2 = 0 \) is excluded (although it can be treated along the same lines).

### APPENDIX B

For large arguments, we can use the Hankel asymptotic expansions for the Bessel functions (see, e.g., Abramowitz and Stegun, 1970) to obtain the following expressions
\( J_0(x) = A_0(x) \sin x + B_0(x) \cos x; \quad J_1(x) = -A_1(x) \sin x - B_1(x) \cos x. \)  
\( Y_0(x) = B_0(x) \sin x - A_0(x) \cos x; \quad Y_1(x) = -A_1(x) \sin x - B_1(x) \cos x. \)

The functions \( A_0(x), A_1(x), B_0(x), B_1(x) \) are given in terms of \( t/t_1(0) = 1, 2, 3, \ldots \) as follows:

\[
J_1(k) = \sum_{k=0}^{\infty} \frac{(-1)^k \psi_1(k)}{(2k)! (8k)_{2k}} \left[ 1 - \frac{16kx}{(4k-1)(4k+1)} \right], \\
Y_1(k) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \psi_1(k)}{(2k)! (8k)_{2k}} \left[ 1 + \frac{16kx}{(4k-1)(4k+1)} \right], \\
J_0(k) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \psi_1(k)}{(2k)! (8k)_{2k}} \left[ 1 + \frac{16kx}{(4k-3)(4k+1)} \right], \\
Y_0(k) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \psi_1(k)}{(2k)! (8k)_{2k}} \left[ 1 - \frac{16kx}{(4k-3)(4k+1)} \right].
\]

In the numerical calculations of the foregoing series, the summation process is terminated at the point where the term becomes less than a specified small number, which is taken here to be \( 10^{-16} \). In this way, the above series of the Hankel asymptotic expansion can be used to calculate the Bessel functions for values of the argument \( x \geq 18 \). The series converges rapidly and the number of terms required in the summation over \( k \) is at most 13 at \( x = 18.0 \), being smaller for larger values of the argument.

### Appendix C

For \( C_{11} \neq C_{22} \), the expression for \( E_0 \) in equation (24a), is:

\[
E_0(r_1, r_2) = \frac{q_0 \left( r_2^2 d_2 + r_1^2 d_1 \right)}{2(C_{11} - C_{22})} \left[ \left( C_{33} + C_{11} \right) \ln \left( \frac{r_2}{r_1} \right) + \frac{(r_2^2 - r_1^2)}{4(C_{11} - C_{22})} \right] + \frac{q_0 d_2 - 2C_{11} q_1 (d_2 + d_1)}{C_{11} - C_{22}},
\]

For \( C_{11} = C_{22} \), the expression for \( E_0 \) is:

\[
E_0(r_1, r_2) = \frac{q_0 \left( r_2^2 d_2 - r_1^2 d_1 \right)}{8C_{11}} \left( C_{33} + C_{11} \right) \ln \left( \frac{r_2}{r_1} \right) + \frac{\ln \left( \frac{r_2}{r_1} \right)}{8C_{11}} \left( r_1^2 d_2 + r_2^2 d_1 \right) (C_{13} - C_{33}) q_1 + \frac{r_1^2}{C_{11}} (C_{13} + C_{11} \ln \left( \frac{r_2}{r_1} \right) + \frac{(r_2^2 - r_1^2)}{4C_{11}} (C_{13} - C_{33}) \left[ q_1 (d_2 + d_1) + \frac{(r_2^2 - r_1^2)}{4C_{11}} \right].
\]

The constant \( A_0^2 \) and \( B_0^2 \) are defined by the recursive formulas:

\[
A_0^2 = A_{0-1}^2 - \frac{1}{(2k-1/2)(2k-3/2)} A_{0-1}^2 + \frac{1}{(2k-1/2)(2k-3/2)} A_{0-2}^2 + \sum_{n=1}^{\infty} (-1)^n \frac{(r_2 r_3)^{2k-1/2} \sin(r_2 r_3)}{(2k-1/2)(2k-3/2)} - \frac{(r_2 r_3)^{2k+1/2} \cos(r_2 r_3)}{2k-1/2}.
\]

The initial values are:

\[
A_0^2 = \int_{r_{1n}}^{r_{2n}} p^{-1/2} \sin p dp, \quad B_0^2 = \int_{r_{1n}}^{r_{2n}} p^{-1/2} \cos p dp.
\]

The expressions for \( E_n, n = 1, \infty \) in equation (27), (they involve an integration of \( R_{21}^0(p) \)) are for the case of large arguments:

\[
E_n = C_1 E_n^0 - \frac{r_2^2 - r_1^2}{8C_{11}} \left[ q_1 (d_2 + d_1) + \frac{(r_2^2 - r_1^2)}{4C_{11}} \right] + (C_{13} - C_{33}) \sum_{n=1}^{\infty} (-1)^n \frac{\ln \left( \frac{r_2}{r_1} \right)}{8C_{11}} \left( r_1^2 d_2 + r_2^2 d_1 \right) (C_{13} - C_{33}) q_1 + \frac{r_1^2}{C_{11}} (C_{13} + C_{11} \ln \left( \frac{r_2}{r_1} \right) + \frac{(r_2^2 - r_1^2)}{4C_{11}} (C_{13} - C_{33}) \left[ q_1 (d_2 + d_1) + \frac{(r_2^2 - r_1^2)}{4C_{11}} \right].
\]

The initial values are:

\[
A_0^2 = \sum_{n=1}^{\infty} \frac{(r_2 r_3)^{2k+1/2}}{(2k+1/2)(2k-1/2)} \cos(r_2 r_3) - \frac{(r_2 r_3)^{2k+1/2}}{(2k+1/2)(2k-1/2)} \sin(r_2 r_3).
\]

The constants \( F_i, i = 1, 2 \) in equation (C2a) are defined:

\[
F_i = \frac{(r_2 r_3)^{2k+1}}{\lambda_2 + 1} \quad \text{for} \quad C_{11} \neq C_{22}, \quad \text{and} \quad F_i = \ln(r_2 r_3) \quad \text{for} \quad C_{11} = C_{22}.
\]

The expressions of \( E_n \) for small arguments were given by Kardomateas (1989) and are the same for this case of general thermal loading.