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Asymptotic analysis considerations on the initial postbuckling behavior of delaminated composites

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Summary. An asymptotic analysis for the initial postbuckling behavior of delaminated beam/plates is performed. Under the assumptions of inextensional deformation, the exact expressions that govern the plane elastic deformation of the different parts of the system are expanded in a Taylor series in terms of the distortion variable. Order of magnitude arguments are used to relate the distortion variables and these are subsequently used in an asymptotic solution for the deflections and the load. Finally, a comparison with experiments is performed.

1 Introduction

Delaminations or interlayer cracks may result from events during service life such as low velocity impacts with particular implications for aerospace applications [1]. Also, many potential sources for defects to occur in the finished product exist in the manufacturing procedures [2]. A situation of fundamental interest in the current context is the behavior of compressed structural members with the presence of delaminations. Prediction and analysis of the deformation with methods based upon the stability theory is essential for the study of another important related aspect, that of delamination growth by fracture mechanics methods. The problem of delamination buckling is not new (see e.g. [3]—[6]); however this work differs from the other investigations in a number of ways, and, as will be evident in what follows, the approach we have used here leads to explicit quantitative description of the initial postbucking behavior and addresses issues of interest regarding the phenomenological aspects of the postbuckling deformation. As a related paper, in an attempt to include the effect of large deflections, the system was modelled in [7] by employing the large deflection (elastica) equations and solving numericaly the resulting nonlinear equations. This paper will use the large deflection equations aiming at a direct asymptotic solution.

In a delaminated system, which can be thought of as consisting of an aggregate of constitutive parts, the conditions of geometrical continuity play a particularly important part in the realization of equilibrium states which follow non-linear paths. There are exact laws governing the behavior of single members in equilibrium under arbitrary end-restraints. They constitute the exact theory of plane deformation of elements that are elastically restrained at the ends by means of concentrated forces and couples [8]. Generalized coordinates of deformation are the distortion parameter α, which represents the tangent rotation at an inflection point from the straight position and the amplitude variable Φ. We define by Φ_i the generalized amplitude referred to the end at the upper, lower or base part ($i = u, l, b$ respectively). The initial postbuckling deformations are relatively small so that the exact expressions may be expanded in Taylor series in terms of the distortion parameter. Exact dependence of the end moments, end rotations and the flexural contraction is through elliptic functions; however the asymptotic expressions given in this work are in terms of trigonometric functions (in the form of Taylor series). Based on this background, in this paper we discuss further theoretical considerations on the phenomenological aspects of the postbuckling deformation of delaminated beams. We also give an asymptotic solution to the problem and present comparison with tests.

2 Analysis

2.1 Phenomenological aspects of the postbuckling deformation based on flexural elastica assumptions

Consider a clamped plate of length L and unit width with a delamination of length $l = 2a$, as shown in Fig. 1. The delamination is at an *arbitrary* location through the thickness T. Over the delaminated region, the laminate consists of the part above the delamination, of thickness H, referred to as the "upper" part and and the part below the delamination, of thickness $T - H$, referred to as the "lower" part. The section near each end where the laminate is intact and of thickness T, is referred to as the "base" laminate.

At this point our basic postulate is that the system is *inextensional*, so the buckled configuration of each constitutive part is part of an inflectional elastica with end amplitude Φ_i and distortion parameter α_i [8]. At the critical state the end-amplitudes are Φ_i^0. The subscript $i = u, l, b$, refers to the upper part, lower part or base laminate respectively. The upper and lower parts are symmetrically distorted. In the following we shall also denote by D_i the bending stiffness, $D_i = Et_i^3/[12(1 - \nu_{13}\nu_{31})]$, t_i being the thickness of the corresponding part.

Suppose that in the slightly buckled configuration Φ_u has changed by a small amount ϕ_u, then

$$\Phi_u = \Phi_u^0 + \phi_u . \tag{1}$$

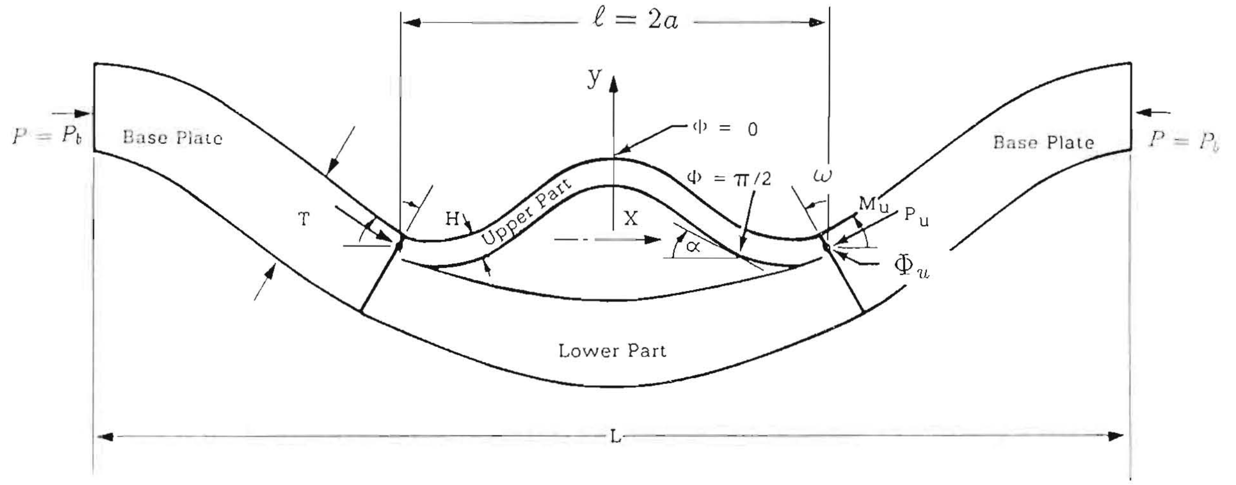

Fig. 1. Definition of the geometry

The change ϕ_u is assumed to be of order α_u, where α_u is the distortion parameter. However, an analogous assumption is not needed for the lower and base parts. Then the end rotation at the common section ω is given by expanding the relevant expression in Taylor series in terms of α_u (notice that at the critical state $\omega^0 = 0$):

$$\omega = (\sin \Phi_u)\,\alpha_u + O(\alpha_u^3) = (\sin \Phi_u^0)\,\alpha_u + (\cos \Phi_u^0)\,\alpha_u \cdot \phi_u + O(\alpha_u \cdot \phi_u^2)\,. \tag{2}$$

By the continuity condition, ω is the same for both the upper and lower part as well as the base plate. The asymptotic expansion of the end moment is similarly (again $\mu_u^0 = 0$)

$$\begin{aligned}\mu_u &= \frac{M_u l}{D_u} = (2\Phi_u \cos \Phi_u)\,\alpha_u + O(\alpha_u^3)\\ &= (2\Phi_u^0 \cos \Phi_u^0)\,\alpha_u + (2 \cos \Phi_u^0 - 2\Phi_u^0 \sin \Phi_u^0)\,\alpha_u \cdot \phi_u + O(\alpha_u \cdot \phi_u^2)\,.\end{aligned} \tag{3}$$

The axial force is given by

$$p_u = \frac{P_u l^2}{D_u} = 4\Phi_u^2 + O(\alpha_u^2) = 4\Phi_u^{02} + 8\Phi_u^0 \cdot \phi_u + O(\phi_u^2)\,. \tag{4}$$

Notice that

$$p_u^0 = \frac{P_u^0 l^2}{D_u} = 4\Phi_u^{02} \tag{5}$$

and the flexural contraction parameter is obtained

$$\begin{aligned}f_u &= \frac{e_u}{l} = \left(\frac{1}{4} - \frac{1}{8}\,\frac{\sin 2\Phi_u}{\Phi_u}\right)\alpha_u^2 + O(\alpha_u^4)\\ &= \left(\frac{1}{4} - \frac{1}{8}\,\frac{\sin 2\Phi_u^0}{\Phi_u^0}\right)\alpha_u^2 + \frac{1}{8}\left(\frac{\sin 2\Phi_u^0}{\Phi_u^{02}} - \frac{2 \cos 2\Phi_u^0}{\Phi_u^0}\right)\alpha_u^2 \cdot \phi_u + O(\alpha_u^4)\,.\end{aligned} \tag{6}$$

Analogous expressions hold for the lower part (subscript l). The expressions for the base plate are somewhat different because this part is unsymmetrically distorted and are outlined in the Appendix.

If at the critical state the axial load in the upper part attains the Euler value, then $\Phi_u^0 = \pi$, and from (2) it is concluded that $\omega = O(\alpha_u^2)$. However, the load at the lower part is below the Euler value (except if the delamination is exactly in the middle of the thickness), $\Phi_l^0 < \pi$ and so from (2), $\omega = O(\alpha_l)$. By a similar argument, $\omega = O(\alpha_b)$. Therefore, we conclude that

$$\alpha_l = O(\alpha_u^2)\,; \qquad \alpha_b = O(\alpha_u^2)\,. \tag{7}$$

Now suppose that at the critical state buckling would occur at $\Phi_u^0 < \pi$. The geometrical compatibility condition reads

$$f_u - f_l = \frac{T}{l}\sin \omega\,. \tag{8}$$

Then from (6) the left hand side of (8) is $O(\alpha_u^2)$ while from (2) the right-hand side is $O(\alpha_u)$. By this argument we conclude that, under the postulates of inextensional elastic behavior, at the critical state the load in the upper part should attain its Euler value. In the postbuckling stages, however, *both* the upper and lower parts as well as the base plate contract flexurally so that *continuity* at the common section is conserved. It follows that the distortion parameters of the lower part and the base plate $\alpha_{l,b}$ are of the order of α_u^2 and furthermore, the rotation at the common section is of the order of α_u^2 whereas both the end moment and change in axial force is of the order of α_u.

2.2 Asymptotic solution

In the following we shall develop an asymptotic solution for the early postbuckling behavior. In formulating the solution, we shall make use of the above arguments. We shall also assume the system of coordinate axes for the upper and lower part located at the middle of the delaminated segment; for the base part, outside the delaminated segment, of length $l_0 = L/2 - a = 2b$, the origin is located at the specimen end. Define the normalized coordinate quantities $\bar{x}$ and $\bar{y}$ by $\bar{x} = x/a$, $\bar{y} = y/a$ for the upper and lower parts, and by $\bar{x} = x/b$, $\bar{y} = y/b$ for the base part. The exact differential equation for the slope function $\theta(\bar{x})$ in each part is

$$\frac{d^2\theta_i(\bar{x})}{d\bar{x}^2} + \bar{\lambda}_i\pi^2 \sin\theta_i(\bar{x}) = 0; \qquad \frac{d\bar{y}_i(\bar{x})}{d\bar{x}} = \sin\theta_i(\bar{x}), \tag{9}$$

where $\bar{\lambda}_i = P_i a^2/(\pi^2 D_i)$ for the upper and lower parts, and $\bar{\lambda}_b = P_b b^2/(\pi^2 D_b)$ for the base part is the normalized load parameter and $\bar{y}_i(\bar{x})$ is the normalized deflection. Expand the load, $\bar{\lambda}_i$, end rotation, ω, and slope functions, θ_i, in terms of a small parameter ε:

$$\bar{\lambda}_i = \bar{\lambda}_{i0} + \varepsilon\bar{\lambda}_{i1} + \varepsilon^2\bar{\lambda}_i^2 + \cdots \tag{10.1}$$

$$\theta_i(x) = \varepsilon\theta_{i1}(x) + \varepsilon^2\theta_{i2}(x) + \cdots \tag{10.2}$$

$$\omega = \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{10.3}$$

Substituting in (9) and making use of $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \cdots$ leads to the following differential equations for the first and second order problem for the upper part

$$\frac{d^2\theta_{u1}(\bar{x})}{d\bar{x}^2} + \bar{\lambda}_{u0}\pi^2\theta_{u1}(\bar{x}) = 0, \tag{11.1}$$

$$\frac{d^2\theta_{u2}(\bar{x})}{d\bar{x}^2} + \bar{\lambda}_{u0}\pi^2\theta_{u2}(\bar{x}) = -\bar{\lambda}_{u1}\pi^2\theta_{u1}(\bar{x}). \tag{11.2}$$

The solution is given in terms of $k_{u0} = \pi\sqrt{\bar{\lambda}_{u0}} = \pi$ by

$$\theta_{u1}(\bar{x}) = c_{u1}\sin k_{u0}\bar{x}, \tag{12.1}$$

$$\theta_{u2}(\bar{x}) = c_{u2}\sin k_{u0}\bar{x} + d_{u,21}\bar{x}\cos k_{u0}\bar{x}; \qquad d_{u,21} = \frac{\pi^2\bar{\lambda}_{u1}c_{u1}}{2k_{u0}}. \tag{12.2}$$

By the arguments made before (see Eq. (7)), for the lower and base parts (assume the base part is clamped at one end),

$$\theta_{l1}(\bar{x}) = 0; \qquad \theta_{b1}(\bar{x}) = 0, \tag{13.1}$$

$$\theta_{l2}(\bar{x}) = c_{l2}\sin k_{l0}\bar{x}; \qquad \theta_{b2}(\bar{x}) = c_{b2}\sin k_{b0}\bar{x}, \tag{13.2}$$

where $k_{i0} = \pi\sqrt{\bar{\lambda}_{i0}}$, $i = l, b$. Notice that $D_l\bar{\lambda}_{l0} = D_u\bar{\lambda}_{u0}(T - H)/H$ or $k_{l0} = \pi\sqrt{\bar{\lambda}_{l0}} = [H/(T - H)]\,\pi$, and by force equilibrium [3] $D_b\bar{\lambda}_{b0}a^2/b^2 = D_u\bar{\lambda}_{u0} + D_l\bar{\lambda}_{l0}$ or $k_{b0} = \pi\sqrt{\bar{\lambda}_{b0}} = \pi Hb/(Ta)$.

Thus the slope at the common section is

$$\omega_1 = 0; \qquad \omega_2 = -d_{u,21} = c_{l2}\sin k_{l0} = -c_{b2}\sin 2k_{b0}. \tag{14}$$

For convenience, define $\varepsilon^2 = \omega$. Then, from (10.3), an additional condition holds,

$$\omega_2 = 1; \qquad \omega_3 = \omega_4 = \cdots = 0. \tag{15}$$

Next the moment equilibrium condition at the common section is considered. The end moments are given by $M_i = -(D_i/a)\,\theta(\bar{x})'|_{\bar{x}=1}$. The equilibrium condition reads for the order ε^1:

$$2D_uc_{u1}k_{u0}a/\pi^2 = D_l\bar{\lambda}_{l1}H - D_u\bar{\lambda}_{u1}(T - H). \tag{16}$$

The conditions (14), (15) with the definition (12.2) for $d_{u,21}$ give an expression between $\bar{\lambda}_{u1}$ and c_{u1}

$$\bar{\lambda}_{u1}c_{u1} = -\frac{2k_{u0}}{\pi^2}. \tag{17.1}$$

Furthermore,

$$d_{u,21} = -1; \qquad c_{l2} = 1/\sin k_{l0}; \qquad c_{b2} = -1/\sin 2k_{b0}. \tag{17.2}$$

Now the geometric compatibility condition is

$$\frac{1}{2}\int_{-1}^{1}\theta_u^2(\bar{x})\,d\bar{x} - \frac{1}{2}\int_{-1}^{1}\theta_l^2(\bar{x})\,d\bar{x} = (T/a)\sin\omega, \tag{18}$$

and by expanding $\sin\omega$, it reads for the order ε^2

$$c_{u1}^2a = 2T. \tag{19}$$

The solution $c_{u1} < 0$ in (19) is retained because of the negative sign of the deflection of the upper part. Thus we obtain from (19) and (17):

$$\bar{\lambda}_{u1}^2 = \frac{2a}{\pi^2T} \tag{20.1}$$

and from (16)

$$\bar{\lambda}_{l1} = \frac{D_u}{D_l}\,\frac{2(T + H)}{\pi Hc_{u1}}. \tag{20.2}$$

We proceed now to the solution to the second order problem. At this point we need the solution $\theta_{i3}(x)$ for the differential equation of order ε_4. For the upper part

$$\frac{d^2\theta_{u3}(\bar{x})}{d\bar{x}^2} + \bar{\lambda}_{u0}\pi^2\theta_{u3}(\bar{x}) = -\bar{\lambda}_{u1}\pi^2\theta_{u2}(\bar{x}) - \bar{\lambda}_{u2}\pi^2\theta_{u1}(\bar{x}) + \frac{1}{6}\bar{\lambda}_{u0}\pi^2\theta_{u1}^3(\bar{x}). \tag{21}$$

Notice that the last term is due to the nonlinear effects. The solution to the above equation is found to be

$$\theta_{u3}(\bar{x}) = c_{u3}\sin k_{u0}\bar{x} + d_{u,31}\bar{x}\cos k_{u0}\bar{x} + d_{u,32}\bar{x}^2\sin k_{u0}\bar{x} + e_{u,31}\sin^3 k_{u0}\bar{x}, \tag{22}$$

where

$$d_{u,32} = -\frac{\tilde{\lambda}_{u1}\pi^2 d_{u,21}}{4k_{u0}}; \qquad e_{u,31} = -\frac{c_{u1}^3}{48}. \tag{23}$$

Moreover, $d_{u,31}$ is given in terms of the unknowns $\tilde{\lambda}_{u2}$ and c_{u2}:

$$2k_{u0}d_{u,31} = \tilde{\lambda}_{u1}c_{u2}\pi^2 + \tilde{\lambda}_{u2}c_{u1}\pi^2 + e_{u,31}6k_{u0}^2 + 2d_{u,32} = f(\tilde{\lambda}_{u2}, c_{u2}). \tag{24}$$

For the lower and base parts ($i = l, b$), the corresponding solution is

$$\theta_{i3}(\tilde{x}) = c_{i3}\sin k_{i0}\tilde{x} + d_{i,31}\tilde{x}\cos k_{i0}\tilde{x}; \qquad d_{i,31} = \frac{\tilde{\lambda}_{i1}c_{i2}\pi^2}{2k_{i0}}. \tag{25}$$

The condition (15) for ω_3 gives from (22) and (25):

$$d_{u,31} = 0; \qquad c_{l3} = -d_{l,31}\cot k_{l0}; \qquad c_{b3} = -2d_{b,31}\cot 2k_{b0}. \tag{26}$$

Taking into account that $\theta_{l1}(x) = 0$ and $\omega_1 = \omega_3 = 0$, the geometric compatibility condition to the order ε^3 is in turn from (18)

$$\int_{-1}^{1} \theta_{u1}(\tilde{x})\,\theta_{u2}(\tilde{x})\,d\tilde{x} = 0, \tag{27}$$

from which we find

$$c_{u2} = \frac{d_{u,21}}{2k_{u0}}. \tag{28}$$

From (28), (26) and (24) we find

$$\tilde{\lambda}_{u2} = \frac{T}{4a}. \tag{29}$$

Now the second order moment equilibrium reads

$$D_u b(c_{u2}k_{u0} + d_{u,21}) - D_l b c_{l2} k_{l0}\cos k_{l0} + D_b a c_{b2} k_{b0}\cos 2k_{b0}$$
$$= \left[D_l\tilde{\lambda}_{l2}\frac{H}{2} - D_u\tilde{\lambda}_{u2}\frac{(T - H)}{2}\right]\frac{\pi^2 b}{a} \tag{30}$$

which gives

$$\tilde{\lambda}_{l2} = \frac{H^2T}{4a(T - H)^2} - \frac{3H^2a}{\pi^2T^3} - \frac{2k_{l0}a}{\pi^2H}\cot k_{l0} - \frac{2k_{b0}T^3a^2}{(T - H)^3\,bH\pi^2}\cot 2k_{b0}. \tag{31}$$

In each case the applied force is given by force equilibrium by $P_j = D_b\pi^2\tilde{\lambda}_{bj}/b^2 = D_u\pi^2\tilde{\lambda}_{uj}/a^2 + D_l\pi^2\tilde{\lambda}_{lj}/a^2$. We can find the solution to higher order problems in a similar fashion. The solution to the fourth order problem is outlined in Appendix II. The deflections are found from (9) by using $\sin\theta_i(\tilde{x}) = \theta_i(\tilde{x}) - \frac{\theta_i^3(\tilde{x})}{6} + \cdots$ and integrating the asymptotic expansions for $\theta_i(\tilde{x})$ from (10.3) to obtain

$$\tilde{y}_i(\tilde{x}) = \varepsilon\int_0^{\tilde{x}}\theta_{i1}(\tilde{x}')\,d\tilde{x}' + \varepsilon^2\int_0^{\tilde{x}}\theta_{i2}(\tilde{x}')\,d\tilde{x}' + \varepsilon^3\int_0^{\tilde{x}}\left[\theta_{i3}(\tilde{x}') - \frac{\theta_{i1}^3(x')}{6}\right]d\tilde{x}' + \cdots. \tag{32}$$

Moreover, the axial displacement is given by

$$\delta = b \int_0^2 \theta_b^2(\bar{x}) \, d\bar{x} + \frac{1}{2} a \int_{-1}^1 \theta_u^2(\bar{x}) \, d\bar{x} - (T - H) \sin \omega . \tag{33}$$

The solution obtained thus far allows obtaining an expression for δ to the order ε^4. For this purpose, the compatibility condition in Appendix II can be used to obtain the fourth order flexural contraction of the upper part from the known solution $\theta_{l2}(\bar{x})$. Moreover, taking into account (15), (18), (19), (27), we can write the normalized shortening $\bar{\delta} = \delta/L$ as follows:

$$\bar{\delta} = \varepsilon^2 \frac{H}{L} + \frac{\varepsilon^4}{2} \left[c_{l2}^2 \left(\frac{a}{L} - \frac{\sin 2k_{l0}a}{2k_{l0}L} \right) + c_{b2}^2 \left(\frac{l_0}{L} - \frac{\sin 2k_{b0}l_0}{2k_{b0}L} \right) \right] + \cdots . \tag{34}$$

3 Discussion of results

The analysis will be used to predict the initial postbuckling behavior in illustrative cases. Consider beam/plates of length/thickness $L/T = 16$, delamination length $l/L = 1/2$, and delaminated layer thicknesses of $H/T = 2/15, 3/15, 4/15$. Let us focus on the strain at the middle of the delaminated layer. Denoting by ϱ the radius of curvature, the strain at the outer surface of the delaminated layer is

$$\varepsilon_u = \frac{H}{2\varrho} = \frac{H}{2a} \left(\varepsilon \theta'_{u1} + \varepsilon^2 \theta'_{u2} + \cdots |_{\bar{x}=0} \right) = \frac{H}{2a} \left[(c_{u1} k_{u0}) \, \varepsilon + (c_{u2} k_{u0} + d_{u21}) \, \varepsilon^2 + \cdots \right] . \tag{35}$$

Figure 2 shows the strain at the middle of the upper delaminated layer as a function of the axial displacement for the cases of varying delamination thickness as described above.

A comparison with experiments is shown in Fig. 3. The specimens were made of 15 plies of unidirectional (0^0 angle ply) prepreg Kevlar 49 of the following specifications; commercial type SP-328, nominal thickness per ply 0.20 mm (0.008 inches), nominal stiffness $E = E_1 = 75.8$ GN/m^2 (11×10^6 psi), $E_2 = 5.5$ GN/m^2 (0.8×10^6 psi), $G_{12} = 2.1$ GN/m^2 (0.3×10^6

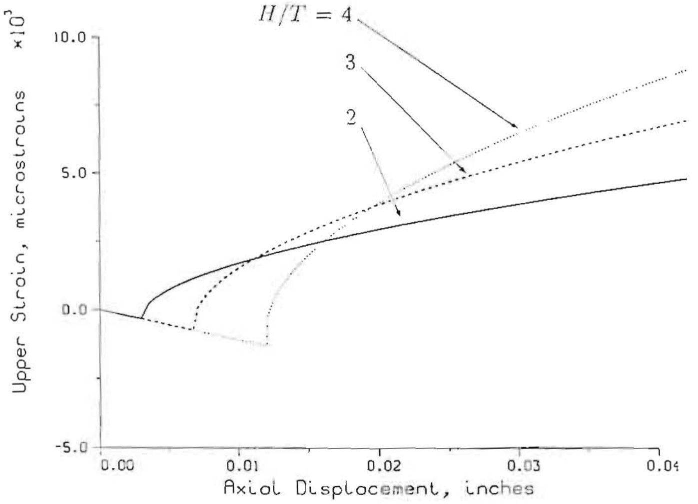

Fig. 2. Strain at the middle of the upper delaminated part vs. axial displacement for the cases of delamination thickness (H)/total thickness (T) of 2/15, 3/15, 4/15

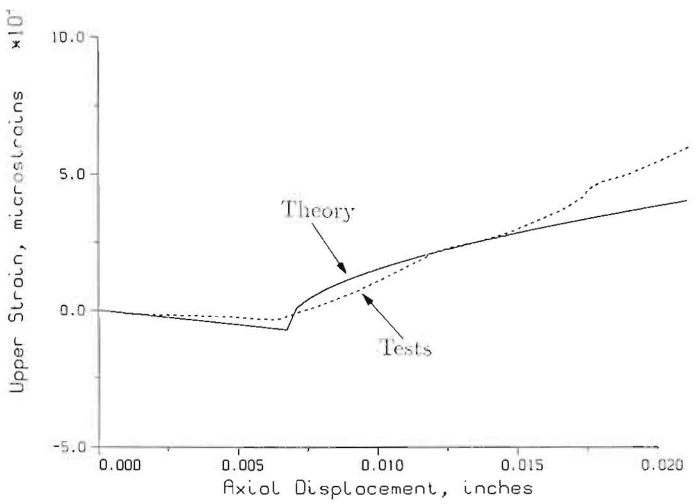

Fig. 3. Theoretical (solid line) vs. experimental (dashed line) strain at the middle of the delaminated part vs. axial displacement

psi), Poisson's ratio $\nu_{12} = 0.34$. A delamination of length $l = 50.8$ mm (2 inches) was introduced by a 0.025 mm (0.001 inch) thick Teflon strip placed in the middle of the length between the third and fourth ply (so that $H/T = 3/15$) and through the width. The length between the grips was $L = 101.6$ mm (4 inches). A width of 12.7 mm (0.5 inch) was used to keep the load level small and prevent any possible bending of the grips. Compression tests were conducted on a 20-kip MTS servohydraulic machine. They were carried out on stroke control with a rate of about 0.8 mm per minute. The specimen was clamped at the upper grip and a special fixture at the lower grip. The latter one was designed so that the specimen slides into it and therefore no bending is introduced by tightening the end. To be able to compare with the theory, the compliance of the testing machine is also needed. It was measured from a simple compression test (without a specimen) and was found to be 0.685×10^{-4} mm/N (0.12×10^{-4} in /lb). Strain was measured with strain gauges at the middle of the delaminated layer. Figure 3 shows the predicted (solid curve) and the experimentally measured (dashed curve) strain as a function of the axial displacement. The levels of strain for both curves are comparable.

Notice that the above analysis predicts that in the initial postbuckling stage the lower part deflects downwards while the upper part deflects upwards. The experiments showed indeed that both parts were deflecting in opposite directions for the values $H/T = 1/15$ to $4/15$ that were tested (this was verified by strain gaging the lower part and observing the sign of the strain).

4 Conclusions

A delaminated beam/plate was analyzed on the basis of an asymptotic expansion in the expressions of the exact theory of elastic deformation (of which the elastica theory is a special case). Under the postulates of inextensional elastic behavior, order of magnitude arguments lead to the conclusion that the distortion parameters of the lower part and the base plate $\alpha_{l,b}$ are of the order α_u^2 (where α_u is the distortion parameter of the upper delaminated part) and furthermore, the end rotation is of the order of α_u^2 whereas both the end moment and change in axial force are of the order of α_u. These considerations are used in an asymptotic solution that gives direct expressions for the load and displacement. Experimental results on the strain at the middle of the delaminated layer are compared with the predictions of the theory.

Appendix I

The distorted curve for the base part is unsymmetric with end amplitudes Φ_{bi}, Φ_{bj}. To illustrate the asymptotic expressions for the base plate, assume the simply supported (pinned end) case. The amplitude at the pinned end takes the value of $\Phi_{bi} = \pi/2$. Suppose that in a slightly buckled configuration the amplitude at the other end (the common section) Φ_b has changed by a small amount ϕ_b, then $\Phi_b = \Phi_b{}^0 + \phi_b$. Then the end rotation is

$$\omega = \left(\sin \Phi_{bj} \quad \frac{\cos \Phi_{bi} - \cos \Phi_{bj}}{\Phi_{bi} - \Phi_{bj}}\right) \alpha_b + O(\alpha_b{}^3) = \left[\sin \Phi_b{}^0 + \frac{\cos \Phi_b{}^0}{\Phi_b{}^0 - (\pi/2)}\right] \alpha_b$$
$$+ \left\{\cos \Phi_b{}^0 - \frac{\sin \Phi_b{}^0}{\Phi_b{}^0 - (\pi/2)} - \frac{\cos \Phi_b{}^0}{[\Phi_b{}^0 - (\pi/2)]^2}\right\} \alpha_b \phi_b + O(\alpha_b \phi_b{}^2). \tag{A1}$$

The expression for the end moment is

$$\mu_b = \frac{M_b l_0}{D_b} = (\Phi_{bi} - \Phi_{bj}) \cos \Phi_b \cdot \alpha_b + O(\alpha_b{}^3)$$
$$= \left(\Phi_b{}^0 - \frac{\pi}{2}\right) \cos \Phi_b{}^0 \cdot \alpha_b + \left[\cos \Phi_b{}^0 - \left(\Phi_b{}^0 - \frac{\pi}{2}\right)\right] \alpha_b \cdot \phi_b + O(\alpha_b \phi_b{}^2). \tag{A2}$$

The axial force for the base plate is the applied axial force which at the critical state is given by

$$p^0 = \frac{P^0 l_0{}^2}{D_b} = \left(\Phi_b{}^0 - \frac{\pi}{2}\right)^2,$$

and the change during the initial postbuckling is

$$\Delta p_1 = (\Phi_{bi} - \Phi_{bj})^2 - (\Phi_{bi}^0 - \Phi_{bj}^0)^2 + O(\alpha_b{}^2) = 2[\Phi_b{}^{20} - (\pi/2)]\, \phi_b + O(\phi_b{}^2). \tag{A3}$$

From (A1) we see that in general $\omega = O(\alpha_b) = O(\alpha_u{}^2)$, and from (A2) the end moment for the base part is of the order of α_u. Furthermore, $\phi_b = O(\phi_u)$ and the change in axial force is of the order of α_u.

Appendix II

The fourth order differential equation for the upper part is

$$\frac{d^2\theta_{u4}(\bar{x})}{d\bar{x}^2} + \bar{\lambda}_{u0}\pi^2\theta_{u4}(\bar{x}) = -\bar{\lambda}_{u1}\pi^2\theta_{u3}(\bar{x}) - \bar{\lambda}_{u2}\pi^2\theta_{u2}(\bar{x}) - \bar{\lambda}_{u3}\pi^2\theta_{u1}(\bar{x})$$
$$+ \bar{\lambda}_{u1}\pi^2 \frac{\theta_{u1}^3(\bar{x})}{6} + \bar{\lambda}_{u0}\pi^2 \frac{\theta_{u1}^2(\bar{x})\, \theta_{u2}(\bar{x})}{2}. \tag{B1}$$

The solution is found to be

$$\theta_{u4}(\bar{x}) = c_{u4}\sin k_{u0}\bar{x} + d_{u,41}\bar{x}\cos k_{u0}\bar{x} + d_{u,42}\bar{x}^2\sin k_{u0}\bar{x} + d_{u,43}\bar{x}^3\cos k_{u0}\bar{x}$$
$$+ e_{u,41}\sin^3 k_{u0}\bar{x} + e_{u,42}\bar{x}\sin^2 k_{u0}\bar{x}\cos k_{u0}\bar{x}, \tag{B 2}$$

where

$$d_{u,43} = \frac{\lambda_{u1}\pi^2 d_{u,32}}{6k_{u0}}; \qquad e_{u,42} = -\frac{c_{u1}^2 d_{u,21}}{2} \tag{B 3.1}$$

$$d_{u,42} = -\frac{1}{4k_{u0}}(\bar{\lambda}_{u2}\pi^2 d_{u,21} + \bar{\lambda}_{u1}\pi^2 d_{u,31} + 6d_{u,43} + 2k_{u0}^2 e_{u,42}), \tag{B 3.2}$$

$$e_{u,41} = -\frac{c_{u1}^2 c_{u2}}{16} - \frac{\bar{\lambda}_{u1}\pi^2}{8k_{u0}^2}\left(\frac{c_{u1}^3}{6} - e_{u,31}\right) - \frac{5e_{u,42}}{8k_{u0}} \tag{B 3.3}$$

and the following expression for $d_{u,41}$, which depends on the unknowns $\bar{\lambda}_{u3}$, c_{u3}:

$$2k_{u0}d_{u,41} = \bar{\lambda}_{u3}\pi^2 c_{u1} + \bar{\lambda}_{u2}\pi^2 c_{u2} + \bar{\lambda}_{u1}\pi^2 c_{u3} + 2d_{u,42} + 6k_{u0}^2 e_{u,41} + 4k_{u0}e_{u,42}. \tag{B 3.4}$$

The condition (15) gives the following condition for $\bar{\lambda}_{u3}$, c_{u3}:

$$6k_{u0}d_{u,41} + \bar{\lambda}_{u1}d_{u,32}\pi^2 = f(\bar{\lambda}_{u3}, c_{u3}) = 0. \tag{B4}$$

Similarly the solution for the lower and base parts, $i = l, b$, is

$$\theta_{i4}(\bar{x}) = c_{i4}\sin k_{u0}\bar{x} + d_{i,41}\bar{x}\cos k_{u0}\bar{x} + d_{i,42}\bar{x}^2\sin k_{u0}\bar{x}, \tag{B 5.1}$$

$$d_{i,42} = -\frac{\bar{\lambda}_{i1}\pi^2 d_{i,31}}{4k_{i0}}; \qquad 2k_{i0}d_{i,41} = \bar{\lambda}_{i1}\pi^2 c_{i3} + \bar{\lambda}_{i2}\pi^2 c_{i2} + 2d_{i,42}. \tag{B 5.2}$$

The geometric compatibility condition (18) for the terms of the order ε^4 is now:

$$\frac{1}{2}\int_{-1}^{1}\theta_{u2}^2(\bar{x})\,d\bar{x} + \int_{-1}^{1}\theta_{u1}(\bar{x})\,\theta_{u3}(\bar{x})\,d\bar{x} - \frac{1}{2}\int_{-1}^{1}\theta_{l2}^2(\bar{x})\,d\bar{x} = 0 \tag{B6}$$

from which we obtain c_{u3} while $\bar{\lambda}_{u3}$ is in turn obtained from (B 4). Now the moment equilibrium equation at the common section for the order ε^3 becomes

$$-D_u\theta'_{u3}(\bar{x})|_{\bar{x}=1} - D_l\theta'_{l3}(\bar{x})|_{\bar{x}=1} + D_b\frac{a}{b}\theta'_{b3}(\bar{x})|_{\bar{x}=2} = \left(D_l\bar{\lambda}_{l3}\frac{H}{2} - D_u\bar{\lambda}_{u3}\frac{T-H}{2}\right)\frac{\pi^2}{a} \tag{B7}$$

from which we obtain $\bar{\lambda}_{l3}$.

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References

[1] Dorey, G.: Impact damage in composites — development, consequences and prevention. In: Proc. ICCM-VI, ECCM-2, 3, pp. 1—26. Elsevier 1987.

[2] Johnson, W., Ghosh, S. K.: Some physical defects arising in composite material fabrication. J. Mater. Sci. **16**, 285—301 (1981).

[3] Kardomateas, G. A., Schmueser, D. W.: Buckling and postbuckling of delaminated composites under compressive loads including transverse shear effects. AIAA Journal **26** (3), 337–343 (1988).
[4] Yin, W. L., Sallam, S. N., Simitses, G. J.: Ultimate axial load capacity of a delaminated beam-plate. AIAA Journal **24**, 123–128 (1986).
[5] Chai, H., Babcock, C. D., Knauss, W. G.: One dimensional modelling of failure in laminated plates by delamination buckling. International Journal of Solids and Structures **17**, 1069–1083 (1981).
[6] Bottega, W. J., Maewal, A.: Delamination buckling and growth in laminates. J. Appl. Mech. **50**, 184–189 (1983).
[7] Kardomateas, G. A.: Large deformation effects in the postbuckling behavior of composites with thin delaminations. AIAA Journal **27** (5), 624–631 (1989).
[8] Britvek, S. J.: The stability of elastic systems. Pergamon Press Inc. 1973.

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