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END FORCE LOADING OF GENERALLY ANISOTROPIC CURVED BEAMS WITH LINEARLY VARYING ELASTIC CONSTANTS

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Abstract—The analytical solution for the stress field is derived for the case of a non-homogeneous curved beam in the form of a segment of a plane circular ring fixed at one end and acted upon by a load at the other end. It is assumed that the beam possesses general anisotropy and that the elastic constants are linear functions (with two non-zero coefficients) of the radial distance. The solution is found in a series form with a complex stress function. Numerical examples for the distribution of stresses in comparison with the homogeneous orthotropic case are presented and the effect of the gradient in the elastic constants is discussed.

INTRODUCTION

Although the problem of bending of anisotropic curved beams has been considered in the literature, for example in Lekhnitskii (1963), the solutions were given for pure bending with general anisotropy or for bending by an end force with orthotropy. Moreover, the material was assumed homogeneous, i.e. non-varying elastic constants. In engineering applications of composite parts in the form of curved beams there is often a gradient in the elastic constants through the thickness. The stress distribution in this case of variable elastic constants is more complicated and a solution for pure bending and assuming orthotropy, is known only when the constants change along the radius according to a power law, and can be found in Lekhnitskii (1968). Such a case does not however reflect the actual distribution of the moduli in practice; instead practical applications are closely represented by a linear variation (with two non-zero coefficients) of the elastic constants through the thickness. A solution for pure bending of cylindrically orthotropic curved beams with such linearly distributed elastic constants was found by Kardomateas (1990). In this paper we present the solution to the more general problem of bending of curved beams by an end force and under the assumption of generalized anisotropy with elastic constants being a linear function of the radial distance with two non-zero coefficients.

The curved beam is considered as a segment of a plane circular ring and it is assumed to be fixed at one end and deformed by forces distributed at the other end which produce a force applied at the center of the cross section (Fig. 1). It is also assumed that there are no body forces. The beam possesses cylindrical anisotropy and its pole is located at the

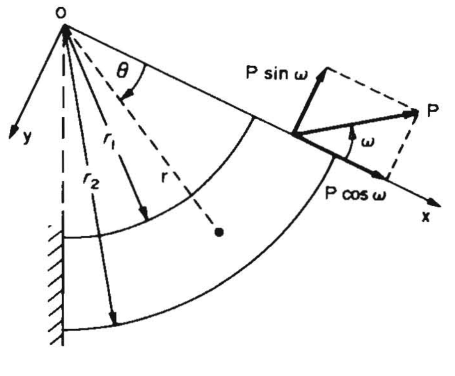

Fig. 1. Definition of the curved beam geometry.

center of the circles, the arcs of which form the beam contour. To describe the stress field we use a complex stress function formulation. After deriving the governing equations, the solution is produced by a series expansion. Results of the analysis for an illustrated example are discussed with emphasis on the effects of the anisotropy and the gradient in the elastic constants.

FORMULATION

The problem considered here is the stress distribution caused by an end force in a curved beam in the shape of a segment of a flat circular ring. We assume that the beam possesses cylindrical anistropy. One pole of anisotropy is located at the common center of the circular arcs which form the outside and inside edges of the beam. Aside from planes of elastic symmetry, which are parallel to the middle surface, there are no other elements of elastic symmetry. Figure 1 shows the cross section of the beam at the middle plane. The beam is fixed at one end and it is deformed by forces distributed at the other end which produce a force P applied at the center of the cross section. It is assumed that the anisotropy pole is the origin of the coordinates, the x-axis runs along the radius at the loaded end, and r_1 and r_2 are the inside and outside radii. We designate by ω the angle between the force and the x axis. The magnitude of the angle between the edge sections is arbitrary, but not larger than 2π. The basic assumption is that the elastic constants have a linear variation through the thickness. Therefore the generalized Hooke's law is written

$$\varepsilon_{rr} = a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta} + a_{16}\tau_{r\theta}, \tag{1a}$$

$$\varepsilon_{\theta\theta} = a_{12}\sigma_{rr} + a_{22}\sigma_{\theta\theta} + a_{26}\tau_{r\theta}, \tag{1b}$$

$$\gamma_{r\theta} = a_{16}\sigma_{rr} + a_{26}\sigma_{\theta\theta} + a_{66}\tau_{r\theta}, \tag{1c}$$

where

$$a_{ij} = a_{ij}(r).$$

Notice that in the special case of orthotropy there would also be radial and tangential planes of symmetry at each point in addition to the plane of elastic symmetry that is parallel to the middle plane and in that case $a_{16} = a_{26} = 0$.

The equilibrium equations are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} = 0, \tag{2a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0. \tag{2b}$$

The equilibrium equations are satisfied by introducing a stress function $F(r, \theta)$ as follows:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}. \tag{3a}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 F}{\partial r^2}; \quad \tau_{r\theta} = -\frac{\partial^2}{\partial r \partial \theta}\left(\frac{F}{r}\right). \tag{3b}$$

Now the compatibility equation needs to be satisfied and is written

$$\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + r\frac{\partial^2 (r\varepsilon_{\theta\theta})}{\partial r^2} - \frac{\partial^2 (r\gamma_{r\theta})}{\partial r \partial \theta} - r\frac{\partial \varepsilon_{rr}}{\partial r} = 0. \tag{4}$$

The elastic constants are not constant throughout but are assumed to be linear functions of r, in the form

$$a_{ij} = a_{ijc} + a_{ijg} r. \tag{5}$$

Using (1), (2), (3), the compatibility equation integrates to the following differential equation which must be satisfied by the stress function:

$$LF = 0, \tag{6a}$$

where L is a differential operator defined by:

$$\begin{aligned} L = {} & (a_{22c}+a_{22g}r)r^2\frac{\partial^4}{\partial r^4} + 2(a_{22c}+2a_{22g}r)r\frac{\partial^3}{\partial r^3} - 2(a_{26c}+a_{26g}r)r\frac{\partial^4}{\partial r^3\partial\theta} \\ & - [a_{11c}-(a_{12g}+2a_{22g}-a_{11g})r]\frac{\partial^2}{\partial r^2} + [(2a_{12c}+a_{66c})+(2a_{12g}+a_{66g})r]\frac{\partial^4}{\partial r^2\partial\theta^2} \\ & - 3a_{26g}r\frac{\partial^3}{\partial r^2\partial\theta} + a_{11c}\frac{1}{r}\frac{\partial}{\partial r} - 2(a_{16c}+a_{16g}r)\frac{1}{r}\frac{\partial^4}{\partial r\partial\theta^3} - (2a_{12c}+a_{66c})\frac{1}{r}\frac{\partial^3}{\partial r\partial\theta^2} \\ & - 2(a_{16c}+a_{26c}+a_{16g}r)\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} + (a_{11c}+a_{11g}r)\frac{1}{r^2}\frac{\partial^4}{\partial\theta^4} + (2a_{16c}+a_{16g}r)\frac{1}{r^2}\frac{\partial^3}{\partial\theta^3} \\ & + [(2a_{11c}+2a_{12c}+a_{66c})+a_{11g}r]\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + [2(a_{16c}+a_{26c})+a_{16g}r]\frac{1}{r^2}\frac{\partial}{\partial\theta}. \end{aligned} \tag{6b}$$

Next, we shall solve the above eqn (6). The solution is set in the form of a series expansion as follows

$$F(r,\theta) = \sum_{k=0}^{\infty} c_k r^{s+k}\,\mathrm{e}^{i\theta} + \bar{c}_k r^{\bar{s}+k}\,\mathrm{e}^{-i\theta} = 2\sum_{k=0}^{\infty}\mathrm{Re}\{c_k r^{s+k}\,\mathrm{e}^{i\theta}\}, \tag{7}$$

where c_k and s are complex numbers, $\bar{c}_k$ and $\bar{s}$ are their conjugates, respectively, and Re denotes the real part. Substituting into the differential equation we obtain:

$$LF = 2\mathrm{Re}\left\{\mathrm{e}^{i\theta}\sum_{k=0}^{\infty} c_k[A(s,k)+B(s,k)r]r^{s+k-2}\right\} = 0, \tag{8}$$

where $A(s,k)$ and $B(s,k)$ are complex functions of s, k and the elastic constants as follows:

$$A(s,k) = [a_{22c}(s+k)(s+k-2)-(a_{11c}+2a_{12c}+a_{66c})-i2a_{26c}(s+k-1)](s+k-1)^2, \tag{9a}$$

$$B(s,k) = [a_{22g}(s+k)(s+k-1)-(a_{11g}+a_{12g}+a_{66g})-ia_{26g}(2s+2k-1)](s+k)(s+k-1). \tag{9b}$$

To determine the constants we equate to zero the coefficient of each power of r in (8). The coefficient of the lowest power of r, which is r^{s-2}, gives the indicial equation $A(s,0)=0$:

$$\{[a_{22c}s(s-2)-(a_{11c}+2a_{12c}+a_{66c})]-i2a_{26c}(s-1)\}(s-1)^2 = 0. \tag{10a}$$

Thus there is one double root s_1 and two simple roots $s_{3,4}$. The complex values $s_{3,4}$ are solutions of the eqn

$$a_{22c}s^2 - 2(a_{22c}+ia_{26c})s - (a_{11c}+2a_{12c}+a_{66c}+i2a_{26c}) = 0. \tag{10b}$$

Therefore,

$$s_1 = 1, \quad s_{3,4} = \frac{a_{22c} \pm z_1 + i(a_{26c} \pm z_2)}{a_{22c}}, \tag{11a}$$

where z_1 and z_2 are defined in terms of

$$\delta_1 = a_{22c}(a_{11c} + 2a_{12c} + a_{22c} + a_{66c}) - a_{26c}^2, \quad \delta_2 = 4a_{22c}a_{26c}, \tag{11b}$$

as follows:

$$z_{1,2} = \sqrt{\frac{\sqrt{\delta_1^2 + \delta_2^2} \pm \delta_1}{2}}. \tag{11c}$$

From the coefficient of r^{s+k-2} we obtain

$$c_k A(s,k) + c_{k-1} B(s,k-1) = 0. \tag{12}$$

This recurrence relation determines successively the coefficients $c_1, c_2, \ldots$ in terms of c_0. Since we have two independent solutions corresponding to the double root s_1, and two corresponding to the single roots s_3, s_4, we shall denote the corresponding coefficients by $c_{i1}, c_{i2}, c_{i3}, c_{i4}$. For example, for the root s_3,

$$c_{31} = -c_{30} \times \frac{\{[a_{22g}(s_3-2)(s_3+1) - (a_{11g} + a_{12g} - 2a_{22g} + a_{66g})] - ia_{26g}(2s_3-1)\}(s_3-1)}{\{[a_{22c}(s_3^2-1) - (a_{11c} + 2a_{12c} + a_{66c})] - 2ia_{26c}s_3\}s_3}. \tag{13}$$

We can arbitrarily set $c_{i0} = 1$.

Although the solutions corresponding to the simple roots $s_{3,4}$ are established in a straightforward manner, the independent solutions corresponding to the double root $s_1 = 1$ are yet to be defined. For the root s_1, the corresponding series solution from (7) is

$$F_1(r,\theta) = 2\mathrm{Re} \sum_{k=0}^{\infty} c_{1k} r^{k+1} \mathrm{e}^{i\theta}. \tag{14}$$

Substituting into the differential equation (6) we get an equation of the form (8) where now $s = s_1 = 1$. However, from eqns (9) we see that

$$B(s_1, 0) = 0, \tag{15a}$$

and therefore from (12) we conclude that only the term corresponding to c_{10} remains, i.e.

$$c_{1k} = 0; \quad k \geqslant 1. \tag{15b}$$

Therefore, (8) becomes

$$LF_1 = 2\mathrm{Re}\,\{c_{10}[A(s,0) + B(s,0)r]r^{s-2}\,\mathrm{e}^{i\theta}\}. \tag{15c}$$

Differentiating both sides with respect to s we obtain:

$$L\left(\frac{\partial F_1}{\partial s}\right) = 2\mathrm{Re}\left\{\mathrm{e}^{i\theta} c_{10}\left[\frac{\partial A(s,0)}{\partial s} + \frac{\partial B(s,0)}{\partial s} r + A(s,0)\ln r + B(s,0) r \ln r\right] r^{s-2}\right\}. \tag{15d}$$

Since $s = 1$ is a double root of $A(s, 0)$, eqn (10a), it follows that

$$A(s_1, 0) = \left(\frac{\partial A(s,0)}{\partial s}\right)_{s=s_1} = 0. \tag{15e}$$

Taking into account (15a) we conclude that

$$L\left(\frac{\partial F_1}{\partial s}\right)_{s=s_1} = c_{10}2\mathrm{Re}\left\{e^{i\theta}\left(\frac{\partial B(s,0)}{\partial s}\right)_{s=s_1} r^{s_1-1}\right\}. \tag{15f}$$

Now we define the second independent solution corresponding to the double root s_1 by

$$F_2(r,\theta) = \left(\frac{\partial F_1}{\partial s}\right)_{s=s_1} + \sum_{k=0}^{\infty} [c_{2k}r^{s_1+k+1}\,e^{i\theta} + \bar{c}_{2k}r^{\bar{s}_1+k+1}\,e^{-i\theta}]. \tag{16}$$

From (8) and (15f) the coefficients of the series in (16) are given by

$$c_{10}\left(\frac{\partial B(s,0)}{\partial s}\right)_{s=s_1} + c_{20}A(s_1, 1) = 0, \tag{17a}$$

and in general,

$$c_{2(k-1)}B(s_1, k) + c_{2k}A(s_1, k+1) = 0, \quad k \geqslant 1. \tag{17b}$$

Thus from (14) and (16) the two solutions corresponding to the double root s_1 are

$$F_1(r,\theta) = 2\mathrm{Re}\,\{c_{10}r\,e^{i\theta}\}, \tag{18a}$$

$$F_2(r,\theta) = 2\mathrm{Re}\left\{e^{i\theta}\left[c_{10}\,r\ln r + \sum_{k=0}^{\infty} c_{2k}r^{k+2}\right]\right\}. \tag{18b}$$

Now let us denote the four series solutions, two corresponding to the double root s_1 and two to the simple roots $s_{3,4}$ by $F_{1,2,3,4}(r,\theta,a_{ij})$. Then the general solution of the homogeneous equation is any linear combination of these four independent solutions, namely

$$F(r,\theta) = \sum_{i=0}^{4} C_iF_i(r,\theta,a_{ij}). \tag{19}$$

Before proceeding to the determination of the constants C_i, let us discuss the convergence of the series (7). We shall use the Gauss test that requires taking the ratio of two consecutive terms

$$\left|\frac{c_kr^{s+k-2}}{c_{k+1}r^{s+k-1}}\right| = \frac{1}{r}\frac{(s+k)}{(s+k-1)}$$

$$\times\frac{\{a_{22c}[(s+k)^2-1]-(a_{11c}+2a_{12c}+a_{66c})-i2a_{26c}(s+k)\}}{\{a_{22g}(s+k-2)(s+k+1)-(a_{11g}+a_{12g}-2a_{22g}+a_{66g})-ia_{26g}(2s+2k-1)\}}$$

$$= \frac{1}{|r|}\left\{\frac{a_{22c}}{a_{22g}} + \frac{a_{22c}}{a_{22g}}\frac{1}{k} + \mathrm{O}\left(\frac{1}{k^2}\right)\right\}. \tag{20}$$

From the Gauss test we conclude that the series (7) is absolutely convergent if $|a_{22c}| > |a_{22g}r|$. This is the only limitation to the solution given here. The same is true for the series (16). In this case the solution is achieved by using a series in descending powers of r, i.e. in the form $\sum c_k r^{s-k} e^{i\theta}$ and the general procedure for formulating a solution would be the same.

Now set

$$s_2 = s_1 + 1. \tag{21}$$

Then from (3) the stresses are given by

$$\sigma_{rr}(r,\theta) = 2\mathrm{Re}\left\{e^{i\theta}C_2 r^{-1} + e^{i\theta}\sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}(s_i+k-1)r^{s_i+k-2}\right\}, \tag{22a}$$

$$\sigma_{\theta\theta}(r,\theta) = 2\mathrm{Re}\left\{e^{i\theta}C_2 r^{-1} + e^{i\theta}\sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}(s_i+k)(s_i+k-1)r^{s_i+k-2}\right\}, \tag{22b}$$

$$\tau_{r\theta}(r,\theta) = -2\mathrm{Re}\left\{ie^{i\theta}C_2 r^{-1} + ie^{i\theta}\sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}(s_i+k-1)r^{s_i+k-2}\right\}. \tag{22c}$$

The constants C_i, $i = 1, 4$, are found from the traction free boundary conditions of

$$\sigma_{rr}(r,\theta) = \tau_{r\theta}(r,\theta) = 0 \quad \text{at} \quad r = r_1, r_2, \tag{23}$$

and the condition that the stresses at the free end be reduced to a force P as follows:

$$\int_{r_1}^{r_2} \sigma_{\theta\theta}(r,\theta)|_{\theta=0}\,\mathrm{d}r = P\sin\omega, \quad \int_{r_1}^{r_2} \tau_{r\theta}(r,\theta)|_{\theta=0}\,\mathrm{d}r = -P\cos\omega, \tag{24a}$$

$$\int_{r_1}^{r_2} r\sigma_{\theta\theta}(r,\theta)|_{\theta=0}\,\mathrm{d}r = 0. \tag{24b}$$

The conditions (23) reduce to the following equations for the complex constants:

$$C_2 r_j^{-1} + \sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}(s_i+k-1)r_j^{s_i+k-2} = 0; \quad j = 1,2. \tag{25}$$

Let us set the complex roots of the indicial equation s_i, the coefficients $c_{i,k}$ and the unknown complex constants C_i in the form

$$C_i = C_{ir} + iC_{im}; \quad c_{ik} = c_{ikr} + ic_{ikm}; \quad s_i = s_{ir} + is_{im}. \tag{26}$$

Moreover, define

$$f_{ji}(k) = (s_{ir}+k)\cos(s_{im}\ln r_j) - s_{im}\sin(s_{im}\ln r_j), \tag{27a}$$

$$g_{ji}(k) = s_{im}\cos(s_{im}\ln r_j) + (s_{ir}+k)\sin(s_{im}\ln r_j). \tag{27b}$$

The conditions (23) reduce to the following four linear equations for the real and imaginary parts, C_{ir}, C_{im}, $i = 2, 4$:

$$C_{2r}r_j^{-1}+\sum_{i=2}^{4} C_{ir}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-2}[c_{ikr}f_{ji}(k-1)-c_{ikm}g_{ji}(k-1)]$$
$$-\sum_{i=2}^{4} C_{im}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-2}[c_{ikm}f_{ji}(k-1)+c_{ikr}g_{ji}(k-1)]=0;\quad j=1,2, \tag{28a}$$

and

$$C_{2m}r_j^{-1}+\sum_{i=2}^{4} C_{ir}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-2}[c_{ikm}f_{ji}(k-1)+c_{ikr}g_{ji}(k-1)]$$
$$+\sum_{i=2}^{4} C_{im}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-2}[c_{ikr}f_{ji}(k-1)-c_{ikm}g_{ji}(k-1)]=0;\quad j=1,2. \tag{28b}$$

Now the second eqn of (24b) becomes

$$C_2(r_2-r_1)+\sum_{i=2}^{4} C_i\sum_{k=0}^{\infty} c_{ik}(s_i+k-1)(r_2^{s_i+k}-r_1^{s_i+k})=0, \tag{29}$$

and it is automatically satisfied once the traction free conditions (23) are satisfied.

Finally, the first of the end conditions (24a) reduces to:

$$\sum_{j=1}^{2}(-1)^j\left\{C_{2r}\ln r_j+\sum_{i=2}^{4} C_{ir}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-1}[c_{ikr}f_{ji}(k)-c_{ikm}g_{ji}(k)]\right.$$
$$\left.-\sum_{i=2}^{4} C_{im}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-1}[c_{ikm}f_{ji}(k)+c_{ikr}g_{ji}(k)]\right\}=(P/2)\sin\omega, \tag{30a}$$

and the second condition in (24a) gives:

$$\sum_{j=1}^{2}(-1)^j\left\{C_{2m}\ln r_j+\sum_{i=2}^{4} C_{ir}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-1}[c_{ikr}\sin(s_{im}\ln r_j)\right.$$
$$+c_{ikm}\cos(s_{im}\ln r_j)]+\sum_{i=2}^{4} C_{im}\sum_{k=0}^{\infty} r_j^{s_{ir}+k-1}[c_{ikr}\cos(s_{im}\ln r_j)$$
$$\left.-c_{ikm}\sin(s_{im}\ln r_j)]\right\}=-(P/2)\cos\omega. \tag{30b}$$

Therefore we have the six eqns (28) and (30) to solve for the six unknowns, the real and imaginary parts of the constants, i.e. C_{ir}, C_{im}, $i=2,4$.

Concerning the displacement field, we use the strain–displacement relations:

$$\frac{\partial u_r}{\partial r}=\varepsilon_{rr}(r,\theta);\quad \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}=\varepsilon_{\theta\theta}(r,\theta);\quad \frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}=\gamma_{r\theta}(r,\theta). \tag{31}$$

Using (1) and (22) and integrating (31) we obtain the displacements. In terms of

$$h_{ic}(k)=a_{11c}+a_{12c}(s_i+k)-ia_{16c}, \tag{32a}$$

$$h_{ig}(k)=[a_{11g}+a_{12g}(s_i+k)-ia_{16g}]\frac{s_i+k-1}{s_i+k}, \tag{32b}$$

we obtain the radial displacement

$$u_r = 2\mathrm{Re}\left\{C_2\, e^{i\theta}[(a_{11c}+a_{12c}-ia_{16c})\ln r+(a_{11g}+a_{12g}-ia_{16g})r] + \sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}r^{s_i+k-1}\, e^{i\theta}[h_{ic}(k)+rh_{ig}(k)] - \frac{iC_2}{2}(a_{11c}+a_{22c}+2a_{12c}+a_{66c})\theta\, e^{i\theta}+d_1\, e^{i\theta}+d_2\, e^{-i\theta}\right\} \quad (33)$$

and in terms of

$$q_{ic}(k) = a_{11c}+a_{12c}-a_{22c}(s_i+k)(s_i+k-1)+ia_{26c}(s_i+k-1)-ia_{16c}, \quad (34a)$$

$$q_{ig}(k) = [a_{11g}-(s_i+k)^2a_{22g}+ia_{26g}(s_i+k)-ia_{16g}]\frac{s_i+k-1}{s_i+k}, \quad (34b)$$

we obtain the tangential displacement

$$u_\theta = 2\mathrm{Re}\left\{C_2 i\, e^{i\theta}[(a_{11c}+a_{12c}-ia_{16c})\ln r-(a_{12c}+a_{22c}-ia_{26c}) -(a_{22g}-a_{11g}+ia_{16c}-ia_{26g})r]+\sum_{i=2}^{4} C_i \sum_{k=0}^{\infty} c_{ik}r^{s_i+k-1} i\, e^{i\theta}[q_{ic}(k)+q_{ig}(k)r] + \frac{iC_2}{2}(a_{11c}+a_{22c}+2a_{12c}+a_{66c})(1-i\theta)\, e^{i\theta}+d_1 i\, e^{i\theta}-d_2 i\, e^{-i\theta}+d_3 r\right\}. \quad (35)$$

Assume the radial boundary of the beam at the built-in end is at $\theta = \phi$. Then take $u_r = u_\theta = 0$ at $\theta = \phi$ and at the middle point, $r = (r_1+r_2)/2$. Also at the built-in end $\partial u_\theta/\partial r = 0$. This gives the necessary conditions for the determination of the constants d_i, $i = 1, 3$ in the displacement field.

RESULTS AND DISCUSSION

Before proceeding to numerical results for an illustrative example case, let us consider the limit of an orthotropic homogeneous material, i.e. the limit of $a_{ijg} = 0$ and $a_{16c} = a_{26c} = 0$. For this limit the roots in (11a) become

$$s_1 = 1, \quad s_{3,4} = 1\pm\beta; \quad \beta = \sqrt{1+\frac{a_{11c}+2a_{12c}+a_{66c}}{a_{22c}}}. \quad (36)$$

The series solutions $F_{3,4}$ degenerate to only the first term for $k = 0$, since $B(s, k) \equiv 0$ in (9b) and (12). Moreover, the solution F_2 in (18b) does not include the sum but only the logarithmic term because $c_{2,k} = 0$ since $B(s, k) \equiv 0$ in (9b) and (17). Therefore, the present solution degenerates in the limit to the solution for the orthotropic homogeneous beam (Lekhnitskii, 1963). In this limit, the complex constants C_i, $i = 2, 4$ are given as follows:

$$2C_{3r}c_{30} = \frac{P\sin\omega}{(r_2-r_1)g_1\beta r_2^\beta}; \quad 2C_{3m}c_{30} = -\frac{P\cos\omega}{(r_2-r_1)g_1\beta r_2^\beta}, \quad (37a)$$

$$2C_{4r}c_{40} = -\frac{P\sin\omega}{(r_2-r_1)g_1\beta}r_1^\beta; \quad 2C_{4m}c_{40} = \frac{P\cos\omega}{(r_2-r_1)g_1\beta}r_1^\beta, \quad (37b)$$

$$2C_{2r} = -\frac{P\sin\omega}{(r_2-r_1)g_1r_2^\beta}(r_1^\beta+r_2^\beta); \quad 2C_{2m} = \frac{P\cos\omega}{(r_2-r_1)g_1r_2^\beta}(r_1^\beta+r_2^\beta). \quad (37c)$$

As an illustrative example consider now a curved beam made of graphite/epoxy with variable elastic constants through the thickness of inside radius $r_1 = 1$ m and $r_2/r_1 = 2$.

Table 1. Convergence of the series solution

	Values of the kth term (at $r = r_2$ and $\theta = 0$) of the stress functions, eqn (7), F_3 and F_4 (in Nt)				
	$s_3 = 2.837 + i(0.064)$			$s_4 = -0.837 - i(1.511)$	
	$k = 0$	$k = 10$	$k = 25$	$k = 50$	$k = 100$
F_3	0.143×10^{2}	0.315×10^{-1}	0.212×10^{-3}	0.213×10^{-6}	0.785×10^{-12}
F_4	0.559×10^{0}	0.369×10^{-2}	0.203×10^{-4}	0.186×10^{-7}	0.664×10^{-13}

Assume that the compliance constants corresponding to the orthotropic lay-up are as follows in $m^2\,Nt^{-1}$ [notice that 1 corresponds to the radial (r) direction and 2 corresponds to the tangential (θ) direction]:

$$a_{11c} = 0.110 \times 10^{-3}, \quad a_{12c} = -0.330 \times 10^{-4}, \quad a_{22c} = 0.680 \times 10^{-5},$$
$$a_{16c} = 0, \quad a_{26c} = 0, \quad a_{66c} = 0.234 \times 10^{-3}.$$

Let us assume that the elastic constants vary according to (5) and the gradient through the thickness is expressed by the parameter p as follows:

$$a_{ijg} = a_{ijc}p. \tag{38}$$

For an example case, assume $p = -0.40$. The constants a_{ij} for any other lay-up angle which is denoted by θ_{el}, and hence general anisotropy, can be found by the known transformation rules (Jones, 1975).

First the convergence of the series solution is illustrated in Table 1 which shows the kth term of the series expansion of the stress functions that correspond to the third, s_3, and fourth, s_4 roots of (10a), $F_3(r, \theta)$ and $F_4(r, \theta)$, evaluated at $r = r_2$ and $\theta = 0$. The series seems to be rapidly converging in a satisfactory manner.

In the following figures the radial distance is normalized as $\tilde{r} = (r - r_1)/(r_2 - r_1)$ and the stresses are normalized as $\tilde{\sigma}_{ij} = \sigma_{ij}(r_2 - r_1)/P$. The effect of a gradient in the elastic constants is illustrated in Figs 2, 3 and 4, which show the distribution through the thickness of the normalized stresses $\tilde{\sigma}_{rr}$, $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\tau}_{r\theta}$ at $\theta = 0$ and for the case of non-homogeneous orthotropy. The curves are compared with the stresses for a beam of constant moduli, i.e. a homogeneous and orthotropic beam (solid curve), given by Lekhnitskii (1963):

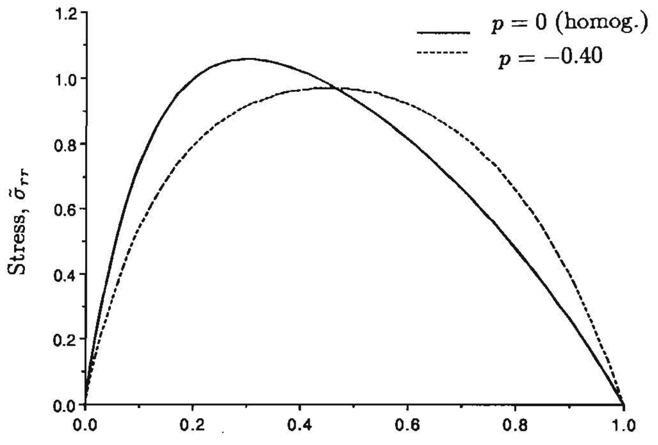

Fig. 2. Radial distribution of the stress σ_{rr} at the section $\theta = 0$ for the orthotropic case ($\theta_{el} = 0$). The broken line is for a gradient parameter $p = -0.40$. The solid line represents the case of a homogeneous orthotropic beam with non-varying elastic constants throughout ($p = 0$).

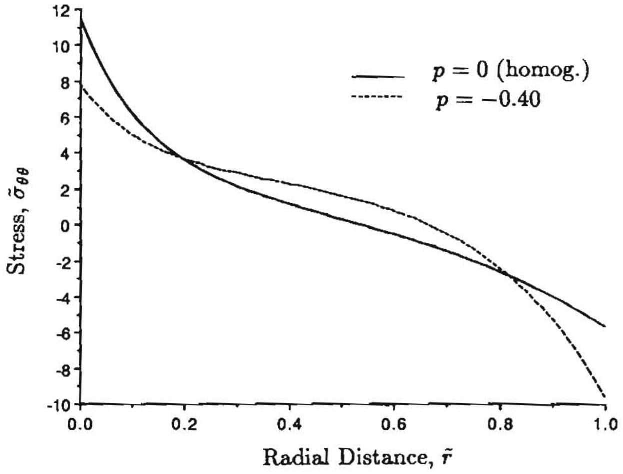

Fig. 3. Stress distribution $\sigma_{\theta\theta}$ through the thickness at the section $\theta = 0$ for the orthotropic case ($\theta_{el} = 0$). The broken line is for a gradient parameter $p = -0.40$. The solid line represents the case of a homogeneous orthotropic beam with non-varying elastic constants throughout ($p = 0$) as given in Lekhnitskii (1968).

$$\sigma_{rr} = \frac{P}{r_2(r_2-r_1)g_1}\frac{r_2}{r}\left[\left(\frac{r}{r_2}\right)^\beta + c^\beta\left(\frac{r_2}{r}\right)^\beta - 1 - c^\beta\right]\sin(\theta+\omega), \tag{39a}$$

$$\sigma_{\theta\theta} = \frac{P}{r_2(r_2-r_1)g_1}\frac{r_2}{r}\left[(1+\beta)\left(\frac{r}{r_2}\right)^\beta + (1-\beta)\left(\frac{r_2}{r}\right)^\beta c^\beta - 1 - c^\beta\right]\sin(\theta+\omega), \tag{39b}$$

$$\tau_{r\theta} = -\frac{P}{r_2(r_2-r_1)g_1}\frac{r_2}{r}\left[\left(\frac{r}{r_2}\right)^\beta + c^\beta\left(\frac{r_2}{r}\right)^\beta - 1 - c^\beta\right]\cos(\theta+\omega), \tag{39c}$$

where β is given by eqn (36) and

$$c = r_1/r_2; \quad g_1 = \frac{2}{\beta}(1-c^\beta) + (1+c^\beta)\ln c. \tag{39d}$$

The stress of highest magnitude is the $\sigma_{\theta\theta}$. Due to the gradient in the elastic constants

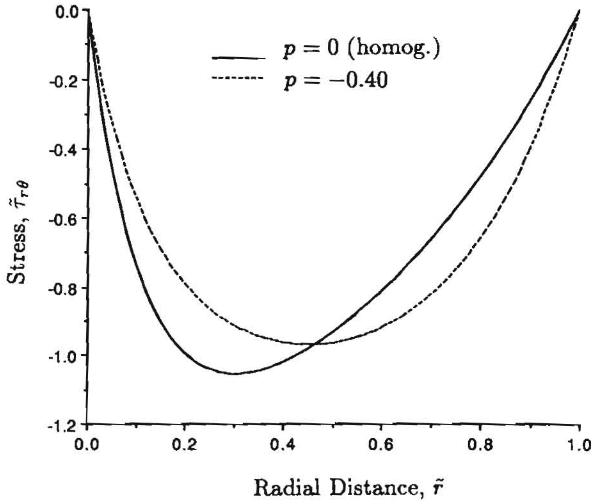

Fig. 4. Stress distribution $\tau_{r\theta}$ through the thickness at the section $\theta = 0$ for the orthotropic case ($\theta_{el} = 0$). The broken line is for a gradient parameter $p = -0.40$. The solid line represents the case of a homogeneous orthotropic beam with non-varying elastic constants throughout ($p = 0$).

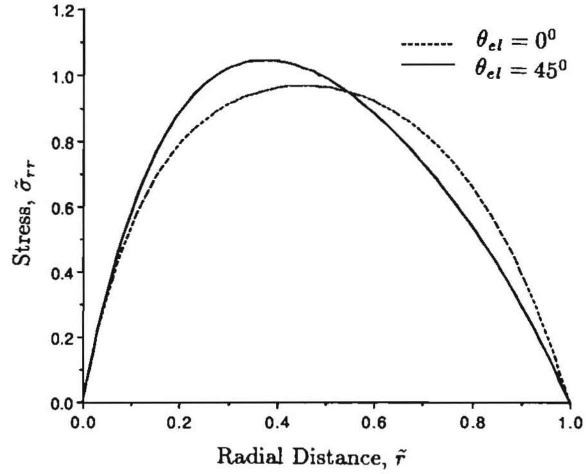

Fig. 5. Stress distribution σ_{rr} through the thickness at the section $\theta = 0$ for a gradient parameter $p = -0.40$, illustrating the effect of anisotropy. The solid line is for the case of a lay-up angle $\theta_{el} = 45°$. The broken line represents the case of a non-homogeneous orthotropic beam ($\theta_{el} = 0$).

the stress is by absolute value 70% higher at the outer surface ($\tilde{r} = 1$) and 30% lower at the inner surface ($\tilde{r} = 0$) relative to the homogeneous case. The important point is that the maximum stress is now at the outer edge as opposed to the homogeneous orthotropic case where the maximum stress occurs at the inner edge (and in this case it is twice that at the outer edge). Notice that the normal stress for an orthotropic homogeneous beam is always larger at the inside edge $\tilde{r} = 0$ and it is equal to (Lekhnitskii, 1968):

$$(\sigma_{\theta\theta})_{r=r_1} = -\frac{P \sin(\theta+\omega)}{r_2(r_2-r_1)g_1}\frac{\beta(1-c^{\beta})}{c}. \tag{40}$$

The other two components of stress σ_{rr} and $\tau_{r\theta}$ are of much smaller magnitude; relative to the homogeneous case the curves are shifted so that the stress is increased at points closer to the outside edge ($\tilde{r} = 1$) and reduced towards the inside ($\tilde{r} = 0$) edge.

The effect of anisotropy is illustrated in Figs 5, 6 and 7, which show the distribution through the thickness of the normalized stresses $\tilde{\sigma}_{rr}$, $\tilde{\sigma}_{\theta\theta}$ and $\tilde{\tau}_{r\theta}$ at $\theta = 0$ and for the case of

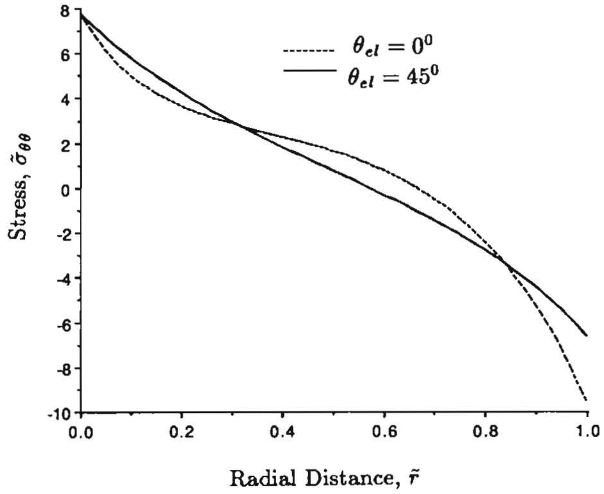

Fig. 6. Stress distribution $\sigma_{\theta\theta}$ through the thickness at the section $\theta = 0$ for a gradient parameter $p = -0.40$, illustrating the effect of anisotropy. The solid line is for the case of a lay-up angle $\theta_{el} = 45°$. The broken line represents the case of a non-homogeneous orthotropic beam ($\theta_{el} = 0$, with gradient parameter $p = -0.40$).

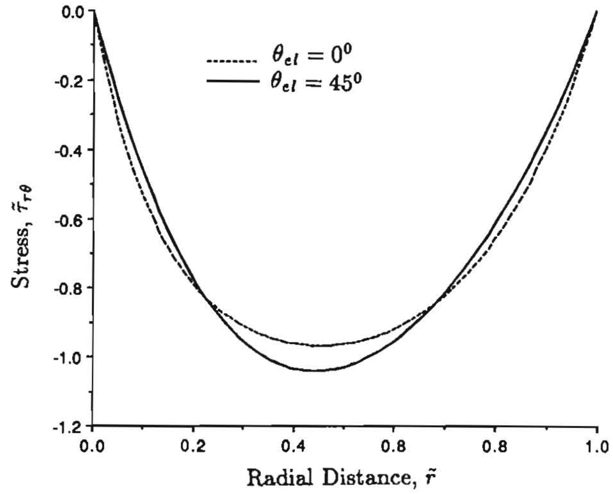

Fig. 7. Stress distribution $\tau_{r\theta}$ through the thickness at the section $\theta = 0$ for a gradient parameter $p = -0.40$, illustrating the effect of anisotropy. The solid line is for the case of a lay-up angle $\theta_{el} = 45°$. The broken line represents the case of a non-homogeneous orthotropic beam ($\theta_{el} = 0$).

a gradient parameter $p = -0.40$. The solid curve represents the case of a lay-up angle $\theta_{el} = 45°$, whereas the dashed curve is for the orthotropic non-homogeneous case. Again, because of the deviation from orthotropy both the location and the maximum stress $\sigma_{\theta\theta}$ occur at the inside edge for the case of $\theta_{el} = 45°$ as opposed to occurring at the outside edge in the non-homogeneous orthotropic case. The maximum stress in the 45° lay-up case is also 20% less than that in the 0° lay-up orthotropic case. Notice also that for an isotropic beam the stresses would be independent of the elastic constants and would depend only on the geometry, i.e. radii r_1, r_2 and on the applied loading P (e.g. Timoshenko and Goodier, 1970).

Finally Fig. 8 illustrates the combined effect of anisotropy and variation of elastic constants through the thickness by showing the ratio of the hoop stress $\sigma_{\theta\theta}$ at the outside and inside fibers as a function of the gradient parameter p for two values of θ_{el}. The ratio becomes greater than unity beyond a certain value of the gradient parameter p. This effect is greater in the orthotropic case than in the 45° lay-up case. The resulting curves are non-linear with higher slope at the larger absolute values of p which means that there is more

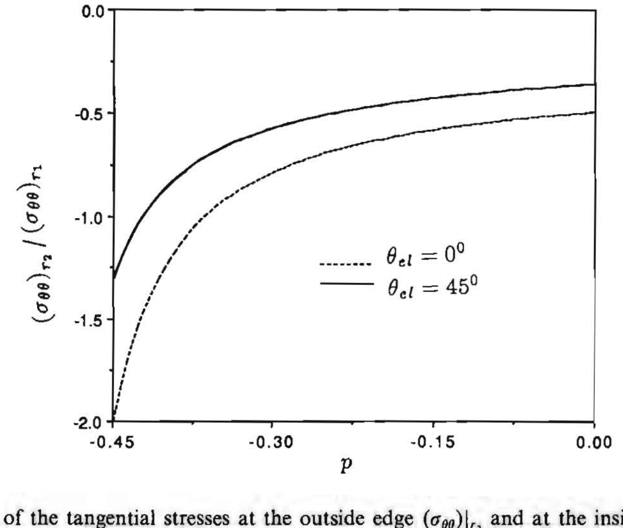

Fig. 8. Ratio of the tangential stresses at the outside edge $(\sigma_{\theta\theta})|_{r_2}$ and at the inside edge $(\sigma_{\theta\theta})|_{r_1}$ at the section $\theta = 0$ vs the gradient parameter p. The solid line is for the case of a lay-up angle $\theta_{el} = 45°$. The broken line represents the case of a non-homogeneous orthotropic beam ($\theta_{el} = 0$).

stress reduction or increase per unit change in the compliance constants at the large gradients of elastic constants through the thickness.

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