Buckling of Thick Orthotropic Cylindrical Shells Under External Pressure

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An elasticity solution to the problem of buckling of orthotropic cylindrical shells subjected to external pressure is presented. In this context, the structure is considered a three-dimensional body. The results show that the shell theory predictions can produce nonconservative results on the critical load of composite shells with moderately thick construction. The solution provides a means of accurately assessing the limitations of shell theories in predicting stability loss.

Introduction

A class of important structural applications of fiber-reinforced composite materials involves the configuration of laminated shells. Although thin plate construction has been the …st of the initial applications, much attention is now being …to configurations classified as moderately thick shell structures. Such designs can be used in components in the aircraft and automobile industries, as well as in the marine industry. Moreover, composite laminates have been considered in space vehicles in the form of circular cylindrical shells as a primary load carrying structure.

In these light-weight shell structures, loss of stability is of primary concern. This subject has been researched to-date through the application of the cylindrical shell theory (e.g., Simitses, Shaw, and Sheinman, 1985). However, previous work (Pagano and Whitney, 1970; Pagano, 1971) has shown that considerable care must be exercised in applying thin shell theory formulations to predict the response of composite cylinders. Besides the anisotropy, composite shells have one other important distinguishing feature, namely extensional-to-shear modulus ratio much larger than that of their metal counterparts.

In order to more accurately account for the aforementioned effects, various modifications in the classical theory of laminated shells have generally been performed (Whitney and Sun, 1974; Librescu, 1975; Reddy and Liu, 1985; see also Noor and Burton, 1990 for a review of shear deformation theories). These higher-order shell theories can be applied to buckling problems with the potential of improved predictions for the critical load (Anastasiadis, 1990).

However, there has not yet been any effort to produce a solution based on three-dimensional elasticity to the problem of buckling of composite shell structures, against which results from various shell theories could be compared.

Towards this objective, this work presents an elasticity solution to the problem of buckling of composite cylindrical orthotropic shells subjected to external pressure. Numerical results for an example case of a fiber-reinforced hollow cylinder under external pressure are derived and compared with shell theory predictions. These results can be used to assess the accuracy of the classical shell theory and the existing improved shell theories for moderately thick construction.

Formulation

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, w_0 the r, θ, and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{1}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r, \theta, z)$, $\alpha v_1(r, \theta, z)$, $\alpha w_1(r, \theta, z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r, \theta, z)$, $v_1(r, \theta, z)$, $w_1(r, \theta, z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

The nonlinear strain displacement equations are

$$\epsilon_{rr} = \frac{\partial u}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right], \tag{2a}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2\right], \tag{2b}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right], \tag{2c}$$

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS.
…sion on this paper should be addressed to the Technical Editor, Pro… …ewis T. Wheeler, Department of Mechanical Engineering, University of …ouston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.
Manuscript received by the ASME Applied Mechanics Division, Sept. 12, 1991; final revision, Jan. 27, 1992. Associate Technical Editor: F. Y. M. Wan.

$$\gamma_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}+\left[\frac{\partial u}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right)+\frac{\partial v}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right)+\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right], \quad (2d)$$

$$\gamma_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}+\left(\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right), \quad (2e)$$

$$\gamma_{\theta z}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}+\left[\frac{\partial u}{\partial z}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right)+\frac{\partial v}{\partial z}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right)+\frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial z}\right]. \quad (2f)$$

Substituting (1) into (2) we find the strain components in the perturbed configuration:

$$\epsilon_{rr}=\epsilon_{rr}^0+\alpha\epsilon_{rr}'+\alpha^2\epsilon_{rr}'' \quad \gamma_{r\theta}=\gamma_{r\theta}^0+\alpha\gamma_{r\theta}'+\alpha^2\gamma_{r\theta}'', \quad (3a)$$

$$\epsilon_{\theta\theta}=\epsilon_{\theta\theta}^0+\alpha\epsilon_{\theta\theta}'+\alpha^2\epsilon_{\theta\theta}'' \quad \gamma_{rz}=\gamma_{rz}^0+\alpha\gamma_{rz}'+\alpha^2\gamma_{rz}'', \quad (3b)$$

$$\epsilon_{zz}=\epsilon_{zz}^0+\alpha\epsilon_{zz}'+\alpha^2\epsilon_{zz}'' \quad \gamma_{\theta z}=\gamma_{\theta z}^0+\alpha\gamma_{\theta z}'+\alpha^2\gamma_{\theta z}'', \quad (3c)$$

where ϵ_{ij}^0 are the values of the strain components in the initial position of equilibrium, ϵ_{ij}' are the strain quantities corresponding to the linear terms, and ϵ_{ij}'' are the ones corresponding to the quadratic terms. These strain quantities are given explicitly in terms of the displacements u_0, v_0, w_0 and u_1, v_1, w_1 in Appendix B.

The stress-strain relations for the orthotropic body are

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{\theta z}\\ \tau_{rz}\\ \tau_{r\theta}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{22}&c_{23}&0&0&0\\ c_{13}&c_{23}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{55}&0\\ 0&0&0&0&0&c_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \gamma_{\theta z}\\ \gamma_{rz}\\ \gamma_{r\theta}\end{bmatrix}, \quad (4)$$

where c_{ij} are the stiffness constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$). Substituting (3) into (4) we get the stresses as

$$\sigma_{rr}=\sigma_{rr}^0+\alpha\sigma_{rr}'+\alpha^2\sigma_{rr}'' \quad \tau_{r\theta}=\tau_{r\theta}^0+\alpha\tau_{r\theta}'+\alpha^2\tau_{r\theta}'', \quad (5a)$$

$$\sigma_{\theta\theta}=\sigma_{\theta\theta}^0+\alpha\sigma_{\theta\theta}'+\alpha^2\epsilon_{\theta\theta}'' \quad \tau_{rz}=\tau_{rz}^0+\alpha\tau_{rz}'+\alpha^2\tau_{rz}'', \quad (5b)$$

$$\sigma_{zz}=\sigma_{zz}^0+\alpha\sigma_{zz}'+\alpha^2\sigma_{zz}'' \quad \tau_{\theta z}=\tau_{\theta z}^0+\alpha\tau_{\theta z}'+\alpha^2\tau_{\theta z}'', \quad (5c)$$

where σ_{ij}^0, σ_{ij}', σ_{ij}'', are expressed in terms of ϵ_{ij}^0, ϵ_{ij}', ϵ_{ij}'', respectively, in the same manner as Eqs. (4) for σ_{ij} in terms of ϵ_{ij}.

In the following we shall keep in (5) and (3) terms up to α, i.e., we neglect the terms which contain α^2.

Governing Equations. The equations of equilibrium are taken in terms of the second Piola-Kirchhoff stress tensor Σ the form

$$\text{div}(\Sigma\cdot\mathbf{F}^T)=0, \quad (6a)$$

where $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F}=\mathbf{I}+\text{grad}\, V, \quad (6b)$$

where V is the displacement vector and $\mathbf{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\mathbf{E}=\frac{1}{2}(\mathbf{F}^T\cdot\mathbf{F}-1). \quad (6c)$$

More specifically, in terms of the linear strains,

$$e_{rr}=\frac{\partial u}{\partial r}, \quad e_{\theta\theta}=\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}, \quad e_{zz}=\frac{\partial w}{\partial z}, \quad (7a)$$

$$e_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}, \quad e_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}, \quad e_{\theta z}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}, \quad (7b)$$

and the linear rotations,

$$2\omega_r=\frac{1}{r}\frac{\partial w}{\partial \theta}-\frac{\partial v}{\partial z}, \quad 2\omega_\theta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}, \quad 2\omega_z=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad (7c)$$

the deformation gradient $\mathbf{F}$ is

$$\mathbf{F}=\begin{bmatrix}1+e_{rr} & \frac{1}{2}e_{r\theta}-\omega_z & \frac{1}{2}e_{rz}+\omega_\theta\\ \frac{1}{2}e_{r\theta}+\omega_z & 1+e_{\theta\theta} & \frac{1}{2}e_{\theta z}-\omega_r\\ \frac{1}{2}e_{rz}-\omega_\theta & \frac{1}{2}e_{\theta z}+\omega_r & 1+e_{zz}\end{bmatrix} \quad (8)$$

and the equilibrium Eq. (6a) gives

$$\frac{\partial}{\partial r}\left[\sigma_{rr}(1+e_{rr})+\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\tau_{rz}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]$$
$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\tau_{r\theta}(1+e_{rr})+\sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\tau_{\theta z}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\tau_{rz}(1+e_{rr})+\tau_{\theta z}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\sigma_{zz}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]$$
$$+\frac{1}{r}\left[\sigma_{rr}(1+e_{rr})-\sigma_{\theta\theta}(1+e_{\theta\theta})+\tau_{rz}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)-\tau_{\theta z}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)-2\tau_{r\theta}\omega_z\right]=0, \quad (9a)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left[\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\sigma_{\theta\theta}(1+e_{\theta\theta})+\tau_{\theta z}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\tau_{rz}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\tau_{\theta z}(1+e_{\theta\theta})+\sigma_{zz}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]$$
$$+\frac{\partial}{\partial r}\left[\sigma_{rr}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\tau_{r\theta}(1+e_{\theta\theta})+\tau_{rz}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]$$
$$+\frac{1}{r}\left[\sigma_{rr}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\tau_{rz}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)+\tau_{\theta z}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)+\tau_{r\theta}(2+e_{rr}+e_{\theta\theta})\right]=0, \quad (9b)$$

$$\frac{\partial}{\partial z}\left[\tau_{rz}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\tau_{\theta z}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\sigma_{zz}(1+e_{zz})\right]$$
$$+\frac{\partial}{\partial r}\left[\sigma_{rr}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\tau_{r\theta}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\tau_{rz}(1+e_{zz})\right]$$
$$+\frac{1}{r}\frac{\partial}{\partial \theta}\left[\tau_{r\theta}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\sigma_{\theta\theta}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\tau_{\theta z}(1+e_{zz})\right]$$
$$+\frac{1}{r}\left[\sigma_{rr}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\tau_{r\theta}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\tau_{rz}(1+e_{zz})\right]=0. \quad (9c)$$

Introducing the linear strains and rotations in the form (3), e.g., $e_{rr} = e_{rr}^0 + \alpha e_{rr}'$, $\omega_z = \omega_z^0 + \alpha\omega_z'$, as well as the stresses from (5) and keeping up to α^1 terms, we obtain a set of equations for the perturbed state in terms of the e_{ij}^0, ω_j^0 and e_{ij}', ω_j'. Notice that in addition to the notations we adopted earlier, e_{ij}^0 and ω_j^0 are the values of e_{ij} and ω_j for $u = u_0$, $v = v_0$ and $w = w_0$ and e_{ij}' and ω_j' are the values for $u = u_1$, $v = v_1$ and $w = w_1$.

Since the displacements u_0, v_0, w_0, correspond to positions of equilibrium, there must exist also equations of the form (9) with the zero superscript, which are obtained by referring (6a) to the initial position of equilibrium.

Thus, after subtracting the equilibrium equations at the perturbed and initial positions, we arrive at a system of homogeneous differential equations which are linear in the derivatives u_1, v_1 and w_1 with respect to r, θ, z. This follows from the fact that σ'_{ij}, e'_{ij}, ω'_j appear linearly in the equation, and are themselves, in virtue of (7), linear functions of these derivatives. The system of equations, corresponding to (9), at the initial position of equilibrium, is, on the other hand, nonlinear in the derivatives of u_0, v_0, w_0. However, if we make the additional assumption to neglect the terms that have e^0_{ij} and ω^0_j as coefficients, i.e., terms $e^0_{ij}\sigma^0_{ij}$ and $\omega^0_j\sigma^0_{ij}$, we can use the linear classical equilibrium equations to solve for the initial position of equilibrium.

Moreover, if we make the assumption to neglect the terms that have e^0_{ij} and ω^0_j as coefficients, i.e., terms $e^0_{ij}\sigma'_{ij}$ and $\omega^0_j\sigma'_{ij}$ and furthermore, since a characteristic feature of stability problems is the shift from positions with small rotations to positions with rotations substantially exceeding the strains, if we neglect the terms $e'_{ij}\sigma^0_{ij}$ thus keeping only the $\omega'_j\sigma^0_{ij}$ terms, we obtain the following buckling equations:

$$\frac{\partial}{\partial r}(\sigma'_{rr}-\tau^0_{r\theta}\omega'_z+\tau^0_{rz}\omega'_\theta)+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{r\theta}-\sigma^0_{\theta\theta}\omega'_z+\tau^0_{\theta z}\omega'_\theta)+\frac{\partial}{\partial z}(\tau'_{rz}-\tau^0_{\theta z}\omega'_z+\sigma^0_{zz}\omega'_\theta)+\frac{1}{r}(\sigma'_{rr}-\sigma'_{\theta\theta}+\tau^0_{rz}\omega'_\theta+\tau^0_{\theta z}\omega'_r-2\tau^0_{r\theta}\omega'_z)=0, \quad (10a)$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\tau^0_{rz}\omega'_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\sigma'_{\theta\theta}+\tau^0_{r\theta}\omega'_z-\tau^0_{\theta z}\omega'_r)+\frac{\partial}{\partial z}(\tau'_{\theta z}+\tau^0_{rz}\omega'_z-\sigma^0_{zz}\omega'_r)+\frac{1}{r}(2\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\sigma^0_{\theta\theta}\omega'_z+\tau^0_{\theta z}\omega'_\theta-\tau^0_{rz}\omega'_r)=0, \quad (10b)$$

$$\frac{\partial}{\partial r}(\tau'_{rz}-\sigma^0_{rr}\omega'_\theta+\tau^0_{r\theta}\omega'_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{\theta z}-\tau^0_{r\theta}\omega'_\theta+\sigma^0_{\theta\theta}\omega'_r)+\frac{\partial}{\partial z}(\sigma'_{zz}-\tau^0_{rz}\omega'_\theta+\tau^0_{\theta z}\omega'_r)+\frac{1}{r}(\tau'_{rz}-\sigma^0_{rr}\omega'_\theta+\tau^0_{r\theta}\omega'_r)=0. \quad (10c)$$

Boundary Conditions. The boundary conditions associated with (6a) can be expressed as

$$(\mathbf{F}\cdot\Sigma^T)\cdot\hat{n}=t(V), \quad (11)$$

where t is the traction vector on the surface which has outward unit normal $\hat{n}=(l, m, n)$ before any deformation. The traction vector t depends on the displacement field $V=(u, v, w)$. Indeed, because of the hydrostatic pressure loading, the magnitude of the surface load remains invariant under deformation, but its direction changes (since hydrostatic pressure is always directed along the normal to the surface on which it acts).

This gives

$$\left[\sigma_{rr}(1+e_{rr})+\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\tau_{rz}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]l+\left[\tau_{r\theta}(1+e_{rr})+\sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\tau_{\theta z}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]m+\left[\tau_{rz}(1+e_{rr})+\tau_{\theta z}\left(\frac{1}{2}e_{r\theta}-\omega_z\right)+\sigma_{zz}\left(\frac{1}{2}e_{rz}+\omega_\theta\right)\right]n=t_r, \quad (12a)$$

$$\left[\sigma_{rr}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\tau_{r\theta}(1+e_{\theta\theta})+\tau_{rz}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]l+\left[\tau_{r\theta}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\sigma_{\theta\theta}(1+e_{\theta\theta})+\tau_{\theta z}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]m+\left[\tau_{rz}\left(\frac{1}{2}e_{r\theta}+\omega_z\right)+\tau_{\theta z}(1+e_{\theta\theta})+\sigma_{zz}\left(\frac{1}{2}e_{\theta z}-\omega_r\right)\right]n=t_\theta, \quad (12b)$$

$$\left[\sigma_{rr}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\tau_{r\theta}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\tau_{rz}(1+e_{zz})\right]l+\left[\tau_{r\theta}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\sigma_{\theta\theta}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\tau_{\theta z}(1+e_{zz})\right]m+\left[\tau_{rz}\left(\frac{1}{2}e_{rz}-\omega_\theta\right)+\tau_{\theta z}\left(\frac{1}{2}e_{\theta z}+\omega_r\right)+\sigma_{zz}(1+e_{zz})\right]n=t_z. \quad (12c)$$

If we write these equations for the initial and the perturbed equilibrium position and then subtract them and use the previous arguments on the relative magnitudes of the rotations ω'_j we obtain

$$(\sigma'_{rr}-\tau^0_{r\theta}\omega'_z+\tau^0_{rz}\omega'_\theta)l+(\tau'_{r\theta}-\sigma^0_{\theta\theta}\omega'_z+\tau^0_{\theta z}\omega'_\theta)m+(\tau'_{rz}-\tau^0_{\theta z}\omega'_z+\sigma^0_{zz}\omega'_\theta)n=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_r(V_0+\alpha V_1)-t_r(V_0)]\right\}, \quad (13a)$$

$$(\tau'_{r\theta}+\sigma^0_{rr}\omega'_z-\tau^0_{rz}\omega'_r)l+(\sigma'_{\theta\theta}+\tau^0_{r\theta}\omega'_z-\tau^0_{\theta z}\omega'_r)m+(\tau'_{\theta z}+\tau^0_{rz}\omega'_z-\sigma^0_{zz}\omega'_r)n=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_\theta(V_0+\alpha V_1)-t_\theta(V_0)]\right\}, \quad (13b)$$

$$(\tau'_{rz}+\tau^0_{r\theta}\omega'_r-\sigma^0_{rr}\omega'_\theta)l+(\tau'_{\theta z}+\sigma^0_{\theta\theta}\omega'_r-\tau^0_{r\theta}\omega'_\theta)m+(\sigma'_{zz}+\tau^0_{\theta z}\omega'_r-\tau^0_{rz}\omega'_\theta)n=\lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_z(V_0+\alpha V_1)-t_z(V_0)]\right\}. \quad (13c)$$

Let $\hat{n}^0$ and $\hat{n}^1$ denote the normal unit vectors to the bounding surface at the initial and perturbed positions of equilibrium, respectively. Before any deformation, this vector is $\hat{n}=(l, m, n)$. For external pressure p loading at the initial position

$$t_r(V_0)=-p\cos(\hat{n}^0,\hat{r});\quad t_\theta(V_0)=-p\cos(\hat{n}^0,\hat{\theta});\quad t_z(V_0)=-p\cos(\hat{n}^0,\hat{z}), \quad (14a)$$

and at the perturbed position

$$t_r(V_0+\alpha V_1)=-p\cos(\hat{n}^1,\hat{r});\quad t_\theta(V_0+\alpha V_1)=-p\cos(\hat{n}^1,\hat{\theta});\quad t_z(V_0+\alpha V_1)=-p\cos(\hat{n}^1,\hat{z}). \quad (14b)$$

But in terms of the deformation gradient

$$\mathbf{F}^{0,1}\cdot\hat{n}=(1+E_n^{0,1})\hat{n}^{0,1}, \quad (15)$$

where E^0_n, E^1_n is the relative elongation normal to the bounding surface at the initial and perturbed equilibrium positions, respectively. More explicitly,

$$\cos(\hat{n}^0,\hat{r})=\frac{1}{1+E^0_n}\left[(1+e^0_{rr})l+\left(\frac{1}{2}e^0_{r\theta}-\omega^0_z\right)m+\left(\frac{1}{2}e^0_{rz}+\omega^0_\theta\right)n\right], \quad (16a)$$

$$\cos(\hat{n}^0,\hat{\theta})=\frac{1}{1+E_n^0}\left[\left(\frac{1}{2}e_{r\theta}^0+\omega_z^0\right)l+(1+e_{\theta\theta}^0)m+\left(\frac{1}{2}e_{\theta z}^0-\omega_r^0\right)n\right]. \qquad (16b)$$

$$\cos(\hat{n}^0,\hat{z})=\frac{1}{1+E_n^0}\left[\left(\frac{1}{2}e_{rz}^0-\omega_\theta^0\right)l+\left(\frac{1}{2}e_{\theta z}^0+\omega_r^0\right)m+(1+e_{zz}^0)n\right]. \qquad (16c)$$

Similar expressions hold true for the perturbed state. For example,

$$\cos(\hat{n}^1,\hat{r})=\frac{1}{1+E_n^1}\Bigg\{(1+e_{rr}^0+\alpha e_{rr}')l+\left[\left(\frac{1}{2}e_{r\theta}^0-\omega_z^0\right)+\alpha\left(\frac{1}{2}e_{r\theta}'-\omega_z'\right)\right]m+\left[\left(\frac{1}{2}e_{rz}^0+\omega_\theta^0\right)+\alpha\left(\frac{1}{2}e_{rz}'+\omega_\theta'\right)\right]n\Bigg\}. \qquad (17)$$

The assumption of small strains allows neglecting E_n^0 and E_n^1 in comparison with unity. Substituting into the expressions (14) for the tractions in terms of the pressure and subtracting the initial and perturbed state and using the same arguments on the magnitude of rotations to neglect e_{ij}' in comparison with ω_k', we arrive at the following expressions:

$$t_r(V_0+\alpha V_1)-t_r(V_0)=p\alpha(\omega_z' m-\omega_\theta' n), \qquad (18a)$$

$$t_\theta(V_0+\alpha V_1)-t_\theta(V_0)=-p\alpha(\omega_z' l-\omega_r' n), \qquad (18b)$$

$$t_z(V_0+\alpha V_1)-t_z(V_0)=p\alpha(\omega_\theta' l-\omega_r' m). \qquad (18c)$$

And in lieu of (13) for the lateral surfaces, i.e., for $m = n = 0$ and $l = 1$,

$$\sigma_{rr}'-\tau_{r\theta}^0\omega_z'+\tau_{rz}^0\omega_\theta'=0, \qquad (19a)$$

$$\tau_{r\theta}'+\sigma_{rr}^0\omega_z'-\tau_{rz}^0\omega_r'=-p\omega_z', \qquad (19b)$$

$$\tau_{rz}'+\tau_{r\theta}^0\omega_r'-\sigma_{rr}^0\omega_\theta'=p\omega_\theta'. \qquad (19c)$$

Prebuckling State. The problem at hand is that of a hollow cylinder rigidly fixed at its ends and deformed by uniformly distributed external pressure p (Fig. 1). The axially symmetric distribution of external forces produces stresses identical to all cross-sections and dependent only on the radial coordinate r. In this manner the forces at the ends are distributed identically over both surfaces and reduce to equal and opposite resultant forces and moments. Let R_1 be the internal and R_2 the external radius and set $c = R_1/R_2$. Lekhnitskii (1963) gave the stress field as follows:

$$\sigma_{rr}^0=-\frac{p}{1-c^{2k}}\left(\frac{r}{R_2}\right)^{k-1}+\frac{pc^{k-1}}{1-c^{2k}}c^{k+1}\left(\frac{R_2}{r}\right)^{k+1}, \qquad (20a)$$

$$\sigma_{\theta\theta}^0=-\frac{p}{1-c^{2k}}k\left(\frac{r}{R_2}\right)^{k-1}-\frac{pc^{k-1}}{1-c^{2k}}kc^{k+1}\left(\frac{R_2}{r}\right)^{k+1}, \qquad (20b)$$

$$\sigma_{zz}^0=\frac{p}{(1-c^{2k})a_{33}}(a_{13}+a_{23}k)\left(\frac{r}{R_2}\right)^{k-1}-\frac{pc^{k-1}}{(1-c^{2k})a_{33}}(a_{13}-a_{23}k)c^{k+1}\left(\frac{R_2}{r}\right)^{k+1}, \qquad (20c)$$

$$\tau_{r\theta}^0=\tau_{rz}^0=\tau_{\theta z}^0=0. \qquad (20d)$$

Equations (4) for the orthotropic constitutive behavior, where c_{ij} are the stiffness constants as well as the inverse relationship where a_{ij} are the compliance constants have been used, i.e.,

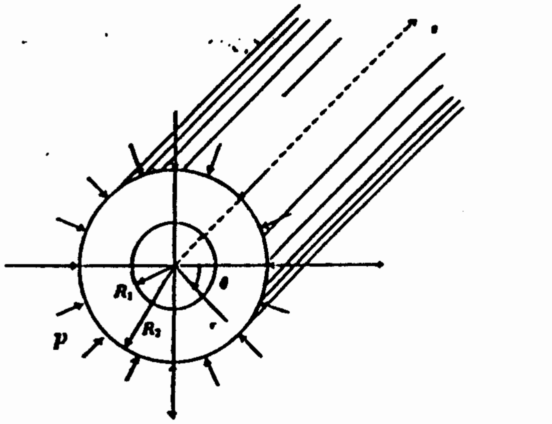

Fig. 1 Hollow cylinder under external pressure

$$\begin{bmatrix}\epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \gamma_{\theta z}\\ \gamma_{rz}\\ \gamma_{r\theta}\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}&0&0&0\\ a_{12}&a_{22}&a_{23}&0&0&0\\ a_{13}&a_{23}&a_{33}&0&0&0\\ 0&0&0&a_{44}&0&0\\ 0&0&0&0&a_{55}&0\\ 0&0&0&0&0&a_{66}\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{\theta z}\\ \tau_{rz}\\ \tau_{r\theta}\end{bmatrix}. \qquad (20e)$$

Since the rotations at the initial position of equilibrium are either zero or of the same order as the strains, the classical linear elasticity equilibrium and strain-displacement equations can ordinarily be applied to the initial position of equilibrium. Hence, integration of the above stress field through linear strain-displacement relations (Eqs. (7)) gives

$$u_0(r)=D_1pr^k+D_2pr^{-k}, \quad v_0=w_0=0, \qquad (21)$$

where

$$k=\sqrt{\frac{a_{11}a_{33}-a_{13}^2}{a_{22}a_{33}-a_{33}^2}}=\sqrt{\frac{c_{22}}{c_{11}}}, \qquad (22)$$

$$D_1=-\frac{1}{(1-c^{2k})kR_2^{k-1}}\left[a_{11}+a_{12}k-\frac{a_{13}}{a_{33}}(a_{13}+a_{23}k)\right]=-\frac{1}{(c_{11}k+c_{12})(1-c^{2k})R_2^{k-1}}, \qquad (23a)$$

$$D_2=-\frac{c^{k-1}R_1^{k+1}}{(1-c^{2k})k}\left[a_{11}-a_{12}k-\frac{a_{13}}{a_{33}}(a_{13}-a_{23}k)\right]=\frac{c^{k-1}R_1^{k+1}}{(-c_{11}k+c_{12})(1-c^{2k})}. \qquad (23b)$$

Perturbed State. In the perturbed position we seek plane equilibrium modes as follows:

$$u_1(r,\theta)=A_n(r)\cos n\theta;$$

$$v_1(r,\theta)=B_n(r)\sin n\theta; \quad w_1(r,\theta)=0. \qquad (24)$$

Substituting in (7) we obtain

$$\epsilon_{rr}^0=kp(C_1r^{k-1}-C_2r^{-k-1}); \quad \epsilon_{\theta\theta}^0=p(C_1r^{k-1}+C_2r^{-k-1}), \qquad (25a)$$

$$\epsilon_{zz}^0=\gamma_{\theta z}^0=\gamma_{rz}^0=\gamma_{r\theta}^0=0. \qquad (25b)$$

The first-order strains are given in Appendix B. However, let us examine the expression for ϵ_{rr}':

$$\epsilon_{rr}'=(1+e_{rr}^0)\frac{\partial u_1}{\partial r}+\left(\frac{1}{2}e_{r\theta}^0+\omega_z^0\right)\frac{\partial v_1}{\partial r}+\left(\frac{1}{2}e_{rz}^0-\omega_\theta^0\right)\frac{\partial w_1}{\partial r}.$$

Since all terms multiplied by e_{ij}^0 or ω_j^0 can be neglected based on the arguments made previously, $\epsilon_{rr}' = \partial u_1/\partial r = e_{rr}'$. It turns out that we can use for the first-order strains the very much simpler linear strains e'_{ij}, i.e., $\epsilon'_{ij} = e'_{ij}$. Therefore,

$$\epsilon'_{rr} = e'_{rr} = A'_n(r)\cos n\theta, \tag{26a}$$

$$\epsilon'_{\theta\theta} = e'_{\theta\theta} = \frac{A_n(r) + nB_n(r)}{r}\cos n\theta, \tag{26b}$$

$$\gamma'_{r\theta} = e'_{r\theta} = \left[B'_n(r) - \frac{B_n(r) + nA_n(r)}{r}\right]\sin n\theta, \tag{26c}$$

$$\epsilon'_{zz} = \gamma'_{\theta z} = \gamma'_{rz} = 0, \tag{26d}$$

and the first-order rotations are

$$2\omega'_z = \left[B'_n(r) + \frac{B_n(r) + nA_n(r)}{r}\right]\sin n\theta, \tag{26e}$$

$$\omega'_\theta = \omega'_r = 0. \tag{26f}$$

Denote $A_n^{(i)}(r)$, $B_n^{(i)}(r)$ the *i*th derivative of $A_n(r)$, $B_n(r)$, respectively, with the notation $A_n^{(0)}(r) = A_n(r)$ and $B_n^{(0)}(r) = B_n(r)$. Substituting in (10) and using (4) and (5), e.g., $\sigma'_{rr} = c_{11}\epsilon'_{rr} + c_{12}\epsilon'_{\theta\theta}$, we obtain the following two linear homogeneous ordinary differential equations of the second order for $A_n(r)$, $B_n(r)$:

$$\sum_{i=0}^{2} A_n^{(i)}(r)\,(d_{i0}r^{i-2} + d_{i1}pr^{k-3+i} + d_{i2}pr^{-k-3+i})$$

$$+\sum_{i=0}^{1} B_n^{(i)}(r)\,(q_{i0}r^{i-2} + q_{i1}pr^{k-3+i} + q_{i2}pr^{-k-3+i}) = 0,$$

$$R_1 \le r \le R_2 \tag{27a}$$

$$\sum_{i=0}^{2} B_n^{(i)}(r)\,(b_{i0}r^{i-2} + b_{i1}pr^{k-3+i} + b_{i2}pr^{-k-3+i})$$

$$+\sum_{i=0}^{1} A_n^{(i)}(r)\,(f_{i0}r^{i-2} + f_{i1}pr^{k-3+i} + f_{i2}pr^{-k-3+i}) = 0,$$

$$R_1 \le r \le R_2 \tag{27b}$$

The boundary conditions (19) are written as follows:

$$A'_n(R_j)c_{11} + [A_n(R_j) + nB_n(R_j)]\frac{c_{12}}{R_j} = 0, \quad j = 1,2 \tag{28a}$$

$$B'_n(R_j)\left[\left(c_{66} + \frac{p_j}{2}\right) + h_{01}pR_j^{k-1} + h_{02}pR_j^{-k-1}\right]$$

$$+[B_n(R_j) + nA_n(R_j)]\left[\left(-c_{66} + \frac{p_j}{2}\right)\frac{1}{R_j} + h_{01}pR_j^{k-2}\right.$$

$$\left. + h_{02}pR_j^{-k-2}\right] = 0, \quad j = 1,2 \tag{28b}$$

where $p_j = p$ for $j = 2$, i.e., $r = R_2$ (outside boundary), and $p_j = 0$ for $j = 1$, i.e., $r = R_1$ (inside boundary).

The constants d_{ij}, q_{ij}, b_{ij}, f_{ij}, h_{ij} in the above equations are given in Appendix A and depend on the material stiffness coefficients c_{ij} and the constants n and k.

Equations (27)–(28) constitute an eigenvalue problem for differential equations, with p the parameter, which can be solved by standard numerical methods (two-point boundary value problem). The relaxation method was used to obtain results which are discussed in the following. The minimum eigenvalue is obtained for $n = 2$. An equally spaced mesh of 241 points was used to derive the results. The procedure is highly efficient with rapid convergence. An investigation of convergence showed that essentially the same results were produced with even three times as many mesh points.

Results and Discussion

As an illustrative example, the critical pressure was determined for a composite circular cylinder of inner radius $R_1 = 1$ m. The moduli in GN/m^2 and Poisson's ratios used (typical for a glass/epoxy material) are listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$.

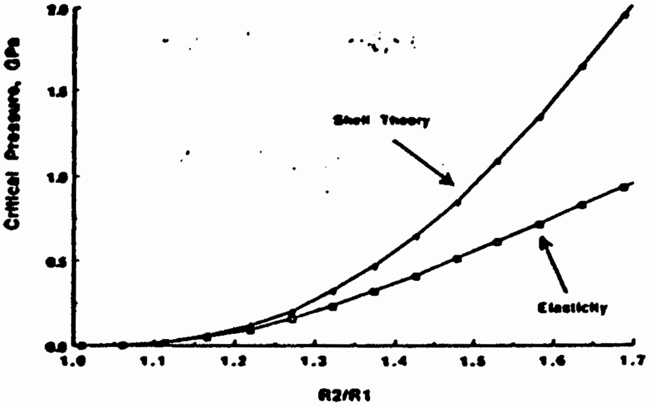

Fig. 2 Critical pressure, p_{cr} versus ratio of outside/inside radius, R_2/R_1. Comparison of the three-dimensional elasticity and the shell theory predictions.

Figure 2 shows the critical pressure as a function of the ratio of outside versus inside radius R_2/R_1. The elasticity solution is compared with the predictions of classical shell theory (e.g., Ambartsumyan, 1961). It is seen that the buckling load predicted by shell theory is 33 percent higher than the elasticity solution for $R_2/R_1 = 1.3$, it is 70 percent higher than the elasticity solution for $R_2/R_1 = 1.5$ and is more than two times the elasticity solution for $R_2/R_1 = 1.65$.

The direct expression for the critical pressure from classical shell theory is

$$p_{cr,sh} = \frac{E_2}{(1 - \nu_{23}\nu_{32})}(n^2 - 1)\frac{h^3}{12R^3} \tag{29a}$$

where $R = (R_1 + R_2)/2$ is the mid-surface radius, and $h = R_2 - R_1$ is the shell thickness.

The previous value can be found by using the Donnell nonlinear shell theory equations (Brush and Almroth, 1975) and seeking the buckled shapes in the form (24) where $A_n(r) = A_n$, i.e., it is now a constant instead of function of r, and $B_n(r) = B_n + (r - R)\beta$ with B_n being a constant, i.e., it admits a linear variation through the thickness. Since $\beta = (v_1 - u_{1,\theta})/R$, the latter can also be written in the form $B_n(r) = B_n + (r - R)(B_n + n)/R$.

As a consequence, we obtain the following shell theory buckling equations:

$$u_{1,\theta} + v_{1,\theta\theta} - \frac{h^2}{12R^2}(u_{1,\theta\theta\theta} - v_{1,\theta\theta}) = 0, \tag{29b}$$

$$\frac{h^2}{12R^2}(u_{1,\theta\theta\theta\theta} - v_{1,\theta\theta\theta}) + (u_1 + v_{1,\theta})$$

$$-\frac{pR(1 - \nu_{23}\nu_{32})}{E_2 h}(v_{1,\theta} - u_{1,\theta\theta}) = 0. \tag{29c}$$

Substituting the displacements from (24) and using the previous expressions for $A_n(r)$ and $B_n(r)$ results in the eigenvalue (29a) and the "eigenvectors" given by

$$A_n = 1; \quad B_n = -\left(1 + \frac{h^2}{12R^2}n^2\right)\Bigg/\left[n\left(1 + \frac{h^2}{12R^2}\right)\right]. \tag{29e}$$

Figures 3 and 4 show the variation of $A_n(r)$ and $B_n(r)$, which define the eigenfunctions, for $R_2/R_1 = 1.5$, as derived from the present elasticity solution, and in comparison with the shell theory assumptions of constant $A_n(r)$ and linear $B_n(r)$.

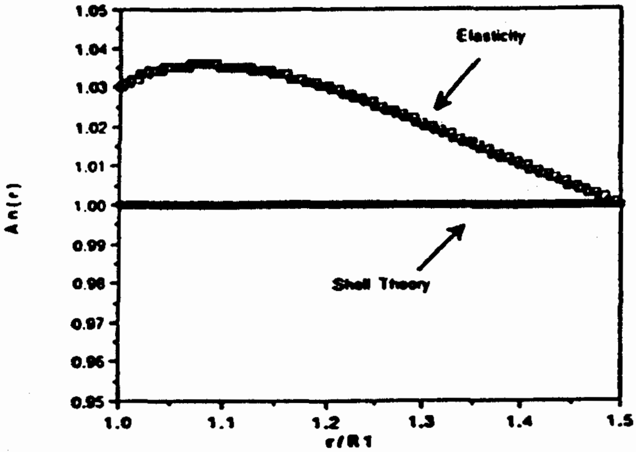

Fig. 3 "Eigenfunction" $A_n(r)$ versus normalized radial distance r/R_1. A unit value at the outside boundary has arbitrarily been set.

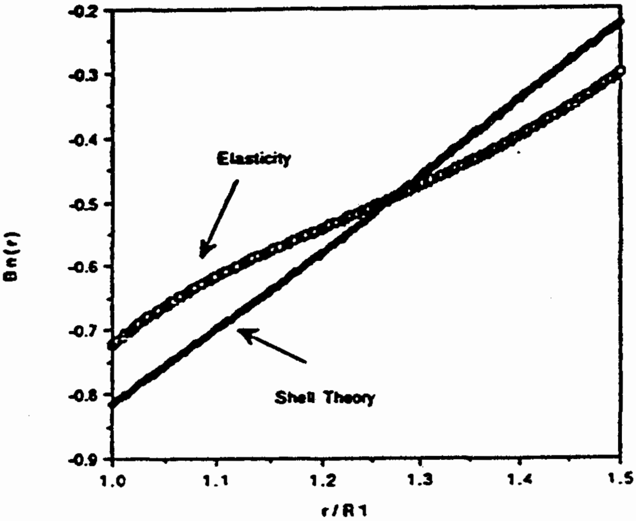

Fig. 4 "Eigenfunction" $B_n(r)$ versus normalized radial distance r/R_1.

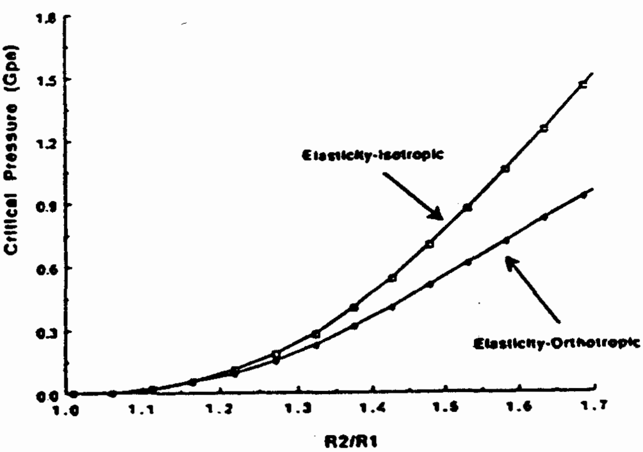

Fig. 5 Critical pressure, p_{cr} versus ratio of outside/inside radius, R_2/R_1, for the orthotropic case, and the isotropic one with $E = E_2$; i.e., the modulus along the periphery and Poisson's ratio $\nu = 0.3$

These values have been normalized by assigning a unit value for A_n at the outside boundary $r = R_2$.

Finally, Fig. 5 shows the effect of material constants by presenting a comparison of the critical load for the orthotropic case with the previously given moduli and Poisson's ratios, and the corresponding one by assuming isotropic material with modulus $E = E_2$, i.e., the modulus along the periphery, and Poisson's ratio $\nu = 0.3$. It is seen that the orthotropy results in significantly lower critical load with increased thickness. For example, at $R_2/R_1 = 1.5$, the isotropic material has 40 percent higher critical load than the orthotropic case. Naturally, the reduction in critical load can be qualitatively attributed to the reduced shear and radial stiffness of the orthotropic material.

Table 1 · Critical pressure, $p_{cr}R_2^3/(E_2h^3)$

Orthotropic, moduli in GN/m^2: $E_2 = 57$, $E_r = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$
Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{21} = 0.277$, $\nu_{31} = 0.400$

R_2/R_1	Elasticity	Shell[1]	Percentage Increase
1.10	0.2728	0.2930	7.4
1.15	0.2768	0.3119	12.7
1.20	0.2784	0.3308	18.8
1.25	0.2780	0.3495	25.7
1.30	0.2762	0.3681	33.3
1.35	0.2733	0.3864	41.4
1.40	0.2696	0.4046	50.1

[1]from Eq. (29a) with $n = 2$

The comparison of our elasticity solution was performed with the Donnell shell theory. It has been known (Danielson and Simmonds, 1969) that the Donnell shell theory can produce in some instances inaccurate results (such as for long tude behavior), as opposed to the more elaborate Flügge theory that provides more accurate predictions. However, for the problem under consideration, due to the assumed two-dimensional buckling modes (i.e., no z component of the displacement field, and no z-dependence of the r and θ displacement components), both the Flügge and Donnell equations would give the same critical load. Indeed, the buckling equations for the Flügge shell theory (see, e.g., Simmonds, 1966) would be: the Eq. (29b) without the term $h^2/12R^2\ (u_{1,\theta\theta\theta} - v_{1,\theta\theta})$, and the Eq. (29c) with the first term being $h^2/12R^2\ (u_{1,\theta\theta\theta\theta} + 2u_{1,\theta\theta} + u_1)$ instead of $h^2/12R^2\ (u_{1,\theta\theta\theta\theta} - v_{1,\theta\theta\theta})$. Substitution of the buckling modes (24) gives the same critical load, Eq. (29a), as the Donnell shell theory. Future work will consider the more complete problem of buckling of cylindrical shells of finite length under axial compression and external pressure, in the context of the present elasticity formulation; in this case the differences among the various shell theories are expected to surface.

It should also be noted that although the equilibrium approach was employed in the present formulation, a variational approach could also be applied. In this case, we can use the principle of virtual displacements by considering virtual displacements of the form $\alpha\delta u_1$, $\alpha\delta v_1$, and $\alpha\delta w_1$. The internal virtual work, which is essentially a volume integral of the product of stresses and strains in (3) and (5), can be written in the form $\delta W_i = \delta W_i^{(0)} + \alpha\delta W_i^{(1)} + \alpha^2\delta W_i^{(2)}$, and the same is true for the external virtual work δW_e due to the applied pressure. Finally, we would obtain the variational formulation in the form of $\delta W_i^{(2)} = \delta W_e^{(2)}$. Such an approach is expected to lead to similar results as the present direct equilibrium approach.

For a more specific comparison of the results for a range of radii ratios that would probably constitute practically moderately thick-to-thick shell construction, Table 1 shows the critical load derived by the present elasticity formulation and the shell theory predictions for orthotropic material and for ratios of outside over inside radius ranging from 1.10 to 1.40. A similar comparison is performed for the isotropic case in Table 2.

From the results presented previously, it can be concluded that predictions of stability loss in composite thick structures can be quite nonconservative if classical approaches are used. Specifically, the previous example showed that the critical load predicted by shell theory is higher than the three-dimensional

Table 2 Critical pressure, $p_{cr}R_2^2/(E_2h^2)$

Isotropic, $E = E_2 = 57$ GN/m^2, $\nu = 0.3$			
c	Elasticity	Shell[a]	Percentage Increase
1.10	0.2999	0.3159	5.3
1.15	0.3109	0.3363	8.2
1.20	0.3209	0.3567	11.2
1.25	0.3301	0.3769	14.2
1.30	0.3384	0.3969	17.3
1.35	0.3459	0.4167	20.5
1.40	0.3528	0.4363	23.7

[a]from Eq. (29a) with $n = 2$

elasticity predictions by more than a factor of two for a ratio of outside over inside radius greater than about 1.6. The present formulation and solution provide a means of accurately assessing the limitations of shell theories in predicting stability loss when the applications involve orthotropy and moderately thick construction. Further work is needed to assess the accuracy of improved higher-order shell theory predictions on the critical load in comparison to the elasticity ones. A comparison of these theories to the classical shell theory has already been performed by Anastasiadis (1990).

Acknowledgment

The financial support of the Office of Naval Research, Mechanics Division, Grant N00014-91-J-1892, and the interest and encouragement of the Project Monitor, Dr. Y. Rajapakse, are both gratefully acknowledged.

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APPENDIX A

Define the constants C_1 and C_2 (to simplify the σ_{ij}^0 expressions) as follows:

$$C_1 = -\frac{1}{(1-c^{2k})R_2^{k+1}};\quad C_2 = \frac{c^{2k}R_2^{k+1}}{(1-c^{2k})}, \tag{A1}$$

where $c = R_1/R_2$. Furthermore, k is defined in (22) and c_{ij} are the stiffness constants from (4).

The coefficients of the first differential Eq. (27a) are

$$d_{20} = c_{11};\quad d_{21} = d_{22} = 0,\quad d_{10} = c_{11};\quad d_{11} = d_{12} = 0, \tag{A2}$$

$$d_{00} = -(c_{22} + c_{66}n^2);\quad d_{01} = -\frac{kn^2}{2}C_1;\quad d_{02} = \frac{kn^2}{2}C_2. \tag{A3}$$

$$q_{00} = -n(c_{22} + c_{66});\quad q_{01} = -\frac{kn}{2}C_1;\quad q_{02} = \frac{kn}{2}C_2. \tag{A4}$$

$$q_{10} = n(c_{12} + c_{66});\quad q_{11} = -\frac{kn}{2}C_1;\quad q_{12} = \frac{kn}{2}C_2. \tag{A5}$$

The coefficients of the second differential Eq. (27b) are given as follows:

$$b_{20} = c_{66};\quad b_{21} = \frac{1}{2}C_1;\quad b_{22} = \frac{1}{2}C_2. \tag{B1}$$

$$b_{10} = c_{66};\quad b_{11} = \frac{1}{2}C_1;\quad b_{12} = \frac{1}{2}C_2. \tag{B2}$$

$$b_{00} = -(c_{22}n^2 + c_{66});\quad b_{01} = -\frac{1}{2}C_1,\quad b_{02} = -\frac{1}{2}C_2. \tag{B3}$$

$$f_{00} = -n(c_{22} + c_{66});\quad f_{01} = -\frac{n}{2}C_1,\quad f_{02} = -\frac{n}{2}C_2. \tag{B4}$$

$$f_{10} = -n(c_{12} + c_{66});\quad f_{11} = \frac{n}{2}C_1,\quad f_{12} = \frac{n}{2}C_2. \tag{B5}$$

Finally, the coefficients of the second boundary condition (28b) are

$$h_{01} = \frac{1}{2}C_1;\quad h_{02} = \frac{1}{2}C_2. \tag{C2}$$

APPENDIX B

The strain components in the initial position of equilibrium are

$$\epsilon_{rr}^0 = \frac{\partial u_0}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_0}{\partial r}\right)^2 + \left(\frac{\partial v_0}{\partial r}\right)^2 + \left(\frac{\partial w_0}{\partial r}\right)^2\right], \tag{D1a}$$

$$\epsilon_{\theta\theta}^0 = \frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta} - \frac{v_0}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}\right)^2\right], \tag{D1b}$$

$$\epsilon_{zz}^0 = \frac{\partial w_0}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_0}{\partial z}\right)^2 + \left(\frac{\partial v_0}{\partial z}\right)^2 + \left(\frac{\partial w_0}{\partial z}\right)^2\right], \tag{D1c}$$

$$\gamma_{r\theta}^0 = \frac{1}{r}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial r} - \frac{v_0}{r} + \left[\frac{\partial u_0}{\partial r}\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta} - \frac{v_0}{r}\right) + \frac{\partial v_0}{\partial r}\left(\frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r}\right) + \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w_0}{\partial \theta}\right], \tag{D1d}$$

$$\gamma_{rz}^0 = \frac{\partial u_0}{\partial z} + \frac{\partial w_0}{\partial r} + \left(\frac{\partial u_0}{\partial r}\frac{\partial u_0}{\partial z} + \frac{\partial v_0}{\partial r}\frac{\partial v_0}{\partial z} + \frac{\partial w_0}{\partial r}\frac{\partial w_0}{\partial z}\right), \tag{D1e}$$

$$\gamma_{\theta z}^0 = \frac{\partial v_0}{\partial z} + \frac{1}{r}\frac{\partial w_0}{\partial \theta} + \left[\frac{\partial u_0}{\partial z}\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta} - \frac{v_0}{r}\right) + \frac{\partial v_0}{\partial z}\left(\frac{1}{r}\frac{\partial v_0}{\partial \theta} + \frac{u_0}{r}\right) + \frac{1}{r}\frac{\partial w_0}{\partial \theta}\frac{\partial w_0}{\partial z}\right]. \tag{D1f}$$

The strain components associated with the linear terms (corresponding to the perturbed position of equilibrium) are

$$\epsilon'_{rr}=\frac{\partial u_1}{\partial r}+\frac{\partial u_0}{\partial r}\frac{\partial u_1}{\partial r}+\frac{\partial v_0}{\partial r}\frac{\partial v_1}{\partial r}+\frac{\partial w_0}{\partial r}\frac{\partial w_1}{\partial r}, \quad (D2a)$$

$$\epsilon'_{\theta\theta}=\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}+\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta}-\frac{v_0}{r}\right)\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)$$
$$+\left(\frac{1}{r}\frac{\partial v_0}{\partial \theta}+\frac{u_0}{r}\right)\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)+\frac{1}{r^2}\frac{\partial w_0}{\partial \theta}\frac{\partial w_1}{\partial \theta}, \quad (D2b)$$

$$\epsilon'_{zz}=\frac{\partial w_1}{\partial z}+\frac{\partial u_0}{\partial z}\frac{\partial u_1}{\partial z}+\frac{\partial v_0}{\partial z}\frac{\partial v_1}{\partial z}+\frac{\partial w_0}{\partial z}\frac{\partial w_1}{\partial z}, \quad (D2c)$$

$$\gamma'_{r\theta}=\frac{1}{r}\frac{\partial u_1}{\partial \theta}+\frac{\partial v_1}{\partial r}-\frac{v_1}{r}+\frac{\partial u_1}{\partial r}\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta}-\frac{v_0}{r}\right)$$
$$+\frac{\partial u_0}{\partial r}\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)+\frac{\partial v_1}{\partial r}\left(\frac{1}{r}\frac{\partial v_0}{\partial \theta}+\frac{u_0}{r}\right)$$
$$+\frac{\partial v_0}{\partial r}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)+\frac{1}{r}\frac{\partial w_1}{\partial r}\frac{\partial w_0}{\partial \theta}+\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial w_1}{\partial \theta}, \quad (D2d)$$

$$\gamma'_{rz}=\frac{\partial u_1}{\partial z}+\frac{\partial w_1}{\partial r}+\frac{\partial u_0}{\partial r}\frac{\partial u_1}{\partial z}+\frac{\partial u_1}{\partial r}\frac{\partial u_0}{\partial z}$$
$$+\frac{\partial v_0}{\partial r}\frac{\partial v_1}{\partial z}+\frac{\partial v_1}{\partial r}\frac{\partial v_0}{\partial z}+\frac{\partial w_0}{\partial r}\frac{\partial w_1}{\partial z}+\frac{\partial w_1}{\partial r}\frac{\partial w_0}{\partial z}, \quad (D2e)$$

$$\gamma'_{\theta z}=\frac{\partial v_1}{\partial z}+\frac{1}{r}\frac{\partial w_1}{\partial \theta}+\frac{\partial u_1}{\partial z}\left(\frac{1}{r}\frac{\partial u_0}{\partial \theta}-\frac{v_0}{r}\right)$$
$$+\frac{\partial u_0}{\partial z}\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)+\frac{\partial v_1}{\partial z}\left(\frac{1}{r}\frac{\partial v_0}{\partial \theta}+\frac{u_0}{r}\right)$$
$$+\frac{\partial v_0}{\partial z}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)+\frac{1}{r}\frac{\partial w_0}{\partial \theta}\frac{\partial w_1}{\partial z}+\frac{1}{r}\frac{\partial w_1}{\partial \theta}\frac{\partial w_0}{\partial z}. \quad (D2f)$$

The strain quantities corresponding to the quadratic terms (associated with the initial post-buckling behavior) are:

$$\epsilon''_{rr}=\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial r}\right)^2+\left(\frac{\partial v_1}{\partial r}\right)^2+\left(\frac{\partial w_1}{\partial r}\right)^2\right], \quad (D3a)$$

$$\epsilon''_{\theta\theta}=\frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)^2+\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)^2+\left(\frac{1}{r}\frac{\partial w_1}{\partial \theta}\right)^2\right], \quad (D3b)$$

$$\epsilon''_{zz}=\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial z}\right)^2+\left(\frac{\partial v_1}{\partial z}\right)^2+\left(\frac{\partial w_1}{\partial z}\right)^2\right], \quad (D3c)$$

$$\gamma''_{r\theta}=\frac{\partial u_1}{\partial r}\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)+\frac{\partial v_1}{\partial r}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)+\frac{1}{r}\frac{\partial w_1}{\partial r}\frac{\partial w_1}{\partial \theta}, \quad (D3d)$$

$$\gamma''_{rz}=\frac{\partial u_1}{\partial r}\frac{\partial u_1}{\partial z}+\frac{\partial v_1}{\partial r}\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial r}\frac{\partial w_1}{\partial z}, \quad (D3e)$$

$$\gamma''_{\theta z}=\frac{\partial u_1}{\partial z}\left(\frac{1}{r}\frac{\partial u_1}{\partial \theta}-\frac{v_1}{r}\right)+\frac{\partial v_1}{\partial z}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}+\frac{u_1}{r}\right)+\frac{1}{r}\frac{\partial w_1}{\partial \theta}\frac{\partial w_1}{\partial z}. \quad (D3f)$$