Stability Loss in Thick Transversely Isotropic Cylindrical Shells Under Axial Compression

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The stability of equilibrium of a transversely isotropic thick cylindrical shell under axial compression is investigated. The problem is treated by making appropriate use of the three-dimensional theory of elasticity. The results are compared with the critical loads furnished by classical shell theories. For the isotropic material cases considered, the elasticity approach predicts a lower critical load than the shell theories, the percentage reduction being larger with increasing thickness. However, both the Flügge and Danielson and Simmonds theories predict critical loads much closer to the elasticity value than the Donnell theory. Moreover, the values of n, m (number of circumferential waves and number of axial half-waves, respectively, at the critical point) for both the elasticity, and the Flügge and the Danielson and Simmonds theories, show perfect agreement, unlike the Donnell shell theory.

Introduction

Loss of stability is of primary concern in composite structural applications because of the large strength-to-weight ratio of these materials. Shell-like components of modest thickness are considered for possible applications in the marine industry, as well as the automobile and space industries. The accurate prediction of buckling under axial compression is an important consideration because of the uniaxial or, more generally, biaxial compressive fields that can be encountered in such applications.

Buckling of shells has been almost exclusively researched to date through the application of the cylindrical shell theory (e.g., Simitses, Shaw, and Sheinman, 1985). The classical solution, derived through the Donnell formulation, yields for an isotropic shell of radius R_0 and thickness h (Fig. 1), a critical stress directly proportional to h/R_0 and independent of the length, l, of the shell.

It has already been pointed out that this solution can yield incorrect results for very long cylinders, which can buckle as columns with undeformed cross-sections (Timoshenko and Gere, 1961). For very short cylinders the classical formula is also inadequate and a trial-and-error procedure through different combinations of the number of circumferential waves, n, and the number of axial half-waves, m, is needed (Brush and Almroth, 1970).

In this respect, Danielson and Simmonds (1969) obtained a set of accurate shell theory buckling equations for arbitrary cylindrical elastic shells of finite length. Their critical loads for shell of finite length were as simple as those predicted by the simplified Donnell equations, yet as accurate as those predicted by the more elaborate Flügge equations (Flügge, 1960). Simmonds and Danielson (1970) used their set of accurate buckling equations to compute the critical loads of axially compressed circular cylindrical shells subjected to "relaxed" boundary conditions. Their results showed that as $l/R_0 \to 0$ or as $l/R_0 \to \infty$, the buckling load approaches zero, in contrast to the behavior predicted by the Donnell equations. All the previously mentioned studies considered thin shells.

It seems natural that a comprehensive investigation of the performance of the simple classical solution and the various shell theories with respect to the shell thickness would be of interest in view of possible structural applications of modest thickness. An accurate solution for the stability characteristics of moderately thick shells is also needed in order to compare the accuracy of the predictions from various improved shell theories (e.g., Whitney and Sun, 1974; Librescu, 1975; Reddy and Liu, 1985; see also Noor and Burton, 1990, for a review of shear deformation theories).

Kardomateas (1993) presented a three-dimensional elasticity formulation and solution for the buckling of cylindrical orthotropic shells subjected to external pressure. It was shown that the critical load predicted by shell theory can be nonconservative (in particular, for the example case of glass/epoxy material considered, it was 34 percent higher than the elasticity solution for ratio of outside/inside radius $R_2/R_1 = 1.3$). This work was based on a simplified problem definition in that the prebuckling stress and displacement field was axisymmetric, and the buckling modes were assumed two-dimensional, i.e., no z-component of the displacement field, and no z-dependence of the r and θ displacement components.

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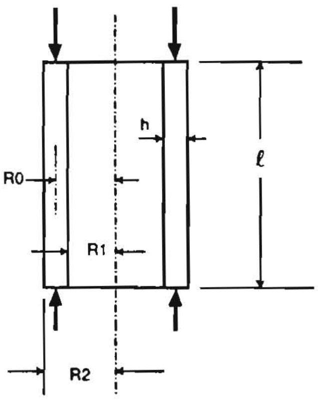

Fig. 1 Cylindrical shell under axial compression

To further assess the thickness effects on the stability of shells, the problem of buckling of a traversely isotropic thick cylindrical shell under axial compression is investigated. No restrictive assumptions are made concerning the thickness. Again, the nonlinear three-dimensional theory of elasticity is appropriately formulated, and closed-form analytical solutions are produced. The formulation is different from the previous work by Kardomateas (1993) because a finite length and a full dependence on r, θ, and z of the buckling modes is now assumed, which increases the complexity of the problem. Specific results will be presented for the critical load and the buckling modes. The results will be compared with the simplified Donnell formula, the critical load from the nonsimplified Donnell shell theory, the Flügge (1960) shell theory, and the Danielson and Simmonds (1969) formulation.

It should be noted that although composite shells may exhibit general anisotropy, we shall restrict ourselves to transverse isotropy, because more general anisotropy would not allow a direct solution of the corresponding three-dimensional elasticity problem. Actually, Elliott (1948) was the first to address the problem of obtaining closed-form solutions for transverse isotropy.

Formulation

The equations of equilibrium are taken in terms of the second Piola-Kirchhoff stress tensor Σ in the form

$$\operatorname{div}(\Sigma . \mathbf{F}^T) = 0, \tag{1a}$$

where $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \operatorname{grad} V, \tag{1b}$$

where V is the displacement vector and $\mathbf{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T . \mathbf{F} - \mathbf{I}) \tag{1c}$$

More specifically, in terms of the linear strains:

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \tag{2a}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \tag{2b}$$

and the linear rotations:

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \quad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{2c}$$

the deformation gradient $\mathbf{F}$ is

$$\mathbf{F} = \begin{bmatrix} 1 + e_{rr} & \frac{1}{2} e_{r\theta} - \omega_z & \frac{1}{2} e_{rz} + \omega_\theta \\ \frac{1}{2} e_{r\theta} + \omega_z & 1 + e_{\theta\theta} & \frac{1}{2} e_{\theta z} - \omega_r \\ \frac{1}{2} e_{rz} - \omega_\theta & \frac{1}{2} e_{\theta z} + \omega_r & 1 + e_{zz} \end{bmatrix}. \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, w_0 the r, θ, and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r, \theta, z)$, $\alpha v_1(r, \theta, z)$, $\alpha w_1(r, \theta, z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r, \theta, z)$, $v_1(r, \theta, z)$, $w_1(r, \theta, z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

Following Kardomateas (1993), we obtain the following buckling equations:

$$\frac{\partial}{\partial r}(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta) + \frac{\partial}{\partial z}(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta) + \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z) = 0, \tag{5a}$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r) + \frac{\partial}{\partial z}(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r) + \frac{1}{r}(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r) = 0, \tag{5b}$$

$$\frac{\partial}{\partial r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r) + \frac{\partial}{\partial z}(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r) + \frac{1}{r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) = 0. \tag{5c}$$

In the previous equations, σ^0_{ij} and ω^0_j are the values of σ_{ij} and ω_j at the initial equilibrium position, i.e., for $u = u_0$, $v = v_0$ and $w = w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position, i.e., for $u = u_1$, $v = v_1$ and $w = w_1$.

The boundary conditions associated with (1a) can be expressed as:

$$(\mathbf{F}.\Sigma^T).\hat{n} = t(V), \tag{6}$$

where t is the traction vector on the surface which has outward unit normal $\hat{n} = (l, m, n)$ before any deformation. The traction vector t depends on the displacement field $V = (u, v, w)$. Again, following Kardomateas (1992), we obtain for the lateral and end surfaces:

$$(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta)l + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)m + (\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)n = 0, \tag{7a}$$

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r)l + (\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)m + (\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)n = 0, \tag{7b}$$

$$(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta)l + (\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta)m + (\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta)n = 0. \tag{7c}$$

Prebuckling State. The problem under consideration is that of a transversely isotropic cylindrical shell compressed by an axial force applied at the ends. The stress-strain relations for the transversely isotropic material are as follows:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}, \quad (8)$$

where c_{ij} are the elastic constants (we have used the notation $1 \equiv r, 2 \equiv \theta, 3 \equiv z$).

Denote the length of the shell by l and the area of the transverse section by A. If we assume that the stresses along the loaded upper end ($z = l$) and the reaction along the lower end ($z = 0$) of the shell are distributed uniformly, and are normal to the bounding planes, then the components of stress tensor that satisfy the equations of equilibrium and the traction conditions on the surfaces are simply

$$\sigma_{zz} = -\frac{P}{A} = -\sigma_0; \quad \sigma^0_{rr} = \sigma^0_{\theta\theta} = \tau^0_{r\theta} = \tau^0_{rz} = \tau^0_{\theta z} = 0. \quad (9a)$$

For a transversely isotropic body, the corresponding displacement field can be found by using (8) and (2):

$$u_0 = \frac{c_{13}}{c_{33}(c_{11}+c_{12}) - 2c_{13}^2}\,\sigma_0 r; \quad v_0 = 0;$$

$$w_0 = -\frac{c_{11}+c_{12}}{c_{33}(c_{11}+c_{12}) - 2c_{13}^2}\,\sigma_0 z. \quad (9b)$$

Perturbed State. Using (5) and (9), the three-dimensional elasticity equilibrium equations for the perturbed position can be written as follows:

$$\frac{\partial \sigma'_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau'_{r\theta}}{\partial \theta} + \frac{\partial}{\partial z}(\tau'_{rz} - \sigma_0 \omega'_\theta) + \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta}) = 0, \quad (10a)$$

$$\frac{\partial \tau'_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma'_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial z}(\tau'_{\theta z} + \sigma_0 \omega'_r) + \frac{2\tau'_{r\theta}}{r} = 0, \quad (10b)$$

$$\frac{\partial \tau'_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau'_{\theta z}}{\partial \theta} + \frac{\partial \sigma'_{zz}}{\partial z} + \frac{\tau'_{rz}}{r} = 0. \quad (10c)$$

In the above equations, σ^0_{ij}, σ'_{ij} are expressed in terms of ϵ^0_{ij}, ϵ'_{ij}, respectively, in the same manner as Eqs. (8) for σ_{ij} in terms of ϵ_{ij}. The strains ϵ'_{ij}, are in turn expressed in terms of the displacements, u_1, v_1, w_1, in the same manner as the linear strain displacement relations (2). Substituting, we obtain the equations of equilibrium in terms of the displacements at the perturbed state, u_1, v_1, w_1 as follows:

$$c_{11}\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} - \frac{u_1}{r^2}\right) + \frac{1}{2}(c_{11}-c_{12})\frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} + \left(c_{55} - \frac{\sigma_0}{2}\right)\frac{\partial^2 u_1}{\partial z^2}$$
$$+ \frac{1}{2}(c_{11}+c_{12})\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}\right) - (c_{11}-c_{12})\frac{1}{r^2}\frac{\partial v_1}{\partial \theta} + \left[(c_{13}+c_{55}) + \frac{\sigma_0}{2}\right]\frac{\partial^2 w_1}{\partial r \partial z} = 0, \quad (11a)$$

$$\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2}\right) + c_{11}\frac{1}{r^2}\frac{\partial^2 v_1}{\partial \theta^2} + \left(c_{55} - \frac{\sigma_0}{2}\right)\frac{\partial^2 v_1}{\partial z^2}$$
$$+ \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{1}{2}(c_{11}+c_{12})\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r}\right) + (c_{11}-c_{12})\frac{u_1}{r} + \left[(c_{13}+c_{55}) + \frac{\sigma_0}{2}\right]\frac{\partial w_1}{\partial z}\right\} = 0, \quad (11b)$$

$$c_{55}\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right) + c_{33}\frac{\partial^2 w_1}{\partial z^2} + (c_{13}+c_{55})\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r}\frac{\partial v_1}{\partial \theta}\right) = 0, \quad (11c)$$

(a) We seek a first group of solutions in terms of a function ϕ in the form

$$u_1 = \frac{\partial \phi}{\partial r}; \quad v_1 = \frac{1}{r}\frac{\partial \phi}{\partial \theta}; \quad w_1 = k\frac{\partial \phi}{\partial z}. \quad (12)$$

A similar form had been used by Elliott (1948) for Cartesian coordinates. Then equations (11a) and (11b) are satisfied if

$$c_{11}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) + \left[c_{55} + k(c_{13}+c_{55}) + \frac{\sigma_0}{2}(k-1)\right]\frac{\partial^2 \phi}{\partial z^2} = 0, \quad (13a)$$

and (11c) is satisfied if

$$[(c_{13}+c_{55}) + kc_{55}]\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) + kc_{33}\frac{\partial^2 \phi}{\partial z^2} = 0. \quad (13b)$$

A nonzero solution of these Eqs. (13a) and (13b) can be found only if they are identical; this occurs if

$$\frac{c_{55} + k(c_{13}+c_{55}) + (k-1)(\sigma_0/2)}{c_{11}} = \frac{kc_{33}}{(c_{13}+c_{55}) + kc_{55}} = s^2. \quad (14a)$$

This gives a quadratic equation for s^2 or k. The equation for $x = s^2$, with roots s_1^2 and s_2^2, depending on the compressive stress σ_0, is

$$c_{11}c_{55}x^2 + \left[(c_{13}+c_{55})\left(c_{13}+c_{55}+\frac{\sigma_0}{2}\right) - c_{55}\left(c_{55}-\frac{\sigma_0}{2}\right) - c_{11}c_{33}\right]x + c_{33}\left(c_{55}-\frac{\sigma_0}{2}\right) = 0, \quad (14b)$$

and the corresponding k_i:

$$k_i = \frac{s_i^2 c_{11} - c_{55} + (\sigma_0/2)}{c_{13} + c_{55} + (\sigma_0/2)} \quad i = 1,2. \quad (14c)$$

(b) A second group of solutions is sought in terms of the function ψ in the form:

$$u_1 = \frac{1}{r}\frac{\partial \psi}{\partial \theta}; \quad v_1 = -\frac{\partial \psi}{\partial r}; \quad w_1 = 0. \quad (15)$$

Then Eqs. (11a) and (11b) are satisfied if

$$\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) + \left(c_{55} - \frac{\sigma_0}{2}\right)\frac{\partial^2 \psi}{\partial z^2} = 0, \quad (16)$$

and Eq. (11c) is identically satisfied.

(c) Finally, an obvious third group of solutions is the rigid-body displacement field with components V_x, V_y, and V_z along the Cartesian x, y, z coordinate system:

$$u_1 = V_x\cos\theta + V_y\sin\theta; \quad v_1 = -V_x\sin\theta + V_y\cos\theta; \quad w_1 = V_z. \quad (17)$$

The displacement is a superposition of the fields (a), (b), and (c).

Now the functions ϕ and ψ are sought in a separable form:

$$\phi_i(r, \theta, z) = Z(z)A_i(\lambda r)\cos n\theta;$$

$$i = 1,2 \text{ corresponding to } s_1, s_2, \tag{18a}$$

$$\psi(r, \theta, z) = Z(z)B(\lambda r)\sin n\theta. \tag{18b}$$

Notice that the decomposition in terms of the three functions ϕ_1, ϕ_2, and ψ may be considered complete in the sense that it results in the number of constants needed for the formulation of the eigenvalue problem as will be seen in the following.

Now set

$$\rho = \lambda r. \tag{18c}$$

Substituting in (13a), we obtain the ordinary differential equations:

$$A_i''(\rho) + \frac{1}{\rho}A_i'(\rho) - \left(s_i^2 + \frac{n}{\rho^2}\right)A_i(\rho) = 0, \tag{19a}$$

where s_i^2 are given in (14a). In a similar fashion, substituting in (16), we obtain the ordinary differential equation:

$$B''(\rho) + \frac{1}{\rho}B'(\rho) - \left(q^2 + \frac{n}{\rho^2}\right)B(\rho) = 0 \quad \text{where } q^2 = \frac{2c_{55} - \sigma_0}{c_{11} - c_{12}}, \tag{19b}$$

Moreover, $Z(z)$ is found to satisfy

$$Z''(z) + \lambda^2 Z(z) = 0. \tag{19c}$$

The assumption

$$Z(z) = \sin\lambda z \tag{19d}$$

satisfies the third differential Eq. (19c).

For a hollow cylinder, since we do not have a restriction of finite values at $r = 0$, the solution to the two Eqs. (19a) and (19b) involves the modified Bessel functions of both the first and the second kind.

$$A_i(\rho) = C_i I_n(s_i\rho) + D_i K_n(s_i\rho); \quad B(\rho) = C_0 I_n(q\rho) + D_0 K_n(q\rho), \tag{20}$$

where the constants $\{C_1, C_2\}$ and $\{D_1, D_2\}$ are in general complex conjugate pairs and C_0 and D_0 are real.

Before satisfying the boundary conditions at the lateral surfaces, we shall discuss the boundary conditions at the ends. From (7), the boundary conditions on the ends are

$$\tau_{rz}' + \sigma_{zz}^0\omega_\theta' = 0; \quad \tau_{\theta z}' - \sigma_{zz}^0\omega_r' = 0; \quad \sigma_{zz}' = 0, \quad \text{at } z = 0, l \tag{21}$$

Since, σ_{zz}' varies as $\sin\lambda z$, the condition $\sigma_{zz}' = 0$ on both the lower end $z = 0$, and the upper end $z = l$, is satisfied if

$$\lambda = \frac{m\pi}{l}. \tag{22}$$

In a cartesian coordinate system (x, y, z), the first two of the conditions in (21) can be written as follows:

$$\tau_{xz}' + \sigma_{zz}^0\omega_y' = 0; \quad \tau_{yz}' - \sigma_{zz}^0\omega_x' = 0. \tag{23}$$

It will be proved now that these remaining two conditions are satisfied on the average.

The lateral surface boundary conditions in the cartesian coordinate system (analogous to (7)), with $\hat{n}$ the normal to the circular contour are (notice that from the prebuckling state $\sigma_{xx}^0 = \sigma_{yy}^0 = \tau_{xy}^0 = 0$):

$$\sigma_{xx}'\cos(\hat{n}, x) + \tau_{xy}'\cos(\hat{n}, y) = 0, \tag{24a}$$

$$\tau_{xy}'\cos(\hat{n}, x) + \sigma_{yy}'\cos(\hat{n}, y) = 0. \tag{24b}$$

Using the equilibrium equation in cartesian coordinates (analogous to (5)), gives

$$\frac{\partial}{\partial z}\int\int_A(\tau_{xz}' + \sigma_{zz}^0\omega_y')dA = -\int\int_A\left(\frac{\partial\sigma_{xx}'}{\partial x} + \frac{\partial\tau_{xy}'}{\partial y}\right)dA. \tag{25a}$$

Using now the divergence theorem for transformation of an area integral into a contour integral, and the condition (24a) on the contour, gives the previous integral as

$$-\int_\gamma[\sigma_{xx}'\cos(\hat{n}, x) + \tau_{xy}'\cos(\hat{n}, y)]ds = 0,$$

where A denotes the area of the annular cross section and γ the corresponding contour. Therefore,

$$\int\int_A(\tau_{xz}' + \sigma_{zz}^0\omega_y')dA = \text{const.} \tag{25b}$$

Since based on the buckling modes, τ_{rz}', ω_θ', $\tau_{\theta z}'$ and ω_r' and hence τ_{xz}', ω_y', τ_{yz}', and ω_x', all have a $\cos(m\pi z/l)$ variation, they become zero at $z = l/(2m)$. Therefore, it is concluded that the constant in (25b) is zero. Similar arguments hold for τ_{yz}'.

Moreover, it can also be proved that the system of resultant stresses (23) would produce no torsional moment. Indeed,

$$\frac{\partial}{\partial z}\int\int_A[x(\tau_{yz}' - \sigma_{zz}^0\omega_x') - y(\tau_{xz}' + \sigma_{zz}^0\omega_y')]dA$$

$$= -\int\int_A\left\{x\left(\frac{\partial\tau_{xy}'}{\partial x} + \frac{\partial\sigma_{yy}'}{\partial y}\right) - y\left(\frac{\partial\sigma_{xx}'}{\partial x} + \frac{\partial\tau_{xy}'}{\partial y}\right)\right\}dA.$$

Again, using the divergence theorem, the previous integral becomes

$$-\int_\gamma\{x[\tau_{xy}'\cos(\hat{n}, x) + \sigma_{yy}'\cos(\hat{n}, y)]$$

$$-y[\sigma_{xx}'\cos(\hat{n}, x) + \tau_{xy}'\cos(\hat{n}, y)]\}ds = 0. \tag{26a}$$

Hence,

$$\int\int_A[x(\tau_{yz}' - \sigma_{zz}^0\omega_x') - y(\tau_{xz}' + \sigma_{zz}^0\omega_y')]dA = \text{const}, \tag{26b}$$

and this constant is again zero since $\tau_{xz}' = \tau_{yz}' = \omega_x' = \omega_y' = 0$ at $z = l/(2m)$.

Finally, it should be noted that since u_1 varies as $\sin\lambda z$,

$$u_1 = \frac{d^2u_1}{dz^2} = 0 \text{ at } z = 0, l, \tag{26c}$$

which is the condition of simply-supported ends; furthermore, w_1 and v_1 can be made equal to zero at some point at the end $z = 0$ by the choice of the constants V_i in (17).

Notice that based on the previous analysis, we have found that

$$\phi_i(r, \theta, z) = [C_i I_n(s_i\lambda r) + D_i K_n(s_i\lambda r)]\cos n\theta\sin\lambda z; \quad i = 1,2 \tag{26d}$$

and

$$\psi(r, \theta, z) = [C_0 I_n(q\lambda r) + D_0 K_n(q\lambda r)]\sin n\theta\sin\lambda z. \tag{26e}$$

Now we proceed to the boundary conditions on the lateral surfaces $r = R_j$, $j = 1, 2$, which will ultimately give the characteristic equation for the critical load.

From (7), we obtain

$$\sigma_{rr}' = 0; \quad \tau_{r\theta}' = 0; \quad \tau_{rz}' = 0, \quad \text{at } r = R_1, R_2. \tag{27}$$

Substituting in (27), (15), (12), (2), and (8), and using the identities for the derivatives of Bessel functions, we obtain a system of six linear homogeneous equations in C_i, D_i, $i = 1$, 2, and C_0, D_0. In particular, the boundary condition $\sigma_{rr}' = 0$ at $r = R_j$, $j = 1, 2$ gives

$$(c_{11} - c_{12})\left\{C_0\left[n(n-1)\frac{I_n}{R_j^2} + n\lambda q\frac{I_{n+1}}{R_j}\right]\right.$$

$$\left. + D_0\left[n(n-1)\frac{K_n}{R_j^2} - n\lambda q\frac{K_{n+1}}{R_j}\right]\right\}$$

$$+\sum_{i=1,2} C_i\left\{c_{11}\left[\lambda^2 s_i^2 I_n+n(n-1)\frac{I_n}{R_j^2}-\lambda s_i\frac{I_{n+1}}{R_j}\right]\right.$$

$$+c_{12}\left[-n(n-1)\frac{I_n}{R_j^2}+\lambda s_i\frac{I_{n+1}}{R_j}\right]$$

$$\left.-c_{13}k_i\lambda^2 I_n\right\}+\sum_{i=1,2} D_i\left\{c_{11}\left[\lambda^2 s_i^2 K_n+n(n-1)\frac{K_n}{R_j^2}+\lambda s_i\frac{K_{n+1}}{R_j}\right]\right.$$

$$-c_{12}\left[n(n-1)\frac{K_n}{R_j^2}+\lambda s_i\frac{K_{n+1}}{R_j}\right]$$

$$\left.-c_{13}k_i\lambda^2 K_n\right\}=0,\quad j=1,2. \qquad (28a)$$

The boundary condition $\tau'_{r\theta} = 0$ at $r = R_j$, $j = 1, 2$ gives

$$C_0\left[2\lambda q\frac{I_{n+1}}{R_j}-2n(n-1)\frac{I_n}{R_j^2}-\lambda^2 q^2 I_n\right]$$

$$-D_0\left[2\lambda q\frac{K_{n+1}}{R_j}+2n(n-1)\frac{K_n}{R_j^2}+\lambda^2 q^2 K_n\right]$$

$$-\sum_{i=1,2} C_i\left[2n(n-1)\frac{I_n}{R_j^2}+2n\lambda s_i\frac{I_{n+1}}{R_j}\right]$$

$$+\sum_{i=1,2} D_i\left[-2n(n-1)\frac{K_n}{R_j^2}+2n\lambda s_i\frac{K_{n+1}}{R_j}\right]=0,\ j=1,2. \qquad (28b)$$

In a similar fashion, the condition $\tau'_{rz} = 0$ at $r = R_j$, $j = 1$, 2 gives

$$C_0\frac{nI_n}{R_j}+D_0\frac{nK_n}{R_j}+\sum_{i=1,2} C_i(k_i+1)\left[\frac{nI_n}{R_j}+\lambda s_i I_{n+1}\right]$$

$$+\sum_{i=1,2} D_i(k_i+1)\left[\frac{nK_n}{R_j}-\lambda s_i K_{n+1}\right],\quad j=1,2. \qquad (28c)$$

The Bessel functions $I_{\{n,n+1\}}$ and $K_{\{n,n+1\}}$ are assumed to be evaluated at $\lambda s_i R_j$ if they are inside the sum, Σ, and hence belong to the coefficients of C_i and D_i, i.e., $I_{\{n,n+1\}} \equiv I_{\{n,n+1\}}(\lambda s_i R_j)$; they are evaluated at $\lambda q R_j$ if they are outside, and hence belong to the coefficients of C_0 and D_0, i.e., $I_{\{n,n+1\}} \equiv I_{\{n,n+1\}}(\lambda q R_j)$.

By equating the determinant of the above system (28) to zero, we obtain an equation for σ_0 (characteristic equation) which can be solved to obtain the critical load. In general, the roots s_1 and s_2 are either both real or complex conjugates, whereas q, defined in (19*b*), is normally a real variable. In the case of real s_1, s_2, the determinant of the linear system (28) is real.

In the case of a complex conjugate pair $\{s_1, s_2\}$, the Bessel functions have complex arguments and the constants C_1, C_2 and D_1, D_2 are complex conjugates, whereas C_0 and D_0 are real variables. Furthermore, $\{I_n(\lambda s_1 R_j),\ I_n(\lambda s_2 R_j)\}$ and $\{I_{n+1}(\lambda s_1 R_j),\ I_{n+1}(\lambda s_2 R_j)\}$ are also complex conjugate pairs and the same holds true for $\{K_n(\lambda s_1 R_j),\ K_n(\lambda s_2 R_j)\}$ and $\{K_{n+1}(\lambda s_1 R_j),\ K_{n+1}(\lambda s_2 R_j)$. The 6 × 6 matrix of coefficients of the linear system (28) has two real columns (corresponding to C_0 and D_0 and the remaining four are two pairs of complex conjugates. Therefore, it turns out that in this case the determinant of (28) is also real. In either case, equating to zero the determinant results in a nonlinear real equation for σ_0.

The modified Bessel functions of zero and first order are evaluated from polynomial coefficients given by Abramowitz and Stegun (1964) and those of the higher order from the associated recurrence relations. It should be noted that due to these polynomial approximations for the Bessel functions, the equation for zero determinant turns out to have a large number of very closely spaced roots, for R_2/R_1 less than about 1.15, which renders this procedure unsuitable for finding the bifurcation load for thin shell construction; for moderate thickness, however, a single, well-defined root is found

Discussion of Results

In a cylindrical shell subjected to axial compression, a large number of instability modes correspond to a single bifurcation point. By setting

$$\bar{m}=\frac{m\pi R_0}{l}, \qquad (29)$$

we obtain the simple formula for the eigenvalues of isotropic shells from the Donnell theory as follows (Timoshenko and Gere, 1961):

$$\sigma_{0,S-\text{Donnell}}=\frac{E}{(1-\nu^2)}\frac{h^2}{12R_0^2}\frac{(\bar{m}^2+n^2)^2}{\bar{m}^2}+E\frac{\bar{m}^2}{(\bar{m}^2+n^2)^2}, \qquad (30)$$

where E is the modulus of elasticity and ν is the Poisson's ratio. A distinct eigenvalue corresponds to each pair of the positive integers m and n. The pair corresponding to the smallest eigenvalue can be determined by trial. Analytical minimization of σ_0 from (30) with respect to the quantity $[(\bar{m}^2 + n^2)/\bar{m}]^2$ gives the well-known classical result:

$$\sigma_{0cr,\ C-\text{Donnell}}=\frac{Eh}{R_0\sqrt{3(1-\nu^2)}}. \qquad (31)$$

As discussed in the Introduction, this formula holds true for shells of intermediate length; for short shells the trial-and-error procedure is necessarily used (Batdorf, 1947). In the classical shell theory solution, the radial displacement is constant through the thickness and the axial and circumferential ones have a linear variation, i.e., the displacements are in the form:

$$u_1(r,\theta,z)=U_0\cos n\theta\sin\lambda z,$$

$$v_1(r,\theta,z)=\left[V_0+\frac{r-R}{R}(V_0+nU_0)\right]\sin n\theta\sin\lambda z. \qquad (32a)$$

$$w_1(r,\theta,z)=[W_0-(r-R)\lambda U_0]\cos n\theta\cos\lambda z. \qquad (32b)$$

where U_0, V_0, W_0 are constants (these displacement field variations would satisfy the classical assumptions of $e_{rr} = e_{r\theta} = e_{rz} = 0$).

Two other shell theories, namely the Flügge (1960) and the Danielson and Simmonds (1969), have produced results for the critical loads in shells and should therefore be compared with the present elasticity solution. The expression for the eigenvalues derived from the Flügge equations (Flügge, 1960), $\sigma_{0,F}$ and the more simplified but just as accurate one by Danielson and Simmonds (1969), $\sigma_{0,DS}$ are

$$\sigma_{0,\{F,DS\}}=E\frac{Q_{F,DS}}{\bar{m}^2[(\bar{m}^2+n^2)^2+n^2]}, \qquad (33a)$$

where the numerator for the Flügge theory is

$$Q_F=\frac{h^2}{12R_0^2(1-\nu^2)}\{(\bar{m}^2+n^2)^4-2[\nu\bar{m}^2+3\bar{m}^4n^2$$

$$+(4-\nu)\bar{m}^2n^4+n^6]+2(2-\nu)\bar{m}^2n^2+n^4\}+\bar{m}^4; \qquad (33b)$$

and for the Danielson and Simmonds equations,

$$Q_{DS}=\frac{h^2}{12R_0^2(1-\nu^2)}(\bar{m}^2+n^2)^2(\bar{m}^2+n^2-1)^2+\bar{m}^4. \qquad (33c)$$

Again, a distinct eigenvalue corresponds to each pair of the positive integers m and n, the critical load being for the pair that renders the lowest eigenvalue.

Concerning the present elasticity formulation, the critical load is obtained by finding the solution σ_0 of the determinant of (28) for a range of n and m, and keeping the minimum value. Table 1 shows the critical load and the corresponding n, m, as predicted by the present three-dimensional elasticity formulation, and the critical load and the corresponding n, m, as predicted by the simple Donnell shell formula, Eq. (30). A length ratio $l/R_2 = 5$, and isotropic material with $E = 14$ GPa and Poisson's ratio $\nu = 0.3$ have been assumed. In all cases, a value of $R_2 = 1$ m is taken, and a range of outside versus inside radius ratios, R_2/R_1, which would probably constitute moderately thick shells is examined. In general, the number of axial half-waves, m, is less for the elasticity solution than that of the shell theory, although the number of circumferential waves, n, is mostly the same. The shell theory predicts a higher load than the elasticity approach, the nonconservatism increasing with thicker shells. For a ratio of outside/inside radius $R_2/R_1 = 1.20$, the simple Donnell shell formula prediction is higher than the elasticity one by more than 50 percent. The last column in Table 1 gives the predictions of the elasticity formulation for the same n, m as in the shell theory solution; again the elasticity results are lower than the corresponding shell theory ones, even for the same n, m values.

One other point should also be mentioned. In order to produce a closed-form expression, the simple Donnell shell formula is based on approximations regarding several terms in the final system of the Donnell shell equations, for example it neglects $h^2/(12R_0^2)$. If these approximations are not enforced, which would result in a more involved formula, we obtain expressions which are referred here as the eigenvalues from the "nonsimplified Donnell shell theory," and these are given in the Appendix. The minimum values (critical loads) from this procedure are compared in Table 2. In general, the difference between the elasticity and the Donnell shell theory becomes smaller for the non-simplified formulas.

Table 2 gives the predictions of the different shell theories for $l/R_2 = 5$ and isotropic material in comparison with the elasticity one. It is clearly seen that:
(1) the values of n, m (number of circumferential waves and number of axial half-waves, respectively, at the critical point) for both the elasticity, and the Flügge, and the Danielson and Simmonds theories agree, unlike the Donnell shell theory.
(2) all shell theories predict higher critical values than the elasticity solution, the percentage increase being larger with thicker shells. However, both the Flügge and Danielson and Simmonds theories predict critical loads much closer to the elasticity value than the Donnell theory. For instance, at $R_2/R_1 = 1.20$, the Flügge theory predicts a 5.7 percent higher value and the Danielson and Simmonds theory a 7.7 percent higher value than the elasticity solution, whereas the Donnell theory gives a 51.1 percent and a 20.7 percent higher value for the simple and the nonsimplified formulas, respectively. Especially noteworthy is the good performance of the Danielson and Simmonds theory despite its simplicity relative to the Flügge theory.

To further examine the eigenvalues for different n, m, Figs. 2(a, b) show the solution σ_0 of the determinant of (28) for values of m (number of axial half-waves) ranging from 1 to 20 and for n (number of circumferential waves) equal to 1, 2, 3. These curves are for $R_2/R_1 = 1.20$, $l/R_2 = 5$ and isotropic material. For this case, the minimum σ_0 (critical stress) is obtained for $n = 2$ and $m = 2$.

If the displacements of the elasticity solution are set in a form analogous to (32), where instead of the constants U_0, V_0, W_0, we have functions of r, we can write these r-dependences as follows:

$$U(r)=C_0 n\frac{I_n}{r}+D_0 n\frac{K_n}{r}\sum_{i=1,2}C_i\left[n\frac{I_n}{r}+\lambda s_i I_{n+1}\right] + D_i\left[n\frac{K_n}{r}-\lambda s_i K_{n+1}\right], \quad (34a)$$

$$V(r)=-\left\{C_0\left[n\frac{I_n}{r}+\lambda q I_{n+1}\right]+D_0\left[n\frac{K_n}{r}-\lambda q K_{n+1}\right] + \sum_{i=1,2}C_i n\frac{I_n}{r}+D_i n\frac{K_n}{r}\right\}, \quad (34b)$$

$$W(r)=\lambda\sum_{i=1,2}k_i(C_i I_n+D_i K_n), \quad (34c)$$

where again the Bessel functions are assumed to be evaluated at $\lambda s_i r$ if they are inside the sum, and hence belong to the coefficients of C_i and D_i; they are evaluated at λqr if they are outside and hence belong to the coefficients of C_0 and D_0. As an illustration, for the critical load of the previous example of $R_2/R_1 = 1.20$, Fig. 3 shows the eigenfunction $U_1(r)$, which seems to be nearly constant (shell theory would have a constant value $U_1 = 1$)

Concerning the effect of material data, it can be proved that for an isotropic material which is characterized by the two

Table 1 Comparison With Classical Donnell Shell Theory Critical Loads, $\sigma_0 R_2/(E_3 h)$ Isotropic, $E = E_3$, $\nu = 0.3$ $l/R_2 = 5$

R_2/R_1	Elasticity (n, m)	Classical Shell[1] (Simplified Donnell) (n, m)	Percent Increase	[Elasticity, at same (n, m) as Shell]
1.15	0.454 (2,1)	0.648 (1,8)	42.7%	(0.627)
1.20	0.437 (2,2)	0.660 (2,5)	51.0%	(0.580)
1.25	0.443 (2,2)	0.675 (2,4)	52.3%	(0.545)
1.30	0.449 (1,1)	0.685 (1,6)	52.6%	(0.635)

[1]From Eq. (30)

Table 2 Comparison With Various Shell Theories Critical Loads, $\sigma_0 R_2/(E_3 h)$ Isotropic, $E = E_3$, $\nu = 0.3$ $l/R_2 = 5$

R_2/R_1	Elasticity (n, m)	Classical[1] (Simplified Donnell) (n, m) Percent Increase	Non-Simplified[2] Donnell (n, m) Percent Increase	Flügge[3] (n, m) Percent Increase	Danielson[3] and Simmonds (n, m) Percent Increase
1.15	0.454 (2,1)	0.648 (1,8) 42.7%	0.549 (2,2) 20.9%	0.471 (2,1) 3.7%	0.481 (2,1) 5.9%
1.20	0.437 (2,2)	0.660 (2,5) 51.0%	0.527 (2,2) 20.6%	0.462 (2,2) 5.7%	0.470 (2,2) 7.6%
1.25	0.443 (2,2)	0.675 (2,4) 52.3%	0.540 (2,2) 21.9%	0.473 (2,2) 6.8%	0.484 (2,2) 9.3%
1.30	0.449 (1,1)	0.685 (1,6) 52.6%	0.571 (2,2) 27.2%	0.492 (1,1) 9.6%	0.499 (1,1) 11.1%

[1]From Eq. (30)
[2]From Eq. (A1)
[3]From Eqs. (33)

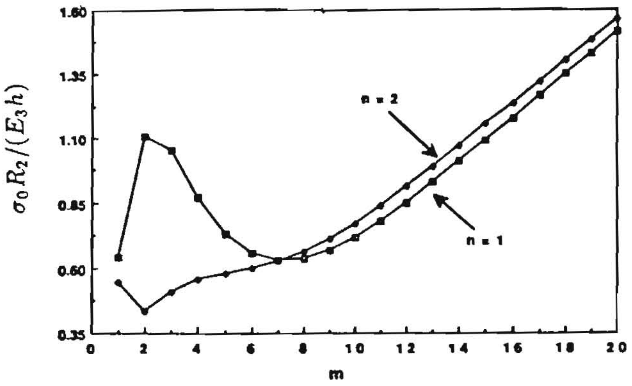

Fig. 2(a) Eigenvalues $\sigma_0 R_2/(E_3 h)$ for $n = 1, 2$ (number of circumferential waves) and for m (number of axial half-waves) ranging from 1 to 20. The curves are for $R_2/R_1 = 1.20$, $l/R_2 = 5$ and isotropic material (minimum σ_0, i.e., critical stress, is for $n = 2$ and $m = 2$).

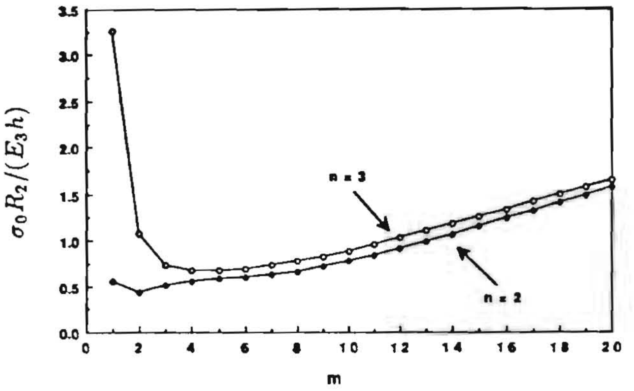

Fig. 2(b) Eigenvalues $\sigma_0 R_2/(E_3 h)$ for $n = 2, 3$ and for m ranging from 1 to 20. The curves are for $R_2/R_1 = 1.20$, $l/R_2 = 5$ and isotropic material (minimum σ_0, i.e., critical stress, is for $n = 2$ and $m = 2$).

stiffness constants $c_{11} = c_{33}$ and c_{55}, $(c_{12} = c_{13} = c_{11} - 2c_{55})$, the roots of (14b) are

$$s_1 = 1 \text{ and } s_2 = q = \left(1 - \frac{\sigma_0}{2c_{55}}\right)^{1/2}. \tag{35}$$

For a transversely isotropic material the roots are in general complex conjugates. We take as an example case the same geometrical dimensions as before, i.e., $l/R_2 = 5.0$, and a transversely isotropic material and with moduli (in GPa): $E_3 = 57$, $E_2 = E_1 = 14$, $G_{31} = G_{23} = 5.7$, and Poisson's ratios: $\nu_{23} = \nu_{13} = 0.068$, $\nu_{12} = 0.400$ (these data can approximate glass/epoxy material with reinforcement along the z-axis). Notice that we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$. Table 3 gives the critical load and the corresponding n, m, as predicted by the present three-dimensional elasticity formulation, and the critical load and the corresponding n, m, as predicted by the nonsimplified transversely isotropic Donnell shell theory (Appendix). Again, the shell theory predicts a higher load than the elasticity approach. Notice that the critical load for the transversely isotropic material (with a reinforced axial direction) has been normalized with a higher value of E_3 with regard to the one for the isotropic case (Table 1); the non-normalized value is actually higher.

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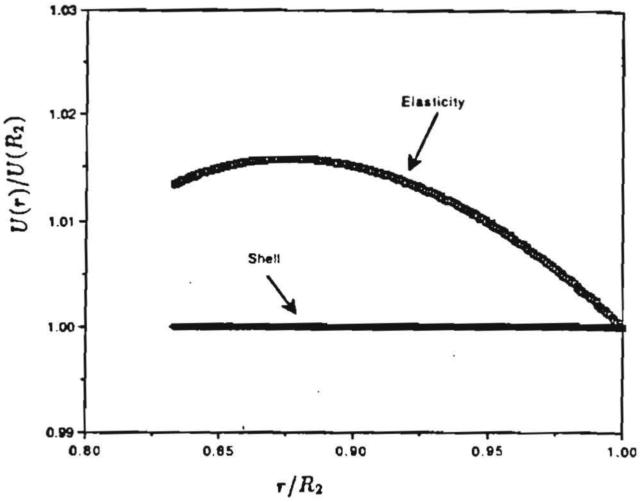

Fig. 3 "Eigenfunction" $U(r)/U(R_2)$ versus normalized radial distance r/R_2, $R_2/R_1 = 1.20$, $l/R_2 = 5$ and isotropic material (shell theory would have a constant value throughout, $U(r)/U(R_2) = 1$)

Table 3 Transversely Isotropic Material Critical Loads, $\sigma_0 R_2/(E_3 h)$
Moduli: (in GPa) $E_3 = 57$, $E_2 = E_1 = 14$, $G_{31} = G_{23} = 5.7$
Poisson's ratios: $\nu_{12} = 0.400$, $\nu_{23} = \nu_{13} = 0.068$
$l/R_2 = 5$

R_2/R_1	Elasticity (n, m)	Nonsimplified[1] Donnell (n, m)	Percent Increase
1.10	0.167 (2,1)	0.193 (3,3)	15.6%
1.15	0.162 (2,1)	0.194 (2,1)	19.7%
1.20	0.164 (2,2)	0.185 (2,2)	12.8%
1.25	0.163 (2,2)	0.186 (2,2)	14.1%
1.30	0.167 (2,2)	0.192 (2,2)	15.0%

[1]From Eq. (A1)

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APPENDIX

Eigenvalues From Nonsimplified Donnell Shell Equations

In terms of

$$\alpha=\frac{h^2}{12R_0^2};\quad \bar{\lambda}=R_0\lambda;\quad \bar{E}_3=\frac{E_3}{1-\nu_{23}\nu_{32}};\quad \bar{E}_2=\frac{E_2}{1-\nu_{23}\nu_{32}},$$

define the following constants that depend on α, the material properties, and the values of n, m:

$$a_{11}=\bar{E}_3\nu_{23}\bar{\lambda},\quad a_{12}=(\bar{E}_3\nu_{23}+G_{23})n\bar{\lambda};\quad a_{13}=-(\bar{E}_3\bar{\lambda}^2+G_{23}n^2)$$

$$a_{21}=-(\bar{E}_2+\bar{E}_2\alpha n^2+\bar{E}_2\nu_{32}\alpha\bar{\lambda}^2+2G_{23}\alpha\bar{\lambda}^2)n$$

$$a_{22}=-(\bar{E}_2n^2+\bar{E}_2\alpha n^2+G_{23}\bar{\lambda}^2+2G_{23}\alpha\bar{\lambda}^2);$$

$$a_{23}=(\bar{E}_3\nu_{23}+G_{23})n\bar{\lambda}$$

$$a_{31}=(\bar{E}_2+\bar{E}_3\alpha\bar{\lambda}^4+2\bar{E}_2\nu_{32}\alpha\bar{\lambda}^2n^2+\bar{E}_2\alpha n^4+4G_{23}\alpha\bar{\lambda}^2n^2)$$

$$a_{32}=(\bar{E}_2+\bar{E}_2\nu_{32}\alpha\bar{\lambda}^2+\bar{E}_2\alpha n^2+4G_{23}\alpha\bar{\lambda}^2)n;\quad a_{33}=-\bar{E}_3\nu_{23}\bar{\lambda}.$$

The eigenvalues for a transversely isotropic material are then given by

$$\sigma_{NS-\mathrm{Donnell}}=-\frac{DET}{(a_{22}a_{13}-a_{12}a_{23})\bar{\lambda}^2},\tag{A1}$$

where

$$DET=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}.$$

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