Bifurcation of Equilibrium in Thick Orthotropic Cylindrical Shells Under Axial Compression

G. A. Kardomateas
Assoc. Professor,
School of Aerospace Engineering,
Georgia Institute of Technology,
Atlanta, GA 30332-0150
Mem. ASME.

The bifurcation of equilibrium of an orthotropic thick cylindrical shell under axial compression is studied by an appropriate formulation based on the three-dimensional theory of elasticity. The results from this elasticity solution are compared with the critical loads predicted by the orthotropic Donnell and Timoshenko nonshallow shell formulations. As an example, the cases of an orthotropic material with stiffness constants typical of glass/epoxy and the reinforcing direction along the periphery or along the cylinder axis are considered. The bifurcation points from the Timoshenko formulation are always found to be closer to the elasticity predictions than the ones from the Donnell formulation. For both the orthotropic material cases and the isotropic one, the Timoshenko bifurcation point is lower than the elasticity one, which means that the Timoshenko formulation is conservative. The opposite is true for the Donnell shell theory, i.e., it predicts a critical load higher than the elasticity solution and therefore it is nonconservative. The degree of conservatism of the Timoshenko theory generally increases for thicker shells. Likewise, the Donnell theory becomes in general more nonconservative with thicker construction.

Introduction

The buckling strength of composite structural members is an important design parameter because of the large strength-to-weight ratio and the lack of extensive plastic yielding in these materials. Fiber-reinforced composite materials can be used in the form of laminated shells in several important structural applications. Although thin plate construction has been the thrust of the initial applications, much attention is now being paid to configurations classified as moderately thick shell structures. Such designs can be used, for example, in the marine industry, as well as for components in the aircraft and automobile industries. Moreover, composite laminates have been considered in space vehicles in the form of circular cylindrical shells as a primary load-carrying structure.

Stability equations for cylindrical shells have been available in the literature mainly for isotropic material (e.g., Flügge, 1960; Danielson and Simmonds, 1969) and a number of analyses have been performed for the buckling strength, based on the application of the cylindrical shell theories (e.g., Simitses, Shaw, and Sheinman, 1985). The relatively simple equations suggested by Donnell (1933) have formed the basis for stability analyses in the literature more than any other set of cylindrical shell equations. Besides the original first set of Donnell equations, a second, more accurate set of cylindrical shell equations that are not subject to some of the shallowness limitation of the first set is also well quoted in the literature (Brush and Almroth, 1975). The latter one will be used in the comparison studies in this paper. Furthermore, in presenting a shell theory formulation for isotropic shells, Timoshenko and Gere (1961) included an additional term in the circumferential displacement part of the second equation (these equations are briefly described in Appendix II). Both the isotropic "nonshallow" Donnell and Timoshenko and Gere formulations can be readily extended for the case of orthotropic material.

In view of possible structural applications of anisotropic shells with sizable thickness, it is desirable to conduct a comprehensive study of the performance of both the readily available Donnell and Timoshenko orthotropic shell theories with respect to the shell thickness. An accurate solution for the stability characteristics of moderately thick shells is also needed in order to enable a future comparison of the accuracy of the predictions from various improved shell theories (e.g., Whitney and Sun, 1974; Librescu, 1975; Reddy and Liu, 1985; see also Noor and Burton, 1990 for a review of shear deformation theories).

Elasticity solutions for the buckling of cylindrical shells have been recently presented by Kardomateas (1993a) for the case of uniform external pressure and orthotropic material; a simplified problem definition was used in this study ("ring"

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assumption), in that the prebuckling stress and displacement field was axisymmetric, and the buckling modes were assumed two-dimensional, i.e., no z component of the displacement field, and no z-dependence of the r and θ displacement components. It was shown that the critical load for external pressure loading, predicted by shell theory can be highly nonconservative for moderately thick construction. A more thorough investigation of the thickness effects was conducted by Kardomateas (1993b) for the case of a transversely isotropic thick cylindrical shell under axial compression. In that work, a full dependence on r, θ, and z of the buckling modes was assumed. The reason for restricting the material to transversely isotropic was the desire to produce closed-form analytical solutions.

Regarding numerical treatments of this problem, Bradford and Dong (1978) performed an analysis of laminated orthotropic cylinders using semianalytical finite elements, with the modeling occurring in the thickness direction. Although the main focus was on natural vibrations, the case of elastic buckling was also discussed. Results for the isotropic and transversely isotropic case, based on that finite element model, were computed by Dong (1994) and compared with these in Kardomateas (1993b), with good agreement.

In the present work, a generally cylindrically orthotropic material under axial compression is considered. Again, the nonlinear three-dimensional theory of elasticity is appropriately formulated, and reduced to a standard eigenvalue problem for ordinary linear differential equations in terms of a single variable (the radial distance r), with the applied axial load P the parameter. The formulation employs the exact elasticity solution by Lekhnitskii (1963) for the prebuckling state. A full dependence on r, θ, and z of the buckling modes is assumed. The work by Kartomateas (1993b) included a comprehensive study of the performance of the Donnell (1933), the Flügge (1960), and the Danielson and Simmonds (1969) theories for isotropic material. These theories were all found to be nonconservative in predicting bifurcation points, the Donnell theory being the most nonconservative. In addition to considering general orthotropy for the material constitutive behavior, the present work extends the latter work by investigating the performance of another classical formulation, i.e., the Timoshenko and Gere (1961) shell theory. In this paper specific results will be presented for the critical load and the buckling modes; these will be compared with both the orthotropic "nonshallow" Donnell and Timoshenko shell formulations. As an example, the cases of an orthotropic material with stiffness constants typical of glass/epoxy and the reinforcing direction along the periphery or along the cylinder axis will be considered.

Formulation

Let us consider the equations of equilibrium in terms of the second Piola-Kirchhoff stress tensor Σ in the form

$$\operatorname{div}(\Sigma . \mathbf{F}^{\mathrm{T}}) = 0, \tag{1a}$$

where $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \operatorname{grad} V, \tag{1b}$$

where V is the displacement vector and $\mathbf{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}.\mathbf{F} - \mathbf{I}). \tag{1c}$$

More specifically, in terms of the linear strains,

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \tag{2a}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},$$
$$e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \tag{2b}$$

and the linear rotations,

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \quad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r},$$
$$2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{2c}$$

the deformation gradient $\mathbf{F}$ is

$$\mathbf{F} = \begin{bmatrix} 1 + e_{rr} & \frac{1}{2}e_{r\theta} - \omega_z & \frac{1}{2}e_{rz} + \omega_\theta \\ \frac{1}{2}e_{r\theta} + \omega_z & 1 + e_{\theta\theta} & \frac{1}{2}e_{\theta z} - \omega_r \\ \frac{1}{2}e_{rz} - \omega_\theta & \frac{1}{2}e_{\theta z} + \omega_r & 1 + e_{zz} \end{bmatrix}. \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, w_0 the r, θ, and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r,\theta,z)$, $\alpha v_1(r,\theta,z)$, $\alpha w_1(r,\theta,z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r,\theta,z)$, $v_1(r,\theta,z)$, $w_1(r,\theta,z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

Following Kardomateas (1993a), we obtain the following buckling equations:

$$\frac{\partial}{\partial r}(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta) + \frac{\partial}{\partial z}(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta) + \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z) = 0, \tag{5a}$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r) + \frac{\partial}{\partial z}(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r) + \frac{1}{r}(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r) = 0, \tag{5b}$$

$$\frac{\partial}{\partial r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r) + \frac{\partial}{\partial z}(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r) + \frac{1}{r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) = 0 \tag{5c}$$

In the previous equations, σ^0_{ij} and ω^0_j are the values of σ_{ij} and ω_j at the initial equilibrium position, i.e., for $u = u_0$, $v = v_0$ and $w = w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position, i.e., for $u = u_1$, $v = v_1$ and $w = w_1$.

The boundary conditions associated with (1a) can be expressed as

$$(\mathbf{F}.\mathbf{\Sigma}^{\mathsf{T}}).\hat{N} = t(V), \qquad (6)$$

where t is the traction vector on the surface which has outward unit normal $\hat{N} = (\hat{l}, \hat{m}, \hat{n})$ before any deformation. The traction vector t depends on the displacement field $V = (u, v, w)$. Again, following Kardomateas (1993a), we obtain for the lateral and end surfaces:

$$(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta)\hat{l} + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)\hat{m}$$

$$+ (\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)\hat{n} = 0, \qquad (7a)$$

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r)\hat{l} + (\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)\hat{m}$$

$$+ (\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)\hat{n} = 0, \qquad (7b)$$

$$(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta)\hat{l} + (\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta)\hat{m}$$

$$+ (\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta)\hat{n} = 0. \qquad (7c)$$

Pre-buckling State. The problem under consideration is that of an orthotropic cylindrical shell compressed by an axial force applied at one end. The stress-strain relations for the orthotropic body are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}, \qquad (8)$$

where c_{ij} are the stiffness constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

Let R_1 be the internal and R_2 the external radius (Fig. 1). Lekhnitskii (1963) gave the stress field for an applied compressive load of absolute value P, in terms of the quantities:

$$k = \sqrt{\frac{a_{11}a_{33} - a_{13}^2}{a_{22}a_{33} - a_{23}^2}}, \qquad (9a)$$

$$\bar{h} = \frac{(a_{23} - a_{13})a_{33}}{(a_{11} - a_{22})a_{33} + (a_{23}^2 - a_{13}^2)}. \qquad (9b)$$

$$\bar{T} = \pi(R_2^2 - R_1^2) - \frac{2\pi\bar{h}}{a_{33}}$$

$$\times \left[\frac{R_2^2 - R_1^2}{2}(a_{13} + a_{23}) - \frac{(R_2^{k+1} - R_1^{k+1})^2}{R_2^{2k} - R_1^{2k}}\frac{a_{13} + ka_{23}}{k+1}\right.$$

$$\left. - \frac{(R_2^{k-1} - R_1^{k-1})^2 (R_1R_2)^2}{R_2^{2k} - R_1^{2k}}\frac{a_{13} - ka_{23}}{k-1}\right]. \qquad (9c)$$

Notice that the formula quoted in Lekhnitskii (1963) for $\bar{T}$ has a slight error in the last term.

The stress field for orthotropy is as follows:

$$\sigma^0_{rr} = P(C_0 + C_1 r^{k-1} + C_2 r^{-k-1}), \qquad (10a)$$

$$\sigma^0_{\theta\theta} = P(C_0 + C_1 k r^{k-1} - C_2 k r^{-k-1}), \qquad (10b)$$

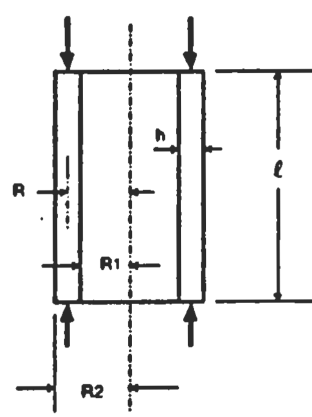

Fig. 1 Cylindrical shell under axial compression

$$\sigma^0_{zz} = -\frac{P}{\bar{T}} - P\left(C_0\frac{a_{13} + a_{23}}{a_{33}} + C_1\frac{a_{13} + ka_{23}}{a_{33}}r^{k-1}\right.$$

$$\left. + C_2\frac{a_{13} - ka_{23}}{a_{33}}r^{-k-1}\right), \qquad (10c)$$

$$\tau^0_{r\theta} = \tau^0_{rz} = \tau^0_{\theta z} = 0, \qquad (10d)$$

where

$$C_0 = -\frac{\bar{h}}{\bar{T}}; \quad C_1 = \frac{R_2^{k+1} - R_1^{k+1}}{R_2^{2k} - R_1^{2k}}\frac{\bar{h}}{\bar{T}}, \qquad (10e)$$

$$C_2 = \frac{R_2^{k-1} - R_1^{k-1}}{R_2^{2k} - R_1^{2k}}(R_1R_2)^{k+1}\frac{\bar{h}}{\bar{T}}. \qquad (10f)$$

Notice that for general orthotropy, both σ^0_{rr} and $\sigma^0_{\theta\theta}$ are nonzero. For an isotropic or transversely isotropic body, these two stress components are zero.

In the previous equations, a_{ij} are the compliance constants, i.e.,

$$\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix}. \qquad (11)$$

The prebuckling solution just described is an exact elasticity solution based on the assumption that the stresses do not vary along the shell axis. Hence, this solution does not take into account the end effects. Recent work by Kollár (1994) has focused on including an axial variation. However, any end effects, being of local nature, would not affect the (global) buckling behavior.

Perturbed State. Using the constitutive relations (8) for the stresses σ'_{ij} in terms of the strains e'_{ij}, the strain-displacement relations (2) for the strains e'_{ij} and the rotations ω'_j in terms of the displacements u_1, v_1, w_1, and taking into account (10), the buckling Eq. (5a) for the problem at hand is written in terms of the displacements at the perturbed state as follows:

$$c_{11}\left(u_{1,rr}+\frac{u_{1,r}}{r}\right)-c_{22}\frac{u_1}{r^2}+\left(c_{66}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta\theta}}{r^2}$$
$$+\left(c_{55}+\frac{\sigma_{zz}^0}{2}\right)u_{1,zz}+\left(c_{12}+c_{66}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,r\theta}}{r}$$
$$-\left(c_{22}+c_{66}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta}}{r^2}+\left(c_{13}+c_{55}-\frac{\sigma_{zz}^0}{2}\right)w_{1,rz}$$
$$+(c_{13}-c_{23})\frac{w_{1,z}}{r}=0. \quad (12a)$$

The second buckling Eq. (5*b*) gives

$$\left(c_{66}+\frac{\sigma_{rr}^0}{2}\right)\left(v_{1,rr}+\frac{v_{1,r}}{r}-\frac{v_1}{r^2}\right)+\left(\frac{\sigma_{rr}^0-\sigma_{\theta\theta}^0}{2}\right)\left(\frac{v_{1,r}}{r}+\frac{v_1}{r^2}\right)$$
$$+c_{22}\frac{v_{1,\theta\theta}}{r^2}+\left(c_{44}+\frac{\sigma_{zz}^0}{2}\right)v_{1,zz}+\left(c_{66}+c_{12}-\frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,r\theta}}{r}$$
$$+\left(c_{66}+c_{22}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta}}{r^2}+\left(c_{23}+c_{44}-\frac{\sigma_{zz}^0}{2}\right)\frac{w_{1,\theta z}}{r}$$
$$+\frac{1}{2}\frac{d\sigma_{rr}^0}{dr}\left(v_{1,r}+\frac{v_1}{r}-\frac{u_{1,\theta}}{r}\right)=0. \quad (12b)$$

In a similar fashion, the third buckling Eq. (5*c*) gives

$$\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right)\left(w_{1,rr}+\frac{w_{1,r}}{r}\right)+\left(c_{44}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{w_{1,\theta\theta}}{r^2}$$
$$+c_{33}w_{1,zz}+\left(c_{13}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)u_{1,rz}+\left(c_{23}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,z}}{r}$$
$$+\left(c_{23}+c_{44}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta z}}{r}+\frac{1}{2}\frac{d\sigma_{rr}^0}{dr}(w_{1,r}-u_{1,z})=0, \quad (12c)$$

In the perturbed position, we seek equilibrium modes in the form

$$u_1(r,\theta,z)=U(r)\cos n\theta\sin\lambda z;$$
$$v_1(r,\theta,z)=V(r)\sin n\theta\sin\lambda z,$$
$$w_1(r,\theta,z)=W(r)\cos n\theta\cos\lambda z, \quad (13)$$

where the functions $U(r)$, $V(r)$, $W(r)$ are uniquely determined for a particular choice of n and λ.

Notice that these modes correspond to the condition of "simply supported" ends since u_1 varies as $\sin\lambda z$ and

$$u_1=\frac{d^2u_1}{dz^2}=0 \quad \text{at} \quad z=0,\ \ell.$$

Denote now $U^{(i)}(r)$, $V^{(i)}(r)$ and $W^{(i)}(r)$ the *i*th derivative of $U(r)$, $V(r)$, and $W(r)$, respectively, with the additional notation $U^{(0)}(r)=U(r)$, $V^{(0)}(r)=V(r)$ and $W^{(0)}(r)=W(r)$.

Substituting in (12*a*), we obtain the following linear homogeneous ordinary differential equation:

$$U(r)''c_{11}+U(r)'\frac{c_{11}}{r}$$
$$+U(r)\left[(b_{00}+b_{01}P)r^{-2}+b_{02}Pr^{k-3}+\right.$$
$$\left.+b_{03}Pr^{-k-3}+(b_{04}+b_{05}P)+b_{06}Pr^{k-1}+b_{07}Pr^{-k-1}\right]$$
$$+\sum_{i=0}^{1}V^{(i)}(r)\left[(d_{i0}+d_{i1}P)r^{i-2}\right.$$
$$\left.+d_{i2}Pr^{k-3+i}+d_{i3}Pr^{-k-3+i}\right]$$
$$+\sum_{i=0}^{1}W^{(i)}(r)\left[(f_{i0}+f_{i1}P)r^{i-1}\right.$$
$$\left.+f_{i2}Pr^{k-2+i}+f_{i3}Pr^{-k-2+i}\right]=0$$
$$R_1\le r\le R_2. \quad (14a)$$

The second differential Eq. (12*b*) gives

$$V(r)\left[(g_{04}+g_{05}P)+g_{06}Pr^{k-1}+g_{07}Pr^{-k-1}\right]$$
$$+\sum_{i=0}^{2}V^{(i)}(r)\left[(g_{i0}+g_{i1}P)r^{i-2}+g_{i2}Pr^{k-3+i}+g_{i3}Pr^{-k-3+i}\right]$$
$$+\sum_{i=0}^{1}U^{(i)}(r)\left[(h_{i0}+h_{i1}P)r^{i-2}\right.$$
$$\left.+h_{i2}Pr^{k-3+i}+h_{i3}Pr^{-k-3+i}\right]$$
$$+W(r)\left[(t_{00}+t_{01}P)r^{-1}+t_{02}Pr^{k-2}+t_{03}Pr^{-k-2}\right]=0$$
$$R_1\le r\le R_2. \quad (14b)$$

In a similar fashion, (12*c*) gives

$$W(r)q_{04}+\sum_{i=0}^{2}W^{(i)}(r)\left[(q_{i0}+q_{i1}P)r^{i-2}\right.$$
$$\left.+q_{i2}Pr^{k-3+i}+q_{i3}Pr^{-k-3+i}\right]$$
$$+\sum_{i=0}^{1}U^{(i)}(r)\left[(s_{i0}+s_{i1}P)r^{i-1}+s_{i2}Pr^{k-2+i}+s_{i3}Pr^{-k-2+i}\right]$$
$$+V(r)\left[(\beta_{00}+\beta_{01}P)r^{-1}+\beta_{02}Pr^{k-2}+\beta_{03}Pr^{-k-2}\right]=0$$
$$R_1\le r\le R_2. \quad (14c)$$

All the previous three Eqs. (14) are linear, homogeneous, ordinary differential equations of the second order for $U(r)$, $V(r)$ and $W(r)$. In these equations, the constants b_{ij}, d_{ij}, f_{ij}, g_{ij}, h_{ij}, t_{ij}, q_{ij}, s_{ij}, and β_{ij} are given in Appendix I and depend on the material stiffness coefficients c_{ij} and k as well as the buckling mode constants n and λ.

Now we proceed to the boundary conditions on the lateral surfaces $r=R_j$, $j=1, 2$. These will complete the formulation of the eigenvalue problem for the critical load.

From (7), we obtain for $\hat{l}=\pm 1$, $\hat{m}=\hat{n}=0$:

$$\sigma_{rr}'=0;\quad \tau_{r\theta}'+\sigma_{rr}^0\omega_z'=0;\quad \tau_{rz}'-\sigma_{rr}^0\omega_\theta'=0,$$
$$\text{at} \quad r=R_1, R_2. \quad (15)$$

Substituting in (8), (2), (13), and (10), the boundary condition $\sigma_{rr}'=0$ at $r=R_j$, $j=1, 2$ gives

$$U'(R_j)c_{11}+\left[U(R_j)+nV(R_j)\right]\frac{c_{12}}{R_j}-c_{13}\lambda W(R_j)=0,$$
$$j=1,2 \quad (16a)$$

The boundary condition $\tau_{r\theta}'+\sigma_{rr}^0\omega_z'=0$ at $r=R_j$, $j=1, 2$ gives

$$V'(R_j)\left[\left(c_{66}+\frac{C_0}{2}P\right)+\frac{C_1}{2}PR_j^{k-1}+\frac{C_2}{2}PR_j^{-k-1}\right]$$
$$+\left[V(R_j)+nU(R_j)\right]\left[\left(-c_{66}+\frac{C_0}{2}P\right)R_j^{-1}\right.$$
$$\left.+\frac{C_1}{2}PR_j^{k-2}+\frac{C_2}{2}PR_j^{-k-2}\right],\quad j=1,2. \quad (16b)$$

In a similar fashion, the condition $\tau'_{rz} - \sigma_{rr}^{0}\omega'_{\theta} = 0$ at $r = R_j$, $j = 1,2$, gives

$$\lambda U(R_j)\left[\left(c_{55} - \frac{C_0}{2}P\right) - \frac{C_1}{2}PR_j^{k-1} - \frac{C_2}{2}PR_j^{-k-1}\right] + $$

$$+ W'(R_j)\left[\left(c_{55} - \frac{C_0}{2}P\right) + \frac{C_1}{2}PR_j^{k-1} + \frac{C_2}{2}PR_j^{-k-1}\right], \quad j = 1,2. \quad (16c)$$

Equations (14) and (16) constitute an eigenvalue problem for differential equations, with the applied compressive load P the parameter, which can be solved by standard numerical methods (two-point boundary value problem).

Before discussing the numerical procedure used for solving this eigenvalue problem, one final point will be addressed. To completely satisfy all the elasticity requirements, we should discuss the boundary conditions at the ends. From (7), the boundary conditions on the ends are

$$\tau'_{rz} + \sigma_{zz}^{0}\omega'_{\theta} = 0; \quad \tau'_{\theta z} - \sigma_{zz}^{0}\omega'_{r} = 0; \quad \sigma'_{zz} = 0, \quad \text{at} \quad z = 0, \ell. \quad (17)$$

Since σ'_{zz} varies as $\sin \lambda z$, the condition $\sigma'_{zz} = 0$ on both the lower end $z = 0$, and the upper end $z = \ell$, is satisfied if

$$\lambda = \frac{m\pi}{\ell}. \quad (18)$$

In a cartesian coordinate system (x, y, z), the first two of the conditions in (17) can be written as follows:

$$\tau'_{xz} + \sigma_{zz}^{0}\omega'_{y} = 0; \quad \tau'_{yz} - \sigma_{xz}^{0}\omega'_{x} = 0. \quad (19)$$

It will be proved now that these remaining two conditions are satisfied on the average.

The lateral surface boundary conditions in the cartesian coordinate system (analogous to (7)), with $\hat{N}$ the normal to the circular contour are

$$\left(\sigma'_{xx} - \tau_{xy}^{0}\omega'_{z}\right)\cos(\hat{N}, x) + \left(\tau'_{xy} - \sigma_{yy}^{0}\omega'_{z}\right)\cos(\hat{N}, y) = 0, \quad (20a)$$

$$\left(\tau'_{xy} + \sigma_{xx}^{0}\omega'_{z}\right)\cos(\hat{N}, x) + \left(\sigma'_{yy} + \tau_{xy}^{0}\omega'_{z}\right)\cos(\hat{N}, y) = 0. \quad (20b)$$

Using the equilibrium equation in cartesian coordinates (analogous to (5)), gives

$$\frac{\partial}{\partial z}\iint_A\left(\tau'_{xz} + \sigma_{zz}^{0}\omega'_{y}\right)dA$$

$$= -\iint_A\left[\frac{\partial}{\partial x}\left(\sigma'_{xx} - \tau_{xy}^{0}\omega'_{z}\right) + \frac{\partial}{\partial y}\left(\tau'_{xy} - \sigma_{yy}^{0}\omega'_{z}\right)\right]dA. \quad (21a)$$

Using now the divergence theorem for transformation of an area integral into a contour integral, and the condition (20a) on the contour, gives the previous integral as

$$+\int_\gamma\left[\left(\sigma'_{xx} - \tau_{xy}^{0}\omega'_{z}\right)\cos(\hat{N}, x) + \left(\tau'_{xy} - \sigma_{yy}^{0}\omega'_{z}\right)\cos(\hat{N}, y)\right]ds = 0,$$

where A denotes the area of the annular cross-section and γ the corresponding contour.

Therefore

$$\iint_A\left(\tau'_{xz} + \sigma_{zz}^{0}\omega'_{y}\right)dA = \text{const.} \quad (21b)$$

Since based on the buckling modes (13) and (18), τ'_{rz}, ω'_{θ}, $\tau'_{\theta z}$, and ω'_{r} and hence τ'_{xz}, ω'_{y}, τ'_{yz}, and ω'_{x}, all have a $\cos(m\pi z/\ell)$ variation, they become zero at $z = \ell/(2m)$. Therefore, it is concluded that the constant in (21b) is zero. Similar arguments hold for τ'_{yz}.

Moreover, it can also be proved that the system of resultant stresses (19) would produce no torsional moment. Indeed,

$$\frac{\partial}{\partial z}\iint_A\left[x\left(\tau'_{yz} - \sigma_{zz}^{0}\omega'_{x}\right) - y\left(\tau'_{xz} + \sigma_{zz}^{0}\omega'_{y}\right)\right]dA$$

$$= -\iint_A\left\{x\left[\frac{\partial\left(\tau'_{xy} + \sigma_{xx}^{0}\omega'_{z}\right)}{\partial x} + \frac{\partial\left(\sigma'_{yy} + \tau_{xy}^{0}\omega'_{z}\right)}{\partial y}\right] - y\left[\frac{\partial\left(\sigma'_{xx} - \tau_{xy}^{0}\omega'_{z}\right)}{\partial x} + \frac{\partial\left(\tau'_{xy} - \sigma_{yy}^{0}\omega'_{z}\right)}{\partial y}\right]\right\}dA.$$

Again, using the divergence theorem, and taking into account (20), the previous integral becomes

$$-\int_\gamma\left\{x\left[\left(\tau'_{xy} + \sigma_{xx}^{0}\omega'_{z}\right)\cos(\hat{N}, x) + \left(\sigma'_{yy} + \tau_{xy}^{0}\omega'_{z}\right)\cos(\hat{N}, y)\right] - y\left[\left(\sigma'_{xx} - \tau_{xy}^{0}\omega'_{z}\right)\cos(\hat{N}, x) + \left(\tau'_{xy} - \sigma_{yy}^{0}\omega'_{z}\right)\cos(\hat{N}, y)\right]\right\}ds = 0, \quad (22a)$$

hence

$$\iint_A\left[x\left(\tau'_{yz} - \sigma_{zz}^{0}\omega'_{x}\right) - y\left(\tau'_{xz} + \sigma_{zz}^{0}\omega'_{y}\right)\right]dA = \text{const}, \quad (22b)$$

and this constant is again zero since $\tau'_{xz} = \tau'_{yz} = \omega'_{x} = \omega'_{y} = 0$ at $z = \ell/(2m)$.

As has already been stated, Eqs. (14) and (16) constitute an eigenvalue problem for ordinary second-order linear differential equations in the r variable, with the applied compressive load P the parameter. This is essentially a standard two-point boundary value problem. The relaxation method was used (Press et al., 1989) which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the shell. For this purpose, an equally spaced mesh of 241 points was employed and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the shell theory solution was used. An investigation of the convergence showed that essentially the same results were produced with even three times as many mesh points. Finding the critical load involves a minimization step in the sense that the eigenvalue is obtained for different combinations of n, m, and the critical load is the minimum. These results are discussed in the following.

Discussion of Results. Results for the critical compressive load, normalized as

$$\hat{P} = \frac{P}{\pi\left(R_2^2 - R_1^2\right)}\frac{R_2}{E_3 h},$$

were produced for a typical glass/epoxy material with moduli in GN/m^2 and Poisson's ratios listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$. It has been assumed that the reinforcing direction is along the periphery.

In the shell theory solutions, the radial displacement is constant through the thickness and the axial and circumferential ones have a linear variation, i.e., they are in the form

Table 1 Comparison with shell theories

Orthotropic with circumferential reinforcement, $\ell/R_2 = 5$

Critical Loads, $\hat{P} = \frac{P}{\pi(R_2^2 - R_1^2)} \frac{R_2}{E_2 h}$

Moduli in GN/m²: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$
Poisson's ration: $\nu_{12} = 0.066$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$

R_2/R_1	Elasticity (n,m)	Donnell Shell† (n,m) (% Increase)	Timoshenko Shell† (n,m) (% Increase)
1.05	0.6764 (2,1)	0.7904 (4,9) (16.9%)	0.6735 (2,1) (-0.4%)
1.10	0.6641 (2,2)	0.7883 (3,6) (18.7%)	0.6461 (2,2) (-2.7%)
1.15	0.6284 (2,2)	0.7716 (2,3) (22.8%)	0.6218 (2,3) (-1.1%)
1.20	0.6134 (2,3)	0.7505 (2,3) (22.4%)	0.5559 (1,1) (-9.4%)
1.25	0.5186 (1,1)	0.7560 (2,4) (45.8%)	0.4549 (1,1) (-12.3%)
1.30	0.4429 (1,1)	0.7771 (1,1) (75.5%)	0.3876 (1,1) (-12.5%)

† See Appendix II

Table 2 Comparison with shell theories

Isotropic, $E = 14$ GN/m², $\nu = 0.3$, $\ell/R_2 = 5$

Critical Loads, $\hat{P} = \frac{P}{\pi(R_2^2 - R_1^2)} \frac{R_2}{E_2 h}$

R_2/R_1	Elasticity (n,m)	Donnell† (n,m) % Increase	Timoshenko† (n,m) % Increase	Flugge‡ (n,m) % Increase	Danielson‡ & Simmonds (n,m) % Increase
1.05	0.4426 (2,1)	0.5474 (2,1) 23.7%	0.4346 (2,1) -1.8%	0.4525 (2,1) 2.2%	0.4559 (2,1) 3.0%
1.10	0.3910 (2,1)	0.4871 (2,1) 24.6%	0.3863 (2,1) -1.2%	0.4019 (2,1) 2.8%	0.4088 (2,1) 4.6%
1.15	0.4547 (2,1)	0.5488 (2,2) 20.7%	0.4373 (2,2) -3.8%	0.4710 (2,1) 3.6%	0.4814 (2,1) 5.9%
1.20	0.4371 (2,2)	0.5272 (2,2) 20.6%	0.4184 (2,2) -4.3%	0.4620 (2,2) 5.7%	0.4705 (2,2) 7.6%
1.25	0.4426 (2,2)	0.5403 (2,2) 22.0%	0.4269 (2,2) -3.5%	0.4726 (2,2) 6.8%	0.4837 (2,2) 9.3%
1.30	0.4487 (1,1)	0.5709 (2,2) 27.2%	0.3895 (1,1) -13.2%	0.4915 (1,1) 9.5%	0.4967 (1,1) 11.1%

† See Appendix II
‡ From equations (24).

$$u_1(r,\theta,z) = U_0 \cos n\theta \sin \lambda z,$$

$$v_1(r,\theta,z) = \left[V_0 + \frac{r-R}{R}(V_0 + nU_0) \right] \sin n\theta \sin \lambda z. \quad (23a)$$

$$w_1(r,\theta,z) = \left[W_0 - (r-R)\lambda U_0 \right] \cos n\theta \cos \lambda z \quad (23b)$$

where U_0, V_0, W_0 are constants (these displacement field variations would satisfy the classical assumptions of $e_{rr} = e_{r\theta} = e_{rz} = 0$).

A distinct eigenvalue corresponds to each pair of the positive integers m and n. The pair corresponding to the smallest eigenvalue can be determined by trial. It should be noted that for isotropic material, some additional shallowness assumptions lead to the well known direct and simple formula: $P_{cr} = E\pi h^2/\sqrt{3(1-\nu^2)}$; the performance of this formula with moderate thickness in isotropic shells was discussed in Kardomateas (1993b).

As noted in the Introduction, there are two sets of the Donnell equations that are most widely used for shell theory solutions. The original first set has been referred as the "shallow" shell formulation, whereas, a second, more accurate set of cylindrical shell equations that are not subject to some of the shallowness limitations of the first set has been referred as the "nonshallow" formulation. The latter has been also called the "nonsimplified" Donnel theory in Kardomateas (1993b). The other benchmark shell theory used in this paper is the one described in Timoshenko and Gere (1961). In this theory, an additional term in the circumferential displacement part of the second equation is included. This additional term is the $RN^0_{z\upsilon,zz} = -P^0\upsilon_{,zz}/2\pi$ where P^0 is the absolute value of the compressive load at the critical point. In the comparison studies we have used an extension of the original, isotropic Donnell and Timoshenko formulations for the case of orthotropic material. The linear algebraic equations for the eigenvalues of both the Donnell and Timoshenko theories are given in Appendix II.

Concerning the present elasticity formulation, the critical load is obtained by finding the solution P for a range of n and m, and keeping the minimum value. Table 1 shows the critical load and the corresponding n, m, as predicted by the present three-dimensional elasticity formulation, and the critical load and the corresponding n, m, as predicted by both the "nonshallow" Donnell and Timoshenko shell equations. A length ratio $\ell/R_2 = 5$ has been assumed. A range of outside versus inside radius, R_2/R_1 from somewhat thin, 1.05, to thick, 1.30, is examined.

Tables 1 and 2 give the predictions of the Donnell and Timoshenko shell theories for the orthotropic and isotropic material, respectively, in comparison with the elasticity one. It is clearly seen that

(1) the bifurcation points from the Timoshenko formulation are always closer to the elasticity predictions than the ones from the Donnell formulation.

(2) For both the orthotropic material cases and the isotropic one, the Timoshenko bifurcation point for the Donnell shell theory, is always higher than the elasticity solution, which means that the Donnell formulation is nonconservative. Moreover, the Donnell theory becomes in general more nonconservative with thicker construction.

(3) On the contrary, the Timoshenko bifurcation point is *lower* than the elasticity one in all cases considered, i.e., the Timoshenko formulation is actually conservative in predicting stability loss. The degree of conservatism of the Timoshenko theory generally increases for thicker shells.

Furthermore, the bifurcation load for the isotropic case (Table 2) is smaller than the corresponding one for the circumferentially reinforced orthotropic case (Table 1), the difference becoming increasingly smaller for thicker construction. This conclusion is true for either the elasticity or the shell theory results (with one exception: for $R_2/R_1 = 1.30$ the Timoshenko prediction is larger for the isotropic case by 1.3 percent). More specifically, based on the elasticity solution, for $R_2/R_1 = 1.10$, the orthotropic case shows a 70 percent higher bifurcation load than the isotropic material, whereas for $R_2/R_1 = 1.25$, the orthotropic material shows only a 17 percent higher bifurcation load than the isotropic case. Therefore, the effect of the circumferential reinforcement in raising the critical load relative to the isotropic case is diminished with thicker construction.

For isotropic materials, two other shell theories, namely the Flügge (1960) and the Danielson and Simmonds (1969) can easily produce results for the critical loads in shells and should, therefore, be compared with the present elasticity solution. The expression for the eigenvalues derived from the Flügge (1960) equations, P_F^0 and the more simplified but just as accurate one by Danielson and Simmonds (1969), P_{DS}^0 are

$$P^0_{(F,DS)} = E\frac{Q_{F,DS}}{\bar{m}^2\left[(\bar{m}^2+n^2)^2+n^2\right]}, \qquad (24a)$$

where the numerator for the Flügge theory is

$$Q_F = \frac{\pi h^3}{6R(1-\nu^2)}\left\{(\bar{m}^2+n^2)^4 - 2\left[\nu\bar{m}^2 + 3\bar{m}^4n^2 + (4-\nu)\bar{m}^2n^4 + n^6\right] + 2(2-\nu)\bar{m}^2n^2 + n^4\right\} + \bar{m}^4, \qquad (24b)$$

and for the Danielson and Simmonds equations,

$$Q_{DS} = \frac{\pi h^3}{6R(1-\nu^2)}(\bar{m}^2+n^2)^2(\bar{m}^2+n^2-1)^2 + \bar{m}^4. \qquad (24c)$$

where R is the mean radius and h the shell thickness Again, a distinct eigenvalue corresponds to each pair of the positive integers m and n, the critical load being for the pair that renders the lowest eigenvalue.

A comparison of the data in Table 2 shows that the values of n, m at the critical point for the elasticity, as well as the

Table 3 Comparison with shell theories

Orthotropic with axial reinforcement, $\ell/R_2 = 5$

Critical Loads, $\hat{P} = \frac{P}{\pi(R_2^2 - R_1^2)}\frac{R_2}{E_2 h}$

Moduli in GN/m²: $E_3 = 57$, $E_2 = E_1 = 14$, $G_{31} = G_{23} = 5.7$, $G_{12} = 5.0$
Poisson's ratios: $\nu_{13} = 0.400$, $\nu_{23} = 0.068$, $\nu_{31} = 0.277$

R_2/R_1	Elasticity (n,m)	Donnell Shell† (n,m) % Increase	Timoshenko Shell† (n,m) % Increase
1.05	0.7666 (4,4)	0.7913 (4,4) (3.2%)	0.7517 (4,4) (-1.9%)
1.10	0.6794 (2,1)	0.7879 (3,3) (16.0%)	0.6473 (2,1) (-4.7%)
1.15	0.6575 (2,1)	0.7877 (2,1) (19.8%)	0.6287 (2,1) (-4.4%)
1.20	0.6686 (2,2)	0.7547 (2,2) (12.9%)	0.6157 (2,2) (-7.9%)
1.25	0.6646 (2,2)	0.7563 (2,2) (13.8%)	0.6140 (2,2) (-7.6%)
1.30	0.6801 (2,2)	0.7823 (2,2) (15.0%)	0.6319 (2,2) (-7.1%)

† See Appendix II

Table 4 Results for thin shells

Critical Loads, $\hat{P} = \frac{P}{\pi(R_2^2 - R_1^2)}\frac{R_2}{E_3 h}$

1. Orthotropic with circumferential reinforcement

R_2/R_1	Elasticity (n,m)	Donnell Shell† (n,m) (% Increase)	Timoshenko Shell† (n,m) (% Increase)
1.04	0.6872 (2,1)	0.8049 (4,8) (17.1%)	0.6811 (2,1) (-0.9%)
1.02	0.7822 (6,13)	0.7957 (6,13) (1.7%)	0.7786 (6,13) (-0.5%)
1.01	0.7904 (9,20)	0.7971 (9,20) (0.9%)	0.7895 (9,20) (-0.1%)

2. Isotropic

R_2/R_1	Elasticity (n,m)	Donnell† (n,m) % Increase	Timoshenko† (n,m) % Increase	Flügge‡ (n,m) % Increase	Danielson‡ & Simmonds (n,m) % Increase
1.04	0.5034 (2,1)	0.5723 (3,2) 13.7%	0.4940 (2,1) -1.8%	0.5143 (2,1) 2.2%	0.5170 (2,1) 2.7%
1.02	0.4999 (3,1)	0.5548 (3,1) 11.0%	0.4983 (3,1) -0.3%	0.5033 (3,1) 0.7%	0.5052 (3,1) 1.1%
1.01	0.5517 (3,1)	0.5977 (7,5) 8.3%	0.5493 (3,1) -0.4%	0.5549 (3,1) 0.6%	0.5559 (3,1) 0.8%

† See Appendix II
‡ From equations (24).

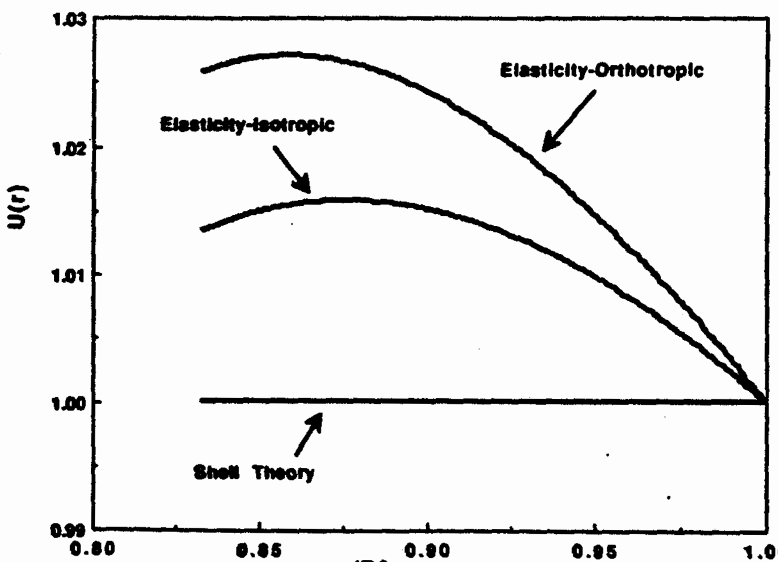

Fig. 2(a) "Eigenfunction" $U(r)$ versus normalized radial distance r/R_2, for the orthotropic with circumferential reinforcing direction case and the isotropic one (shell theory would have a constant value throughout, $U(r) = 1$ for both cases)

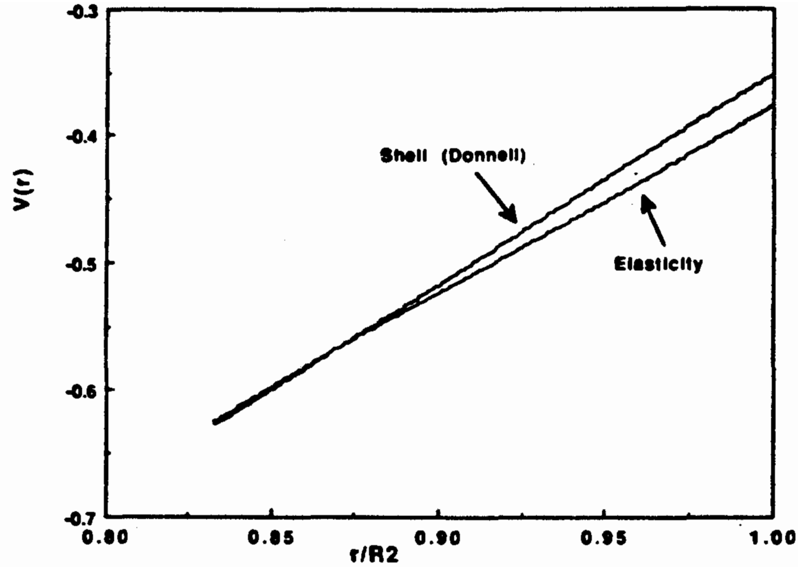

Fig. 2(*b*) "Eigenfunction" $V(r)$ versus normalized radial distance r/R_2 from the elasticity solution and the Donnell shell theory, which would show linear variation. The results are for the orthotropic with circumferential reinforcing direction case.

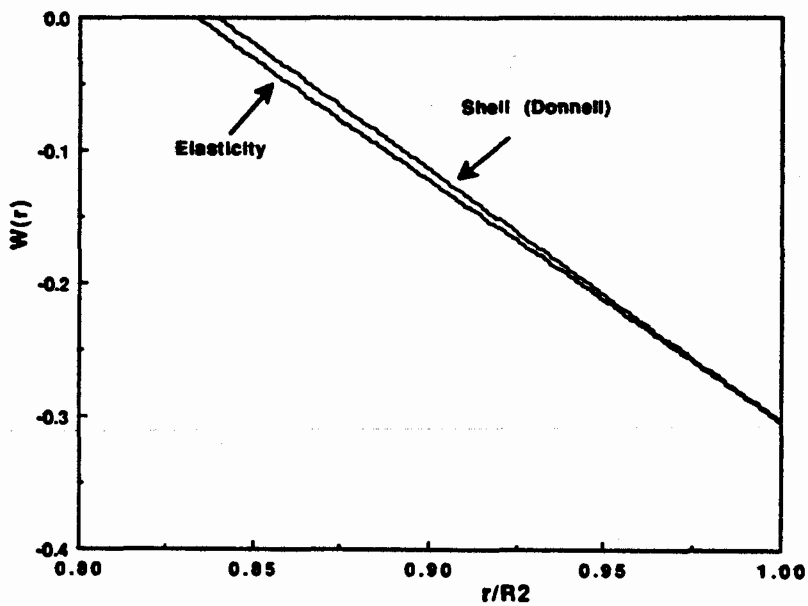

Fig. 2(*c*) "Eigenfunction" $W(r)$ versus normalized radial distance r/R_2, from the elasticity solution and the Donnell shell theory (the latter has a linear variation). The results are for the orthotropic with circumferential reinforcing direction case.

Flügge and the Danielson and Simmonds theories show perfect agreement, and that both Flügge, and the Danielson and Simmonds theories are nonconservative, the degree of nonconservatism increasing with thicker shells. We may now rank these theories for isotropic materials by concluding that the best estimates are provided by the Timoshenko theory, followed by the Flügge and the Danielson and Simmonds theories and finally the Donnell theory. Of these, only the Timoshenko theory is conservative.

Table 3 presents the results for the bifurcation load in the case of the same orthotropic material (typical of glass/epoxy), which is now positioned so that the reinforcement is axial. To be able to perform direct comparisons, the load has now been normalized with E_2, which is the same as E_3 in the other two cases (Tables 1, 2). It can be seen that the bifurcation load now is in general higher than both the isotropic and the orthotropic with circumferential reinforcement cases. Again, based on the elasticity solution, for $R_2/R_1 = 1.10$, the axially reinforced case shows a 74 percent higher bifurcation load than the isotropic material, whereas for $R_2/R_1 = 1.25$, the axially reinforced material shows a 49 percent higher bifurcation load than the isotropic case. Therefore, the effect of the axial reinforcement in raising the critical load relative to the isotropic case is much less sensitive to the thickness than with circumferential reinforcement. Another interesting observation is that in all cases, n, m at the critical load for the elasticity theory are always less or equal to the corresponding values of the Donnell shell theory.

It should also be mentioned that the elasticity results of Table 2 for isotropic material that were produced through the present formulation, confirm the results from the closed-form analytical isotropic solution of Kardomateas (1993b). Moreover, this work complements the latter by including a comparison with the Timoshenko and Gere shell theory, which is actually found to be the only shell theory that results in conservative estimates of the critical load.

Although the focus of this work is the study of moderately thick shells, one would expect buckling to be even more important for very thin shell construction. Therefore, Table 4 shows the bifurcation load from the three-dimensional elasticity analysis for thin shells in order of decreasing thickness (thickness over mean radius, h/R, up to 1/100), in comparison with these shell theories. The results are for the mildly orthotropic glass/epoxy material, as well as the isotropic case. In all cases, it is seen that the Timoshenko theory renders conservative estimates for the critical load, and it is again much more closer to the elasticity prediction than the Donnell theory. Moreover, the values of (n, m) at the critical point for both the elasticity and the Timoshenko theory agree perfectly for the thin shells of Table 4, unlike the Donnell theory. For the isotropic material, the Flügge and Danielson and Simmonds theories have also been examined and are shown to provide much better (although nonconservative) estimates than the Donnell theory, with perfect agreement with the elasticity results on the values of (n, m) at the critical point.

Finally, to obtain more insight into the displacement field, Figs. 2(a,b,c) show the variation of $U(r)$, $V(r)$, and $W(r)$, which define the eigenfunctions, for $R_2/R_1 = 1.2$, as derived from the present elasticity solution, and in comparison with the Donnell shell theory assumptions of constant $U(r)$, and linear $V(r)$ and $W(r)$. These values have been normalized by assigning a unit value for U at the outside boundary $r = R_2$. These plots illustrate graphically the deviation of U from constant and the deviation of V and W from linearity. Although the Donnell shell theory eigenfunction has been plotted for $V(r)$ and $W(r)$, the Timoshenko theory line would nearly coincide with the latter.

Acknowledgment

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References

Bradford, L. G., and Dong, S. B., 1978, "Natural Vibrations of Orthotropic Cylinders Under Initial Stress," *Journal of Sound and Vibration*, Vol. 60, No. 2, pp. 157–175.

Brush, D. O., and Almroth, B. O., 1975, *Buckling of Bars, Plates, and Shells*, McGraw-Hill, New York.

Danielson, D. A., and Simmonds, J. G., 1969, "Accurate Buckling Equations for Arbitrary and Cylindrical Elastic Shells," *Int. J. Eng. Sci.*, Vol. 7, pp. 459–468.

Dong, S. B., 1994, discussion of the two papers by Kardomateas (1993a,b), ASME JOURNAL OF APPLIED MECHANICS, Vol. 61, p. 225.

Donnell, L. H., 1933, "Stability of Thin-Walled Tubes Under Torsion," NACA Rep. 479.

Flügge, W., 1960, *Stresses in Shells*, Springer, Berlin, pp. 426–432.

Kardomateas, G. A., 1993a, "Buckling of Thick Orthotropic Cylindrical Shells Under External Pressure," ASME JOURNAL OF APPLIED MECHANICS, Vol. 60, pp. 195–202.

Kardomateas, G. A., 1993b, "Stability Loss in Thick Transversely Isotropic Cylindrical Shells Under Axial Compression," ASME JOURNAL OF APPLIED MECHANICS, Vol. 60, pp. 506–513.

Kollár, L. P., 1994, "Three Dimensional Analysis of Composite Cylinders Under Axially Varying Hygrothermal and Mechanical Loads," *Computers and Structures*, Vol. 17, No. 50-4, pp. 525–540.

Lekhnitskii, S. G., 1963, *Theory of Elasticity of an Anisotropic Elastic Body*, Holden Day, San Francisco; also Mir Publishers, Moscow, 1981.

Librescu, L., 1975, *Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures*, Nordhoff International, Leyden.

Noor, A. K., and Burton, W. S., 1990, "Assessment of Computational Models for Multilayered Composite Shells," ASME *Appl. Mech. Rev.*, Vol. 43, No. 4, pp. 67–97.

Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., 1989, *Numerical Recipes*, Cambridge University Press, Cambridge, U.K.

Reddy, J. N., and Liu, C. F., 1985, "A Higher-Order Shear Deformation Theory of Laminated Elastic Shells," *Int. J. Eng. Sci.*, Vol. 23, No. 3, pp. 319–330.

Simitses, G. J., Shaw, D., and Sheinman, I., 1985, "Stability of Cylindrical Shells by Various Nonlinear Shell Theories," *ZAMM, Z. Angew. Math. u. Mech.*, Vol. 65, No. 3, pp. 159–166.

Timoshenko, S. P., and Gere, J. M., 1961, *Theory of Elastic Stability*, McGraw-Hill, New York.

Whitney, J. M., and Sun, C. T., 1974, "A Refined Theory for Laminated Anisotropic Cylindrical Shells," ASME JOURNAL OF APPLIED MECHANICS, Vol. 41, No. 2, pp. 471–476.

APPENDIX I

For convenience define

$$D_0 = -\frac{1}{\bar{T}} - C_0\frac{a_{13} + a_{23}}{a_{33}}. \quad \text{(A1)}$$

$$D_1 = -C_1\frac{a_{13} + ka_{23}}{a_{33}}; \quad D_2 = -C_2\frac{a_{13} - ka_{23}}{a_{33}}. \quad \text{(A2)}$$

The coefficients of the first differential Eq. (14a) are

$$b_{00} = -(c_{22} + c_{66}n^2); \quad b_{01} = -n^2C_0/2.0;$$
$$b_{02} = -C_1kn^2/2; \quad b_{03} = C_2kn^2/2;$$
$$b_{04} = -c_{55}\lambda^2; \quad b_{05} = -D_0\lambda^2/2;$$
$$b_{06} = -D_1\lambda^2/2; \quad b_{07} = -D_2\lambda^2/2, \quad \text{(A3)}$$

$$d_{10} = n(c_{12} + c_{66}); \quad d_{11} = -nC_0/2; \quad d_{12} = -nkC_1/2;$$
$$d_{13} = nkC_2/2; \quad d_{00} = -n(c_{22} + c_{66}); \quad d_{01} = -nC_0/2;$$
$$d_{02} = -nkC_1/2; \quad d_{03} = nkC_2/2. \quad \text{(A4)}$$

$$f_{10} = -\lambda(c_{13} + c_{55}); \quad f_{11} = \lambda D_0/2; \quad f_{12} = \lambda D_1/2;$$
$$f_{13} = \lambda D_2/2; \quad f_{00} = \lambda(c_{23} - c_{13}); \quad f_{01} = f_{02} = f_{03} = 0. \quad \text{(A5)}$$

The coefficients of the second differential Eq. (14b) are given as follows:

$$g_{20} = c_{66}; \quad g_{21} = C_0/2; \quad g_{22} = C_1/2; \quad g_{23} = C_2/2$$
$$g_{10} = c_{66}; \quad g_{11} = C_0/2; \quad g_{12} = C_1/2; \quad g_{13} = C_2/2$$
$$g_{00} = -(c_{22}n^2 + c_{66}); \quad g_{01} = -C_0/2; \quad g_{02} = -C_1/2;$$
$$g_{03} = -C_2/2; \quad g_{04} = -c_{44}\lambda^2; \quad g_{05} = -D_0\lambda^2/2;$$
$$g_{06} = -D_1\lambda^2/2; \quad g_{07} = -D_2\lambda^2/2, \quad \text{(A6)}$$

$$h_{10} = -(c_{66} + c_{12})n; \quad h_{11} = nC_0/2; \quad h_{12} = nC_1/2;$$
$$h_{13} = nC_2/2; \quad h_{00} = -(c_{22} + c_{66})n; \quad h_{01} = -nC_0/2;$$
$$h_{02} = -nC_1/2; \quad h_{03} = -nC_2/2, \quad \text{(A7)}$$

$$t_{00} = (c_{23} + c_{44})n\lambda; \quad t_{01} = -n\lambda D_0/2;$$
$$t_{02} = -n\lambda D_1/2; \quad t_{03} = -n\lambda D_2/2. \quad \text{(A8)}$$

Finally, the coefficients of the third differential Eq. (14c) are

$$q_{20} = c_{55}; \quad q_{21} = C_0/2; \quad q_{22} = C_1/2; \quad q_{23} = C_2/2$$
$$q_{10} = c_{55}; \quad q_{11} = C_0/2; \quad q_{12} = kC_1/2; \quad q_{13} = -kC_2/2$$
$$q_{00} = -c_{44}n^2; \quad q_{01} = -C_0n^2/2; \quad q_{02} = -kn^2C_1/2;$$
$$q_{03} = kn^2C_2/2; \quad q_{04} = -c_{33}\lambda^2, \quad \text{(A9)}$$

$$s_{10} = (c_{55} + c_{13})\lambda; \quad s_{11} = -\lambda C_0/2; \quad s_{12} = -\lambda C_1/2;$$
$$s_{13} = -\lambda C_2/2; \quad s_{00} = (c_{23} + c_{55})\lambda; \quad s_{01} = -\lambda C_0/2;$$
$$s_{02} = -k\lambda C_1/2; \quad s_{03} = k\lambda C_2/2, \quad \text{(A10)}$$

$$\beta_{00} = (c_{23} + c_{44})n\lambda; \quad \beta_{01} = -n\lambda C_0/2;$$

$$\beta_{02} = -kn\lambda C_1/2; \quad \beta_{03} = kn\lambda C_2/2. \qquad \text{(A11)}$$

APPENDIX II

Eigenvalues From Nonshallow Donnell and Timoshenko Shell Theories

In the shell theory formulation, the displacements are in the form

$$u_1 = U_0 \cos n\theta \sin \lambda z, \quad v_1 = V_0 \sin n\theta \sin \lambda z,$$

$$w_1 = W_0 \cos n\theta \cos \lambda z,$$

where U_0, V_0, W_0 are constants.

The equations for the nonshallow (or nonsimplified) Donnell shell theory for $N_\theta^0 = N_{z\theta}^0 = 0$, $N_z^0 = -P^0/(2\pi R)$ are (Brush and Almroth, 1975)

$$RN_{z,z} + N_{z\theta,\theta} = 0$$

$$RN_{z\theta,z} + N_{\theta,\theta} + \frac{M_{\theta,\theta}}{R} + M_{z\theta,z} = 0$$

$$N_\theta - RN_z^0 u_{,zz} - RM_{z,zz} - \frac{M_{\theta,\theta\theta}}{R} - 2M_{z\theta,z\theta} = 0.$$

The Timoshenko shell theory has the additional term $RN_z^0 v_{,zz}$ in the second equation. We have denoted by R the mean shell radius and by P^0 the absolute value of the compressive load.

In terms of the "equivalent property" constants

$$C_{22} = E_2 h/(1 - \nu_{23}\nu_{32}); \quad C_{33} = E_3 h/(1 - \nu_{23}\nu_{32})$$

$$C_{23} = \frac{E_3 \nu_{23} h}{1 - \nu_{23}\nu_{32}}; \quad C_{44} = G_{23} h, \quad D_{ij} = C_{ij}\frac{h^2}{12},$$

the coefficient terms in the homogeneous equations system that gives the eigenvalues are

$$\alpha_{11} = C_{23}\lambda; \quad \alpha_{12} = (C_{23} + C_{44})n\lambda;$$

$$\alpha_{13} = -(C_{33}R\lambda^2 + C_{44}n^2/R),$$

$$\alpha_{21} = -\left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 2\frac{D_{44}\lambda^2}{R}\right)n,$$

$$\alpha_{22} = -\left(\frac{C_{22}n^2}{R} + C_{44}R\lambda^2 + \frac{D_{22}n^2}{R^3} + 2\frac{D_{44}\lambda^2}{R}\right),$$

$$\alpha_{23} = (C_{23} + C_{44})n\lambda,$$

$$\alpha_{31} = \frac{C_{22}}{R} + \frac{D_{22}n^4}{R^3} + 2\frac{D_{23}\lambda^2 n^2}{R} + D_{33}\lambda^4 R + 4\frac{D_{44}\lambda^2 n^2}{R},$$

$$\alpha_{32} = \left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 4\frac{D_{44}\lambda^2}{R}\right)n,$$

$$\alpha_{33} = -C_{23}\lambda.$$

Notice that in the above formulas we have used the curvature expression $\kappa_{z\theta} = (v_{,z} - u_{,z\theta})/R$ for both theories.

Then the linear homogeneous equations system that gives the eigenvalues for the Timoshenko shell formulation is

$$\alpha_{11}U_0 + \alpha_{12}V_0 + \alpha_{13}W_0 = 0, \qquad \text{(B1)}$$

$$\alpha_{21}U_0 + \left(\alpha_{22} + \frac{\lambda^2}{2\pi}P^0\right)V_0 + \alpha_{23}W_0 = 0, \qquad \text{(B2)}$$

$$\left(\alpha_{31} - \frac{\lambda^2}{2\pi}P^0\right)U_0 + \alpha_{32}V_0 + \alpha_{33}W_0 = 0. \qquad \text{(B3)}$$

For the Donnell shell formulation, the additional term in the coefficient of V_0 in (B2) is omitted, i.e., the coefficient of V_0 is only α_{22}. The eigenvalues are naturally found by equating to zero the determinant of the coefficients of U_0, V_0, and W_0.