Pergamon

Mechanics Research Communications, Vol. 21, No. 5, pp. 483-492, 1994

0093-6413/94 $6.00 + .00

0093-6413(94)00010-7

ON THE FULLY PLASTIC FLOW PAST A GROWING ASYMMETRIC CRACK AND ITS RELATION TO MACHINING MECHANICS

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(Received 10 February 1994; accepted for print 22 March 1994)

Introduction

Fully plastic flow before fracture is desirable even in structures containing cracks. Such ductility is reduced if plastic flow is limited to one shear band, by a weld, for example [1]. Nonhardening plasticity gives a shear band of infinitesimal thickness. Strain hardening, however, causes the deformation field to fan out, leaving a finite strain except possibly at the crack tip.

In orthogonal machining the geometry is similar, with the cutting tool progressing steadily below the plastic zone. Here, again due to strain hardening, the plastic zone fans out over 10^0-30^0 as opposed to the single plane required by the perfectly plastic solid [2]. Both cases involve plane strain flow.

In the following we shall provide a solution and discuss its implications for steady flow of a rigid plastic solid in the vicinity of a crack tip or a tool tip with assumed straight flanks of finite angle.

The assumption of rigid flanks and the associated stress singularity

Let us postulate a steady flow in rigid-plastic, linearly strain hardening material. The mechanics of the problem should eventually answer the question as to whether the crack tip has a finite angle. Let us start by assuming a crack of finite angle, ω, and rigid body velocity of the material flowing past the flanks. To satisfy incompressibility assume a stream function, ψ, in polar coordinates, r and θ. We seek the form of the stream function in the immediate vicinity of the crack tip where the velocities should be nonzero and finite. From an asymptotic expansion of ψ in the dominant term as $r \to 0$, $r^s F(\theta)$, it is seen that the exponent s must be unity since the velocities at the tip are nonzero and finite. Thus,

$$\psi = rF(\theta) . \tag{1}$$

The corresponding velocities are:

$$\dot{u}_r = \frac{1}{r}\frac{\partial\psi}{\partial r} = F'(\theta) ; \quad \dot{u}_\theta = -\frac{\partial\psi}{\partial r} = -F(\theta) . \tag{2}$$

The strain rates are:

$$\dot{\epsilon}_r = \frac{\partial \dot{u}_r}{\partial r} = 0 = -\dot{\epsilon}_\theta , \tag{3}$$

$$\dot{\gamma}_{r\theta} = \frac{\partial \dot{u}_\theta}{\partial r} + \frac{1}{r}\frac{\partial \dot{u}_r}{\partial \theta} - \frac{\dot{u}_\theta}{r} = \frac{F''(\theta) + F(\theta)}{r} . \tag{4}$$

Thus the only component of strain is shear. The equivalent strain rate is:

$$\dot{\bar{\epsilon}} = \sqrt{\frac{2}{3}\left[\dot{\epsilon}_r^2 + \dot{\epsilon}_\theta^2 + 2\left(\frac{\dot{\gamma}_{r\theta}}{2}\right)^2\right]} = \frac{F''(\theta) + F(\theta)}{r\sqrt{3}} . \tag{5}$$

The stress deviators s_{ij} are found from the stress-strain relations and the equivalent stress $\bar{\sigma}$:

$$\dot{\epsilon}_{ij} = \frac{3}{2}\frac{s_{ij}}{\bar{\sigma}}\dot{\bar{\epsilon}} . \tag{6}$$

Since $\dot{\epsilon}_r = \dot{\epsilon}_\theta = 0$ from (3),

$$s_r = s_\theta = 0 \quad \text{and} \quad | s_{r\theta} | = \bar{\sigma}/\sqrt{3} . \tag{7}$$

The material is rigid-plastic, linearly strain hardening, hence:

$$\bar{\sigma} = Y + H\bar{\epsilon} . \tag{8}$$

The accumulated equivalent strain is calculated by integration along a streamline, where the time increment is expressed in terms of that required for an element to traverse an increment of angle:

$$\bar{\epsilon} = \int_{-\infty}^{t} \dot{\bar{\epsilon}} dt = \int \frac{\dot{\bar{\epsilon}}}{\dot{u}_\theta} r d\theta = -\int_0^\theta \frac{| F''(\theta) + F(\theta) |}{\sqrt{3}F(\theta)} d\theta . \tag{9}$$

Thus the equivalent strain is independent of radius. The same holds for the equivalent stress, $\bar{\sigma}$, by (8), and also for the shear stress, by (7), i.e.

$$s_{r\theta} = s_{r\theta}(\theta) \ . \tag{10}$$

Now, let us turn to the equilibrium equations. In terms of the mean normal stress, σ,

$$\frac{\partial \sigma}{\partial r} + \frac{\partial s_r}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{s_r - s_\theta}{r} = 0 \ , \tag{11}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma}{\partial \theta} + \frac{1}{r}\frac{\partial s_\theta}{\partial \theta} + \frac{2 s_{r\theta}}{r} = 0 \ . \tag{12}$$

Introducing (7) to eliminate s_r, s_θ, (10) to eliminate $\partial s_{r\theta}/\partial r$, and using cross-differentiation to eliminate σ, leads to:

$$\frac{d^2 s_{r\theta}}{d\theta^2} = 0 \ , \quad \text{from which} \quad \frac{d s_{r\theta}}{d\theta} = \text{const.} = D \ . \tag{13}$$

Now, (11) simplifies with (7), and after substituting (13), can be integrated as follows:

$$\frac{\partial \sigma}{\partial r} + \frac{1}{r}\frac{d s_{r\theta}}{d\theta} = 0 \ , \quad \sigma = -D \ln(r/R) + C(\theta) \ . \tag{14}$$

Differentiating (12) with $s_\theta = 0$, and noting that $s_{r\theta} \neq f(r)$ from (10), gives:

$$\frac{\partial^2 \sigma}{\partial \theta^2} = -2\frac{d s_{r\theta}}{d\theta} \ , \quad C(\theta) = -s_{r\theta,\theta}\theta^2 + C_1\theta + C_2 \ . \tag{15}$$

Let us define $\sigma(R, 0)$ as the mean normal stress at a convenient radius R and $\theta = 0$. Then, equation (14) becomes:

$$\sigma(r,\theta) - \sigma(R,0) = -s_{r\theta,\theta}\left[\ln(r/R) + \theta^2\right] \ . \tag{16}$$

Thus, the mean normal stress at the crack tip ($r \to 0$) appears to have a logarithmic singularity. Let us now complete the study of the field specified by (1) for flow past rigid flanks of a finite angle by applying the boundary conditions and deriving its streamlines. Two possible flow fields are consistent with the constant rate of shear stress from (13) and the hardening of the material (increase in equivalent stress $\bar{\sigma}$ from (7) as it flows along the streamline). The first field, shown in Fig. 1a, is for $s_{r\theta,\theta} > 0$. From (16) this model gives a tensile logarithmic singularity in the mean normal stress as $r \to 0$, and thus this field will be called "tensile". The second field, shown in Fig. 1b, is for $s_{r\theta,\theta} < 0$. Here the singularity in the mean normal stress is compressive and, accordingly, this field will be called "compressive". A compressive singularity, however, would require strains of order unity or more for fracture. Since such large strains are not actually observed [1], the "compressive" field is not plausible for the growing crack.

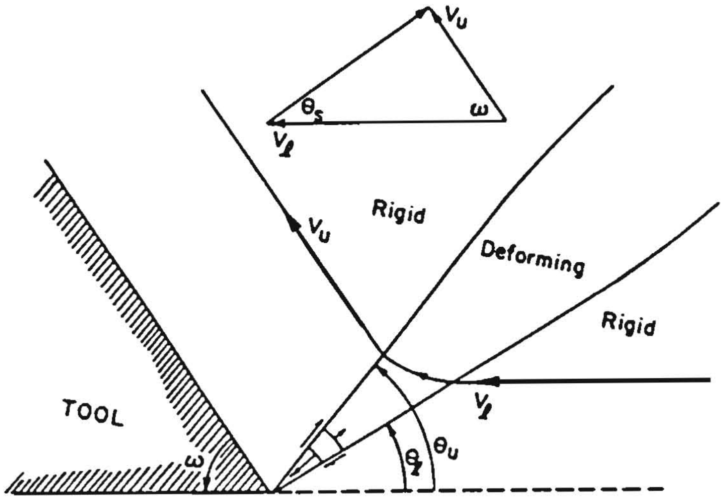

FIG. 1a

The flow field for tension in the band. The machining case is illustrated; otherwise ω is the crack opening angle.

Two other conceivable fields can be excluded. A single band being split by the crack (Fig. 1c) would have shear stresses of the same sign, but increasing in magnitude both above and below the line of advance. This would give tensile and compressive singularities adjacent to each other, and a discontinuity in normal stress. If the shear in a band being split by the crack were to change sign, on the other hand, there would be an intermediate region below yield, and the band would separate into two, corresponding to these in Figs 1a,b. In the limit, the Mode I field would be approached.

Thus the "tensile" field of Fig.1a is the only acceptable. From (7) and (8) for positive shearing,

$$\frac{ds_{r\theta}}{d\theta} = \frac{1}{\sqrt{3}}\frac{d\bar{\sigma}}{d\theta} = \frac{H}{\sqrt{3}}\frac{d\bar{\epsilon}}{d\theta} \,. \tag{17}$$

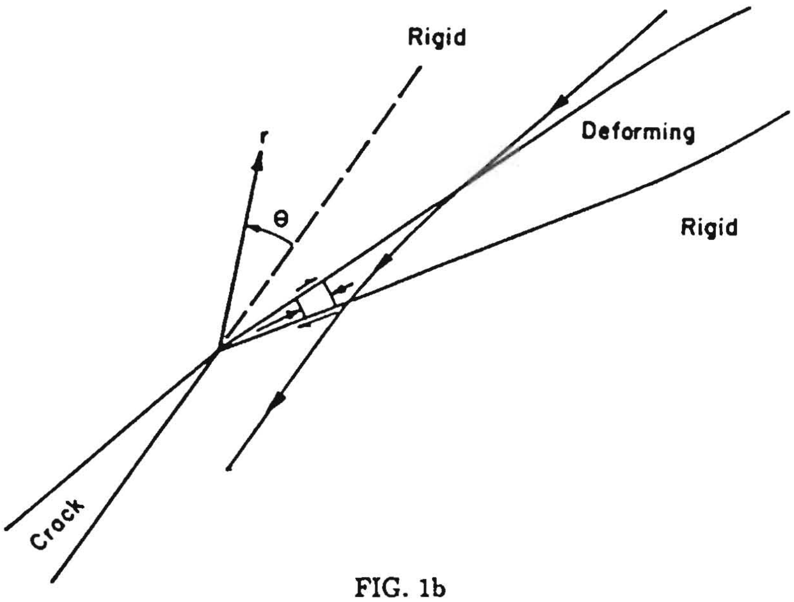

FIG. 1b
The field for compression in the band.

(17) and (13) give,

$$\frac{d\bar{\epsilon}}{d\theta} = \frac{D\sqrt{3}}{H} = \frac{1}{\sqrt{3}}\frac{d\gamma_{r\theta}}{d\theta} \,. \tag{18}$$

By differentiating (9), and taking into consideration that for positive $s_{r\theta}$ then $\dot{\gamma}_{r\theta}$, given by (4), is positive [this can be readily seen from (6)], gives:

$$\frac{d\bar{\epsilon}}{d\theta} = -\frac{F''(\theta) + F(\theta)}{\sqrt{3}F(\theta)} \,. \tag{19}$$

(18) and (19) give finally

$$F''(\theta) + k^2 F(\theta) = 0 \,, \tag{20}$$

where

$$k^2 = 1 + \frac{3s_{r\theta,\theta}}{H} = 1 + \frac{d\gamma_{r\theta}}{d\theta} \,. \tag{21}$$

The solution of (20) is:

$$F(\theta) = A\cos k\theta + B\sin k\theta \ . \tag{22}$$

Referring to Fig. 1a we denote by v_ℓ, v_u the (rigid body) velocities at the lower and upper boundaries of the deforming region, which are at angles θ_ℓ and θ_u respectively. Then the boundary conditions are as follows: At the lower boundary,

$$\dot{u}_r = -v_\ell\cos\theta_\ell \ , \quad \text{and by (2),} \quad F'(\theta_\ell) = -v_\ell\cos\theta_\ell \ . \tag{23}$$

$$\dot{u}_\theta = v_\ell\sin\theta_\ell \ , \quad \text{and by (2),} \quad F(\theta_\ell) = -v_\ell\sin\theta_\ell \ . \tag{24}$$

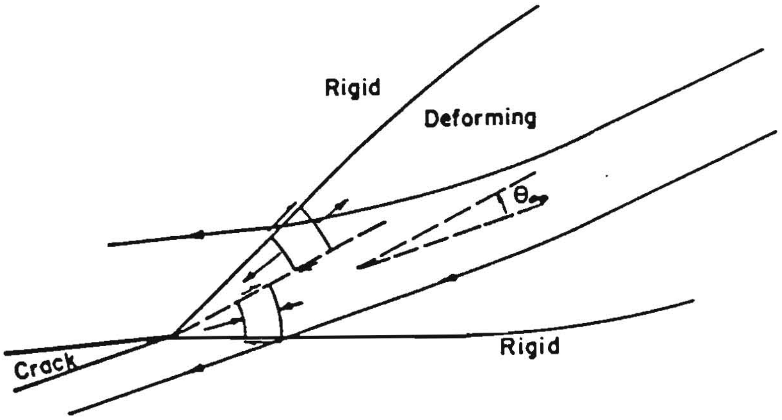

FIG. 1c
The field for a single band being split by the crack.

Similarly, at the upper boundary,

$$F'(\theta_u) = -v_u\cos(\theta_u + \omega) \ ; \quad F(\theta_u) = -v_u\sin(\theta_u + \omega) \ . \tag{25}$$

Using (22) and eliminating v_u/v_ℓ, A, B gives ω from:

$$\omega = \arctan\left[\frac{P - \tan k\theta_\ell}{k(1 + P\tan k\theta_\ell)}\right] - \theta_u \ , \tag{26}$$

where

$$P = \frac{k\tan\theta_\ell + \tan k\theta_u}{1 - k\tan\theta_\ell\tan k\theta_u} \ . \tag{27}$$

Substituting back into the boundary conditions gives:

$$\frac{v_u}{v_\ell} = \frac{k_u\sin\theta_\ell\cos k\theta_u + \cos\theta_\ell\sin k\theta_u}{k\sin(\theta_u + \omega)\cos k\theta_\ell + \cos(\theta_u + \omega)\sin k\theta_\ell} \ , \tag{28}$$

$$\frac{A}{v_\ell} = \frac{(v_u/v_\ell)\sin(\theta_u + \omega)\sin k\theta_\ell - \sin k\theta_u \sin\theta_\ell}{\sin k(\theta_u - \theta_\ell)} , \tag{29}$$

$$\frac{B}{v_\ell} = \frac{\sin\theta_\ell \cos k\theta_u - (v_u/v_\ell)\sin(\theta_u + \omega)\cos k\theta_\ell}{\sin k(\theta_u - \theta_\ell)} . \tag{30}$$

The velocity triangle shown in Fig. 1a defines a "slip" angle θ_s:

$$\theta_s = \arcsin\left\{\frac{(v_u/v_\ell)\sin\omega}{[(v_u/v_\ell)^2 - 2(v_u/v_\ell)\cos\omega + 1]^{1/2}}\right\} . \tag{31}$$

Assume now a critical strain γ_u occurring at the upper boundary. Then

$$d\gamma_{r\theta}/d\theta = \gamma_u/(\theta_u - \theta_\ell) , \tag{32}$$

and

$$k^2 = 1 + \gamma_u/(\theta_u - \theta_\ell) . \tag{33}$$

Finally, the rotation of the material element relative to that of the stress field is important in hole growth and thus it is worth considering. The rotation of the element is:

$$\dot{\phi}_m = \frac{1}{2}\left(\frac{\partial \dot{u}_\theta}{\partial r} + \frac{\dot{u}_\theta}{r} + \frac{1}{r}\frac{\partial \dot{u}_r}{\partial \theta}\right) ,$$

and from (2)

$$\dot{\phi}_m = -\frac{1}{2r}\left[F(\theta) + F''(\theta)\right] ,$$

while that of the stress field is:

$$\dot{\phi}_f = \frac{\dot{u}_\theta}{r} = -\frac{F(\theta)}{r} ,$$

giving a relative rotation:

$$\dot{\phi}_{rel} = \frac{1}{2r}\left[F(\theta) - F''(\theta)\right] .$$

For $F(\theta)$ given by (22) and since $k > 1$ by (21), it is found that rotation and shear strain are of different sign. The effect is to open up the holes and thus to increase the damage. An illustrative example of streamlines for a particular case is given in the next section. Now, according to Hill [3] , the infinite mean normal stress by (16) cannot be sustained at the rigid flank and this will lead to plastic yielding. However, consider now the contribution of elasticity. An elastic plastic strain singularity would perturb the near tip field so that it could wash out the above mean normal stress singularity; nonetheless the characteristics of the flow field at moderate distances could still be described by the above rigid plastic solution, especially at low levels of strain hardening when the elastic plastic singularity is weak [4]. Further discussion in connection with the problem of machining, which follows next, will reveal that certain characteristics of machining nechanics can be explained from this flow field.

Relation to the Mechanics of Machining

In machining, a shear band with an undetermined rigid-plastic boundary breaks through to a free surface. The problem is similar to the mixed mode crack growth, except that the deformation is larger. Christopherson et al [2] tried to assess the effect of work hardening in the mechanics of orthogonal machining. By modifying the slip line equations and estimating roughly the magnitude of the added term, they pointed out that, due to hardening, the hydrostatic stress changes from compressive at the free surface to tensile near the tool point. What they found was essentially the qualitative effect of the logarithmic singularity derived above for fully plastic flow. In fact, we can also deduce that, for a certain change in the yield stress between the chip and the parent material, if the deforming region is narrower, the angular change in the shear stress [i.e., D in equation (13)] is bigger and, consequently, the singularity stronger, in accordance with their observation that the work-hardening effect becomes more pronounced as the plastic zone gets narrower.

It is worth considering now the region of dominance of the logarithmic singularity in the mean normal stress that would characterize the flow past rigid flanks. Using typical data, $d\bar{\sigma}/d\bar{\epsilon} = H \simeq Y$ for 1020 steel and $d\bar{\epsilon}/d\theta \simeq 1.5$, gives from (7):

$$s_{r\theta,\theta} = \frac{1}{\sqrt{3}}\frac{d\bar{\sigma}}{d\bar{\epsilon}}\frac{d\bar{\epsilon}}{d\theta} = \frac{1}{\sqrt{3}}H\frac{d\bar{\epsilon}}{d\theta} \simeq 0.87Y \ . \tag{34}$$

From the fully plastic flow field of Prandtl for tension of grooved plane strain specimens (see e.g. McClintock [5]), $\sigma \simeq 2.8Y$ and, assuming that R is the radius at which $\sigma(r,0)$ changes sign, gives from (16), $r/R \simeq 0.04$. The distance R is within the macroscopic scale, as is evident from the approximate study for the machining field done by Christopherson et al [6]. According to their modified (to include hardening) slip-line theory, the change in the mean normal stress $\Delta\sigma$ from the free surface to a point in the band is roughly estimated in terms of the shear yield strengths in the work-piece and the chip, k_w and k_c, the distance s_1 from the free surface, and the local width s_2 of the slipband:

$$\Delta\sigma \simeq (k_c - k_w)s_1/s_2 \,. \tag{35}$$

In a typical instant in which mild steel is machined, k_c may be 40% more than k_w, so $k_c - k_w \simeq 0.4k$. Since at the free surface $\sigma \simeq -k$, the mean normal stress becomes positive at about $s_1/s_2 = 2.5$, which for 10^0 angular width happens at a radius R approximately 1/2 the total shear band length. Thus the singularity in the mean normal stress would dominate in a significant region and could describe the field at moderate distances.
Taking now a particular example from machining (Fig. 2) for $\gamma_u = 1.3$, $\theta_\ell = 40^0$, $\theta_u = 50^0$, we find by using equations (26)-(33):

$$\omega = 59^0 \,, \quad v_u/v_\ell = 0.73 \,, \quad d\gamma_{r\theta}/d\theta = 7.44 \,, \quad \theta_s = 45.06^0$$

and, for θ in radians:

$$F(\theta)/v_\ell = 0.59\cos(2.9\theta) - 0.459\sin(2.9\theta)$$

A resulting from (1) streamline for this particular example has been sketched in Fig. 2.

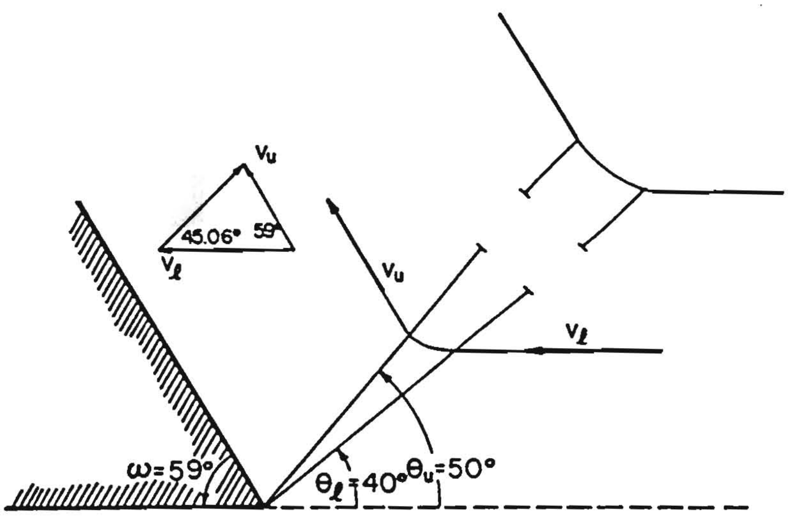

FIG. 2
A typical streamline predicted by the stream function (1) for the singular region in machining.

Since now there is tension near the tool point, it is possible that brittle (or ductile) fracture may occur at a particular history of stress and this could give rise to the characteristic fracture running ahead of the tool point and the formation of a built-up edge or a discontinuous chip [6]. In particular, according to the "tensile" field, the maximum strain occurs at the boundary with the chip (upper boundary of the shear band), where cracking could occur.

Conclusions

The aim of this article is to demonstrate the close relationship that exists between the mechanics of orthogonal machining and that of a growing mixed mode crack of finite angle. Assuming in both cases rigid-plastic, linearly strain hardening material, and steady flow with rigid material flowing past straight flanks gives a tensile logarithmic singularity in the mean normal stress. This indicates that the flanks of the crack tend to deform, and for the machining case it helps to explain the formation of a discontinuous chip, and, with a continuous chip, the tendencies to form a built-up edge or to fracture ahead of the tool.

Acknowledgements. The financial support of the Office of Naval Research, Contract N00014-82K-0025, and the interest of the project monitor, Dr. Y. Rajapakse, are gratefully acknowledged. The author is also grateful to Professor Frank A. McClintock for useful discussions.

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