Three-Dimensional Elasticity Solution for the Buckling of Transversely Isotropic Rods: The Euler Load Revisited

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The bifurcation of equilibrium of a compressed transversely isotropic bar is investigated by using a three-dimensional elasticity formulation. In this manner, an assessment of the thickness effects can be accurately performed. For isotropic rods of circular cross-section, the bifurcation value of the compressive force turns out to coincide with the Euler critical load for values of the length-over-radius ratio approximately greater than 15. The elasticity approach predicts always a lower (than the Euler value) critical load for isotropic bodies; the two examples of transversely isotropic bodies considered show also a lower critical load in comparison with the Euler value based on the axial modulus, and the reduction is larger than the one corresponding to isotropic rods with the same length over radius ratio. However, for the isotropic material, both Timoshenko's formulas for transverse shear correction are conservative; i.e., they predict a lower critical load than the elasticity solution. For a generally transversely isotropic material only the first Timoshenko shear correction formula proved to be a conservative estimate in all cases considered. However, in all cases considered, the second estimate is always closer to the elasticity solution than the first one and therefore, a more precise estimate of the transverse shear effects. Furthermore, by performing a series expansion of the terms of the resulting characteristic equation from the elasticity formulation for the isotropic case, the Euler load is proven to be the solution in the first approximation; consideration of the second approximation gives a direct expression for the correction to the Euler load, therefore defining a new, revised, yet simple formula for column buckling. Finally, the examination of a rod with different end conditions, namely a pinned-pinned rod, shows that the thickness effects depend also on the end fixity.

Introduction

The elastic buckling of slender rods and beams was the first stability problem to be investigated because of its historical importance in construction engineering. Recently, demands in the analysis and the design of light and highly stiff structures of many types, made of advanced composite materials and capable of carrying relatively high loads have been directly responsible for increased interest in extending the theoretical knowledge in this area.

When a bar is initially straight and of perfect geometry, and subjected to the action of a compressive force without eccentricity, it has been called an "ideal column."The case of a slender, ideal column, which is built in vertically at the base, free at the upper end, and subjected to an axial force P, constitutes the first problem of bifurcation buckling, the one that was originally solved by Euler (1744, 1933). The Euler solution is based on the well known Euler-Bernoulli assumptions (i.e., plane sections remain plane after bending, no effect of transverse shear) and for an isotropic elastic material. Nontrivial solutions (nonzero transverse deflections) are then sought for the equations governing bending of the column under an axial compressive load, and subject to the particular set of boundary conditions; thus, the problem is reduced to an eigen boundary value problem (e.g., Simitses, 1986).

As the cross-sectional dimensions of a rod increase relative to the length, it is naturally expected that the classical Euler

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load would deviate from the exact critical load. A similar deviation is expected if the materials laws are not isotropic. The objective of the present paper is to investigate the extent to which the classical Euler load represents the critical load, as derived by three-dimensional elasticity analyses for a generally transversely isotropic rod with no restrictive assumptions regarding the cross-sectional dimensions.

Regarding elasticity solutions to buckling, Kumar and Niyogi (1982) studied the bifurcations in axially compressed thick elastic tubes by using the strain-energy function of Ogden (1972) to represent the material behavior in isotropic solids. A three-dimensional elasticity formulation and solution for buckling of orthotropic composite materials was presented by Kardomateas (1993a) in connection with the problem of buckling of thick cylindrical orthotropic shells subjected to external pressure. It was shown that the critical load predicted by shell theory can be quite non-conservative for thick construction. This work was based on the simplifying assumption that the pre-buckling stress and displacement field was axisymmetric, and the buckling modes were assumed two-dimensional (ring assumption); i.e., no z (axial) component of the displacement field, and no z-dependence of the r and θ displacement components. In a subsequent article, Kardomateas and Chung (1994) presented a solution that relaxes this ring approximation, i.e., based on a nonzero axial displacement and a full dependence of the buckling modes on the three coordinates.

In order to further assess the thickness effects on the stability of shells, Kardomateas (1993b) presented a solution for the case of a transversely isotropic thick cylindrical shell under axial compression. In that work, a full dependence on r, θ, and z of the buckling modes was assumed. The reason for restricting the material to transversely isotropic was the desire to produce closed form analytical solutions. The same problem of an axially loaded moderately thick cylindrical shell was treated in a subsequent study for the case of a generally orthotropic shell (Kardomateas, 1995). A comparison with various shell theories showed that for the isotropic material cases considered, both the Flügge (1960) and Danielson and Simmonds (1969) shell theories predicted critical loads much closer to the elasticity value than the Donnell (Brush and Almroth, 1975) theory; the elasticity approach predicted a lower critical load than all these classical shell theories, the percentage reduction being larger with increasing thickness. However, in this study, an additional shell theory, namely that of Timoshenko and Gere (1961), was examined. It was found that for both the orthotropic and the isotropic material cases, the Timoshenko bifurcation points are lower than the elasticity ones. This means that the Timoshenko formulation is conservative, unlike all the other shell theories examined.

Regarding the stability loss of elastic bars, the only alternative direct expressions to the Euler load that exist in the literature are two formulas suggested by Timoshenko and Gere (1961). These were intended to account for the influence of transverse shearing forces. These load expressions, denoted by P_{T1} and P_{T2}, are given in the Results section. Despite the simplicity of the derivation of these formulas, it will be seen that they perform remarkably well in accounting for the thickness effects as well as for the effects of a low ratio of shear versus extensional modulus. It should be noted that although a study of the buckling of a generally anisotropic rod would be desirable, this work is restricted to the case of transverse isotropy, because more general anisotropy would not allow a direct closed-form solution of the corresponding three-dimensional elasticity problem.

The study conducted in this paper includes specific results for the critical load and the buckling modes of a cylindrical rod under axial compression for various ratios of length over radius l/R. The effect of transverse isotropy is examined by considering two material cases: one that approximates glass/epoxy and the other that approximates graphite/epoxy, both with reinforcing direction along the z-axis. The results from the elasticity formulation will be compared with the classical Euler load predictions and with Timoshenko and Gere's (1961) column buckling with transverse shear correction formulas. Moreover, in addition to the historically important Euler rod of one end fixed and the other free, results for the bifurcation load of a rod with both ends pinned are produced.

Another important contribution of this work is the derivation of a new, simple formula for column buckling. This is achieved by performing a series expansion of the terms of the resulting characteristic equation from the elasticity formulation for the isotropic case. It is also proved that the Euler load is the solution of this elasticity formulation in the first approximation. The direct formula thus produced is shown to provide an excellent agreement to the elasticity results for isotropic columns.

Formulation

The equilibrium of a rod, considered as a three-dimensional elastic body, can be described in terms of the second Piola-Kirchhoff stress tensor $\mathbf{\Sigma}$ as follows (e.g., Ciarlet, 1988):

$$\operatorname{div}(\mathbf{\Sigma} \cdot \mathbf{F}^{\mathrm{T}}) = 0. \tag{1a}$$

Here $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \operatorname{grad}V, \tag{1b}$$

where V is the displacement vector and $\mathbf{I}$ is the identity tensor.

It should be noticed that the strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}). \tag{1c}$$

Since we consider a circular section, we employ cylindrical coordinates, and we can specifically write the components of the deformation gradient $\mathbf{F}$ in terms of the linear strains:

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}, \tag{2a}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \tag{2b}$$

and the linear rotations:

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \quad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r},$$

$$2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \tag{2c}$$

as follows:

$$\mathbf{F} = \begin{bmatrix} 1 + \epsilon_{rr} & \frac{1}{2}\gamma_{r\theta} - \omega_z & \frac{1}{2}\gamma_{rz} + \omega_\theta \\ \frac{1}{2}\gamma_{r\theta} + \omega_z & 1 + \epsilon_{\theta\theta} & \frac{1}{2}\gamma_{\theta z} - \omega_r \\ \frac{1}{2}\gamma_{rz} - \omega_\theta & \frac{1}{2}\gamma_{\theta z} + \omega_r & 1 + \epsilon_{zz} \end{bmatrix}. \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, w_0 the r, θ and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r, \theta, z)$, $\alpha v_1(r, \theta, z)$, $\alpha w_1(r, \theta, z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r, \theta, z)$, $v_1(r, \theta, z)$, $w_1(r, \theta, z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

Following Kardomateas (1993a), we obtain the following buckling equations:

$$\frac{\partial}{\partial r}\left(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta\right) + \frac{\partial}{\partial z}\left(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta\right) + \frac{1}{r}\left(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z\right) = 0, \tag{5a}$$

$$\frac{\partial}{\partial r}\left(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r\right) + \frac{\partial}{\partial z}\left(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r\right) + \frac{1}{r}\left(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r\right) = 0, \tag{5b}$$

$$\frac{\partial}{\partial r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r\right) + \frac{\partial}{\partial z}\left(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r\right) + \frac{1}{r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) = 0. \tag{5c}$$

In these equations, σ^0_{ij} and ω^0_j are the values of σ_{ij} and ω_j at the initial equilibrium position; i.e., for $u = u_0$, $v = v_0$ and $w = w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position; i.e., for $u = u_1$, $v = v_1$ and $w = w_1$.

The boundary conditions associated with (1a) can be expressed as (e.g., Ciarlet, 1988):

$$(\mathbf{F} \cdot \mathbf{\Sigma}^{\mathrm{T}}) \cdot \hat{n} = t(V), \tag{6}$$

where t is the traction vector on the surface which has outward unit normal $\hat{n} = (l, m, n)$ before any deformation. The traction vector t depends on the displacement field $V = (u, v, w)$. Again, following Kardomateas (1993a), we obtain for the lateral and end surfaces:

$$\left(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta\right)l + \left(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta\right)m + \left(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta\right)n = 0, \tag{7a}$$

$$\left(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r\right)l + \left(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r\right)m + \left(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r\right)n = 0, \tag{7b}$$

$$\left(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta\right)l + \left(\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta\right)m + \left(\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta\right)n = 0. \tag{7c}$$

Prebuckling State. We consider the case of a cylindrical rod compressed by an axial force, P, applied at the one end, which is free. The other end of the rod is fixed. Denote the length of the rod by l and the area of the transverse section by A. The material is transversely isotropic, obeying the stress-strain relations:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}, \tag{8}$$

where c_{ij} are the elastic constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

If we assume that the stresses along the loaded upper end ($z = l$) and the reaction along the lower ($z = 0$) end of the rod are distributed uniformly and are normal to the bounding planes, then the components of stress tensor that satisfy the equations of equilibrium and the traction conditions on the surfaces are simply

$$\sigma^0_{zz} = -\frac{P}{A} = -\sigma_0; \quad \sigma^0_{rr} = \sigma^0_{\theta\theta} = \tau^0_{r\theta} = \tau^0_{rz} = \tau^0_{\theta z} = 0. \tag{9}$$

Perturbed State. Using (5) and (9), the three-dimensional elasticity equilibrium equations for the perturbed position can be written as follows (primes denote values at the perturbed position):

$$\frac{\partial \sigma'_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau'_{r\theta}}{\partial \theta} + \frac{\partial}{\partial z}(\tau'_{rz} - \sigma_0\omega'_\theta) + \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta}) = 0, \tag{10a}$$

$$\frac{\partial \tau'_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma'_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial z}(\tau'_{\theta z} + \sigma_0\omega'_r) + \frac{2\tau'_{r\theta}}{r} = 0, \tag{10b}$$

$$\frac{\partial \tau'_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau'_{\theta z}}{\partial \theta} + \frac{\partial \sigma'_{zz}}{\partial z} + \frac{\tau'_{rz}}{r} = 0. \tag{10c}$$

In the above equations, σ^0_{ij}, σ'_{ij} are expressed in terms of ϵ^0_{ij}, ϵ'_{ij}, respectively, in the same manner as the stress-strain law; i.e., Eqs. (8), for σ_{ij} in terms of ϵ_{ij}. The strains ϵ'_{ij}, are in turn expressed in terms of the displacements, u_1, v_1, w_1, in the same manner as the linear strain displacement relations (2). Substituting, we obtain the equations of equilibrium in terms of the displacements at the perturbed state, u_1, v_1, w_1 as follows:

$$c_{11}\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} - \frac{u_1}{r^2}\right) + \frac{1}{2}(c_{11} - c_{12})\frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2} + \left(c_{55} - \frac{\sigma_0}{2}\right)\frac{\partial^2 u_1}{\partial z^2} + \frac{1}{2}(c_{11} + c_{12})\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v_1}{\partial \theta}\right) - (c_{11} - c_{12})\frac{1}{r^2}\frac{\partial v_1}{\partial \theta} + \left[(c_{13} + c_{55}) + \frac{\sigma_0}{2}\right]\frac{\partial^2 w_1}{\partial r \partial z} = 0, \tag{11a}$$

$$\frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2}\right) + c_{11}\frac{1}{r^2}\frac{\partial^2 v_1}{\partial \theta^2} + \left(c_{55} - \frac{\sigma_0}{2}\right)\frac{\partial^2 v_1}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\frac{1}{2}(c_{11} + c_{12})\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r}\right) + (c_{11} - c_{12})\frac{u_1}{r} + \left[(c_{13} + c_{55}) + \frac{\sigma_0}{2}\right]\frac{\partial w_1}{\partial z}\right\} = 0, \tag{11b}$$

$$c_{55}\left(\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_1}{\partial \theta^2}\right)+c_{33}\frac{\partial^2 w_1}{\partial z^2}$$
$$+(c_{13}+c_{55})\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial r}+\frac{u_1}{r}+\frac{1}{r}\frac{\partial v_1}{\partial \theta}\right)=0. \quad (11c)$$

We seek a *first group of solutions* in terms of a function ϕ in the form

$$u_1=\frac{\partial \phi}{\partial r};\quad v_1=\frac{1}{r}\frac{\partial \phi}{\partial \theta};\quad w_1=k\frac{\partial \phi}{\partial z}. \quad (12)$$

A similar form had been used by Elliott (1948) for Cartesian coordinates. Then equations (11a) and (11b) are satisfied if

$$c_{11}\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$
$$+\left[c_{55}+k(c_{13}+c_{55})+\frac{\sigma_0}{2}(k-1)\right]\frac{\partial^2 \phi}{\partial z^2}=0, \quad (13a)$$

and (11c) is satisfied if

$$[(c_{13}+c_{55})+kc_{55}]\left(\frac{\partial^2 \phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$
$$+kc_{33}\frac{\partial^2 \phi}{\partial z^2}=0. \quad (13b)$$

A nonzero solution of these Eqs. (13a) and (13b) can be found only if they are identical; this occurs if

$$\frac{c_{55}+k(c_{13}+c_{55})+(k-1)(\sigma_0/2)}{c_{11}}$$
$$=\frac{kc_{33}}{(c_{13}+c_{55}+kc_{55})}\equiv s^2. \quad (14a)$$

This gives a quadratic equation for s^2 or k. The equation for $x=s^2$, with roots s_1^2 and s_2^2, depending on the compressive stress σ_0, is

$$c_{11}c_{55}x^2+\left[(c_{13}+c_{55})\left(c_{13}+c_{55}+\frac{\sigma_0}{2}\right)\right.$$
$$\left.-c_{55}\left(c_{55}-\frac{\sigma_0}{2}\right)-c_{11}c_{33}\right]x+c_{33}\left(c_{55}-\frac{\sigma_0}{2}\right)=0, \quad (14b)$$

and the corresponding k_i:

$$k_i=\frac{s_i^2c_{11}-c_{55}+(\sigma_0/2)}{c_{13}+c_{55}+(\sigma_0/2)}\, i=1,2. \quad (14c)$$

A *second group of solutions* is sought in terms of the function ψ in the form

$$u_1=\frac{1}{r}\frac{\partial \psi}{\partial \theta};\quad v_1=-\frac{\partial \psi}{\partial r};\quad w_1=0. \quad (15)$$

Then Eqs. (11a) and (11b) are satisfied if

$$\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right)$$
$$+\left(c_{55}-\frac{\sigma_0}{2}\right)\frac{\partial^2 \psi}{\partial z^2}=0, \quad (16)$$

and Eq. (11c) is identically satisfied.

Finally, an obvious *third group of solutions* is the rigid-body displacement field with components V_x, V_y, and V_z along the Cartesian x, y, z coordinate system:

$$u_1=V_x\cos\theta+V_y\sin\theta;\quad v_1=-V_x\sin\theta+V_y\cos\theta;$$
$$w_1=V_z. \quad (17)$$

The displacement is a superposition of these three fields. Now, the functions ϕ and ψ are sought in a separable form:

$$\phi_i(r,\theta,z)=Z(z)A_i(\lambda r)\cos\theta;$$
$$i=1,2 \text{ corresponding to } s_1, s_2, \quad (18a)$$
$$\psi(r,\theta,z)=Z(z)B(\lambda r)\sin\theta. \quad (18b)$$

Set

$$\rho=\lambda r. \quad (18c)$$

Substituting in (13a), we obtain the ordinary differential equations:

$$A_i''(\rho)+\frac{1}{\rho}A_i'(\rho)-\left(s_i^2+\frac{1}{\rho^2}\right)A_i(\rho)=0 \quad (19a)$$

where s_i^2 are given in (14a). In a similar fashion, substituting in (16), we obtain the ordinary differential equation

$$B''(\rho)+\frac{1}{\rho}B'(\rho)-\left(q^2+\frac{1}{\rho^2}\right)B(\rho)=0$$
$$\text{where } q^2=\frac{2c_{55}-\sigma_0}{c_{11}-c_{12}}. \quad (19b)$$

Moreover, $Z(z)$ is found to satisfy

$$Z''(z)+\lambda^2Z(z)=0. \quad (19c)$$

The assumption

$$Z(z)=\cos\lambda z \quad (19d)$$

satisfies the third differential equation, (19c).

The solution to the two Eqs. (19a) and (19b) involves only the modified Bessel functions of first order of the first kind:

$$A_i(\rho)=C_iI_1(s_i\rho);\quad B(\rho)=C_0I_1(q\rho), \quad (20)$$

where the constants C_i are in general complex conjugates and C_0 is real.

Before satisfying the boundary conditions at the lateral surface, which will ultimately provide the system of equations for the eigenvalue problem, we shall discuss the boundary conditions at the ends. From (7) the boundary conditions on the ends are

$$\tau'_{rz}+\sigma^0_{zz}\omega'_\theta=0;\quad \tau'_{\theta z}-\sigma^0_{zz}\omega'_r=0;\quad \sigma'_{zz}=0,\quad \text{at } z=0, l. \quad (21)$$

Since σ'_{zz} varies as $\cos\lambda z$, the condition $\sigma_{zz}=0$ on the upper end $z=l$ is satisfied if

$$\lambda=\frac{\pi}{2l}. \quad (22)$$

In a cartesian coordinate system (x, y, z), the first two of the conditions in (21) can be written as follows:

$$\tau'_{xz}+\sigma^0_{zz}\omega'_y=0;\quad \tau'_{yz}-\sigma^0_{zz}\omega'_x=0. \quad (23)$$

It will be proved that these remaining two conditions are satisfied on the average. At this point, it should be noted that for some of the boundary conditions, a form of resultant instead of pointwise conditions has been frequently used in elasticity treatments, and can be considered as based on some form of the Saint-Venant's principle. For this reason, they are sometimes referred to as relaxed end conditions of the Saint-Venant's type (Horgan, 1989). Now the lateral surface boundary conditions in the cartesian coordinate system, with $\hat{n}$ the normal to the circular contour are from (7):

$$\sigma'_{xx}\cos(\hat{n},x)+\tau'_{xy}\cos(\hat{n},y)=0, \tag{24a}$$

$$\tau'_{xy}\cos(\hat{n},x)+\sigma'_{yy}\cos(\hat{n},y)=0, \tag{24b}$$

with $\hat{n}$ the normal to the circular contour. Using the equilibrium equation in cartesian coordinates, Green's theorem for transformation of an area integral into a contour integral, and the above conditions on the contour, we obtain

$$\frac{\partial}{\partial z}\iint_A\left(\tau'_{xz}+\sigma^0_{zz}\omega'_y\right)dA=-\iint_A\left(\frac{\partial\sigma'_{xx}}{\partial x}+\frac{\partial\tau'_{xy}}{\partial y}\right)dA. \tag{25a}$$

The previous integral then becomes

$$-\int_\gamma\left[\sigma'_{xx}\cos(\hat{n},x)+\tau'_{xy}\cos(\hat{n},y)\right]ds=0,$$

where A denotes the area of the annular cross-section and γ the corresponding contour. Therefore

$$\iint_A\left(\tau'_{xz}+\sigma^0_{zz}\omega'_y\right)dA=\text{const.} \tag{25b}$$

Since at $z=0$, $\tau'_{xz}=\omega'_y=0$ because τ'_{xz}, τ'_{yz} and ω'_y, ω'_x all have a $\sin[\pi z/(2l)]$ variation (and so do τ'_{rz}, $\tau'_{\theta z}$ and ω'_θ, ω'_r), it is concluded that this constant is zero. Similar arguments hold for τ'_{yz}.

Moreover, it can also be proved that this system of stresses would produce no torsional moment. Indeed,

$$\frac{\partial}{\partial z}\iint_A\left[x\left(\tau'_{yz}-\sigma^0_{zz}\omega'_x\right)-y\left(\tau'_{xz}+\sigma^0_{zz}\omega'_y\right)\right]dA$$

$$=-\iint_A\left\{x\left(\frac{\partial\tau'_{xy}}{\partial x}+\frac{\partial\sigma'_{yy}}{\partial y}\right)-y\left(\frac{\partial\sigma'_{xx}}{\partial x}+\frac{\partial\tau'_{xy}}{\partial y}\right)\right\}dA.$$

Again, using the divergence theorem, the previous integral becomes

$$-\int_\gamma\left\{x\left[\tau'_{xy}\cos(\hat{n},x)+\sigma'_{yy}\cos(\hat{n},y)\right]\right.$$

$$\left.-y\left[\sigma'_{xx}\cos(\hat{n},x)+\tau'_{xy}\cos(\hat{n},y)\right]\right\}ds=0, \tag{25c}$$

hence

$$\iint_A\left[x\left(\tau'_{yz}-\sigma^0_{zz}\omega'_x\right)-y\left(\tau'_{xz}+\sigma^0_{zz}\omega'_y\right)\right]dA=\text{const}, \tag{25d}$$

and this constant is again zero since $\tau'_{xz}=\tau'_{yz}=\omega'_x=\omega'_y=0$ at $z=0$.

Finally, it should be noted that concerning the variation along z,

$$w_1,\quad\frac{\partial u_1}{\partial z},\quad\frac{\partial v_1}{\partial z}\sim Z'(z)\sim\sin\lambda z, \tag{25e}$$

therefore at $z=0$, $w_1=\partial u_1/\partial z=\partial v_1/\partial z=0$ and u_1 and v_1 and can be made equal to zero at some point of the end $z=0$ by the choice of the constants V_i in (17). In this sense, the end $z=0$ is "clamped."

Notice that based on the previous analysis, we have found that

$$\phi_i(r,\theta,z)=C_iI_1(s_i\lambda r)\cos\theta\cos\lambda z;\quad i=1,2 \tag{26a}$$

and

$$\psi(r,\theta,z)=C_0I_1(q\lambda r)\sin\theta\cos\lambda z. \tag{26b}$$

Now we proceed to the boundary conditions on the lateral surface $r=R$. From (7), we obtain

$$\sigma'_{rr}=0;\quad\tau'_{r\theta}=0;\quad\tau'_{rz}=0,\quad\text{at}\quad r=R. \tag{27}$$

Substituting in (27), (15), (12), (2), and (8), and using the identities for the derivatives of Bessel functions, we obtain a system of three linear homogeneous equations in C_i, $i=1,2$ and C_0:

$$C_0(c_{11}-c_{12})q\frac{I_2(\lambda qR)}{R}$$

$$+\sum_{i=1,2}C_i\left\{c_{11}\left[\lambda s_i^2I_1(\lambda s_iR)-s_i\frac{I_2(\lambda s_iR)}{R}\right]\right.$$

$$\left.+c_{12}s_i\frac{I_2(\lambda s_iR)}{R}-c_{13}k_i\lambda I_1(\lambda s_iR)\right\}=0, \tag{28a}$$

$$C_0\left[2q\frac{I_2(\lambda qR)}{R}-\lambda q^2I_1(\lambda qR)\right]-\sum_{i=1,2}C_i2s_i\frac{I_2(\lambda s_iR)}{R}=0, \tag{28b}$$

$$C_0\frac{I_1(\lambda qR)}{R}$$

$$+\sum_{i=1,2}C_i(k_i+1)\left[\frac{I_1(\lambda s_iR)}{R}+\lambda s_iI_2(\lambda s_iR)\right]=0. \tag{28c}$$

By equating the determinant of the above system (28) to zero, we obtain an equation for σ_0 (characteristic equation) which can be solved to obtain the critical load. In general, the roots s_1 and s_2 are either both real or complex conjugates, whereas q, defined in (19b), is normally a real variable. In the case of real s_1, s_2 the determinant of the linear system (28) is real.

In the case of a complex conjugate pair $\{s_1, s_2\}$, the Bessel functions have complex arguments and the constants C_1, C_2 are complex conjugates, whereas C_0 is a real number. Furthermore, $\{I_1(\lambda s_1R), I_1(\lambda s_2R)\}$ and $\{I_2(\lambda s_1R), I_2(\lambda s_2R)\}$ are also complex conjugate pairs. The 3×3 matrix of coefficients of the linear system (28) has one real column (corresponding to C_0) and the remaining two are two pairs of complex conjugates. Therefore, it turns out that in this case the determinant of (28) is pure imaginary. In either case, equating the determinant to zero results in a nonlinear real equation for σ_0.

In performing the numerical calculations, the modified Bessel function of zero and first order can be evaluated from polynomial coefficients given by Abramowitz and Stegun (1964) and that of the second order from the associated recurrence relations.

A New, Improved, Direct Formula for Isotropic Column Buckling: The Euler Load Revisited

In the case of isotropy, in which $s_1=1$ and $s_2=q$, expansion of the determinant for the system of linear homogeneous Eqs. (28) and use of the series expansion of the Bessel functions can lead to direct formulas for the critical load. Notice that for isotropy, $c_{12}=c_{13}=c_{11}-2c_{55}$.

Set

$$\bar{\lambda}=\lambda2\sqrt{I/A}=\lambda R=\frac{\pi R}{2l}, \tag{29a}$$

where $A=\pi R^2$ is the cross-sectional area and $I=\pi R^4/4$ is the moment of inertia of the cross-section. Furthermore, we can define the quantity ϵ by using (19b):

$$\epsilon=1-q^2=\frac{\sigma_0}{2c_{55}}. \tag{29b}$$

From (14c), since for isotropy $s_1=1$ and $s_2=q$, we find that $k_1=1$, and $k_2=k$ is expanded by substituting $q^2=1-\epsilon$:

$$k = \frac{q^2(c_{11} - c_{55})}{c_{11} - c_{55}q^2} = 1 - \frac{c_{11}}{c_{11} - c_{55}}\epsilon + \frac{c_{11}c_{55}}{(c_{11} - c_{55})^2}\epsilon^2 - \frac{c_{11}c_{55}^2}{(c_{11} - c_{55})^3}\epsilon^3 + \ldots \quad (29c)$$

The Bessel functions in (28) are then expanded in powers of $\bar{\lambda}$ or $\bar{\lambda}q$ by using the series expansions: $I_1(x) = x/2 + x^3/16 + x^5/16.24 + x^7/16.24.48 + \ldots$, and $I_2(x) = x^2/8 + x^4/6.16 + x^6/8.16.24 + x^8/10.16.24.48 + \ldots$.

Now since $\epsilon = O(\bar{\lambda}^2)$, in the first approximation the characteristic equation would include the terms: ϵ, $\bar{\lambda}^2$, ϵ^2, $\bar{\lambda}^2\epsilon$, and $\bar{\lambda}^4$; in the second approximation it would also include ϵ^3, $\bar{\lambda}^2\epsilon^2$, $\bar{\lambda}^4\epsilon$, and $\bar{\lambda}^6$. Therefore, in performing the calculations, only terms up to $\bar{\lambda}^6$ need be retained. The detailed expressions for the resulting coefficients of the system (28) are given in the Appendix. Equating the 3×3 determinant to zero results in the following equation:

$$c_{55}\frac{q^2\bar{\lambda}^5}{32}\left\{4\epsilon^2 + \left[\frac{c_{55}}{2(c_{11} - c_{55})} - \frac{3}{2}\right]\bar{\lambda}^2\epsilon - 4\frac{c_{11}}{c_{11} - c_{55}}\epsilon^3 + \left[\frac{19}{6} + \frac{4c_{11}}{3(c_{11} - c_{55})} - \frac{c_{11}c_{55}}{2(c_{11} - c_{55})^2}\right]\bar{\lambda}^2\epsilon^2 + \left[\frac{7c_{55}}{48(c_{11} - c_{55})} - \frac{25}{48}\right]\bar{\lambda}^4\epsilon\right\} = 0. \quad (30)$$

Notice that the coefficients of ϵ, $\bar{\lambda}^2$, $\bar{\lambda}^4$, and $\bar{\lambda}^6$ turn out to be zero.

In terms of the Poisson's ratio, ν, and the Young's modulus, E,

$$c_{11} = E\frac{1 - \nu}{(1 + \nu)(1 - 2\nu)}; \quad c_{55} = \frac{E}{2(1 + \nu)}. \quad (31)$$

Substituting into (30) gives in the first approximation, i.e., terms $O(\epsilon^2)$:

$$\epsilon = \frac{1 + \nu}{4}\bar{\lambda}^2. \quad (32a)$$

Notice that the parameter $\epsilon = \sigma_0/(2c_{55})$ comes naturally from the formulation of the problem. It can also be written as $\epsilon = \sigma_0(1 + \nu)/E$, hence it is a suitable small parameter since the axial stress, σ_0, is much smaller than the modulus, E.

Now, using the definitions (29*a*,*b*) gives

$$\sigma_0 = E\frac{\lambda^2 R^2}{4}. \quad (32b)$$

Hence, the critical load is

$$P_{cr} = \sigma_0 A = E\lambda^2\frac{\pi R^4}{4} = \lambda^2 EI; \quad (32c)$$

i.e., from the first approximation we recover the classic Euler load.

In the second approximation, we keep all the terms in (30) and, substituting (31), we obtain the following quadratic equation in ϵ:

$$8(1 - \nu)\epsilon^2 - \left[4 + \frac{\bar{\lambda}^2}{6}(29 + 2\nu - 12\nu^2)\right]\epsilon + \bar{\lambda}^2(1 + \nu) + \frac{\bar{\lambda}^4}{24}(9 + 7\nu) = 0. \quad (33)$$

The solution to this equation can be written in the form

$$\epsilon = \frac{1 + \nu}{4}\bar{\lambda}^2 - \frac{\epsilon_2}{16(1 - \nu)}, \quad (34a)$$

where

$$\epsilon_2 = \sqrt{\Delta} - 4 - \frac{\bar{\lambda}^2}{6}(5 + 2\nu + 12\nu^2), \quad (34b)$$

and

$$\Delta = 16 + \frac{\bar{\lambda}^2}{3}(20 + 8\nu + 48\nu^2) + \frac{\bar{\lambda}^4}{36}(409 + 212\nu - 356\nu^2 - 48\nu^3 + 144\nu^4). \quad (34c)$$

Using (29*a*, *b*) gives now the critical load as follows:

$$P_{cr} = \lambda^2 EI - \frac{\epsilon_2}{16(1 - \nu^2)}EA. \quad (35)$$

Therefore, *in the second approximation, the Euler load, $\lambda^2 EI$, is reduced by the quantity $\epsilon_2 EA/[16(1 - \nu^2)]$*. Equation (35) provides a new, improved, direct formula for isotropic column buckling. As with the Euler load, the end fixity is accounted for in λ, which for the fixed-free case is: $\lambda = \pi/2l$. Results that will be presented in the next section will show that formula (35) provides estimates that almost coincide with the elasticity results, therefore providing an excellent account of the thickness effects for isotropic columns.

Discussion of Results

The Euler critical load for a compressed fixed-free column is

$$P_{\text{Euler}} = \frac{\pi^2 E_3 I}{4l^2} = E_3 I\lambda^2 = E_3 A\frac{R^2\lambda^2}{4} \quad (36)$$

where I is the moment of inertia of the cross section.

Concerning the present elasticity formulation, the critical load is obtained by finding the solution σ_0 of the determinant of (28). The ratio $P_{\text{Euler}}/P_{\text{elast}} = E_3\pi^3 R^4/(\sigma_0 4l^2)$ of Table 1 compares the critical load, as predicted by the present three-dimensional elasticity formulation, with the Euler load. The other numbers in brackets and in parentheses represent the predictions of the two Timoshenko transverse shear correction formulas that will be discussed later.

Three material cases have been considered with the following properties (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$): (*a*) Isotropic material with modulus $E = E_3$ and Poisson's ratio $\nu = 0.3$. (*b*) Material No. 1 (which approximates glass/epoxy with reinforcement along the z-axis) with moduli in GPa: $E_3 = 57$, $E_2 = E_1 = 14$, $G_{31} = G_{23} = 5.7$ and Poisson's ratios: $\nu_{12} = 0.400$, $\nu_{23} = \nu_{13} = 0.068$. (c) Material No. 2 (which approximates graphite/epoxy with reinforcement along the z-axis) with moduli in GPa: $E_3 = 140$, $E_2 = E_1 = 10$, $G_{31} = G_{23} = 5.0$ and Poisson's ratios: $\nu_{12} = 0.53$, $\nu_{23} = \nu_{13} = 0.02$.

It can be proved that for an isotropic material, which is characterized by the two stiffness constants $c_{11} = c_{33}$ and c_{55}, ($c_{12} = c_{13} = c_{11} - 2c_{55}$), the roots of (14*b*) are

$$s_1 = 1 \quad \text{and} \quad s_2 = q = \left(1 - \frac{\sigma_0}{2c_{55}}\right)^{1/2}. \quad (37)$$

For a transversely isotropic material, like the two cases (b) and (c), the roots are in general complex conjugates.

The critical load from the elasticity solution, is presented graphically in Fig. 1. From both Fig. 1 and Table 1, it can be concluded that the critical load for isotropic bodies is in general lower than the Euler value. The difference is, how-

Table 1 One end fixed, the other free

	Critical Loads, $P^{\dagger}_{\mathrm{Euler}}/P_{\mathrm{elast}}$ $[P^{\ddagger}_{T1}/P_{\mathrm{elast}}]$ $(P^{\ddagger}_{T2}/P_{\mathrm{elast}})$		
l/R	Isotropic	Material 1	Material 2
15.0	1.000 [0.992] (0.992)	1.017 [0.988] (0.988)	1.050 [0.968] (0.973)
13.0	1.001 [0.991] (0.991)	1.022 [0.982] (0.984)	1.065 [0.956] (0.966)
11.0	1.006 [0.991] (0.991)	1.031 [0.975] (0.978)	1.093 [0.943] (0.960)
9.0	1.012 [0.990] (0.990)	1.047 [0.965] (0.971)	1.138 [0.921] (0.950)
7.0	1.017 [0.981] (0.982)	1.080 [0.947] (0.961)	1.228 [0.883] (0.944)
5.0	1.036 [0.968] (0.971)	1.159 [0.910] (0.947)	1.449 [0.820] (0.961)
3.0	1.101 [0.919] (0.941)	1.439 [0.817] (0.955)	2.227 [0.713] (1.091)

$^{\dagger}P_{\mathrm{Euler}}/P_{\mathrm{elast}} = E_3\pi^3R^4/(\sigma_0 4l^2)$, where σ_0 is from the elasticity solution (the determinant of (28)).

‡Timoshenko's first and second transverse shear correction formulas, Eqs. (40a, b).

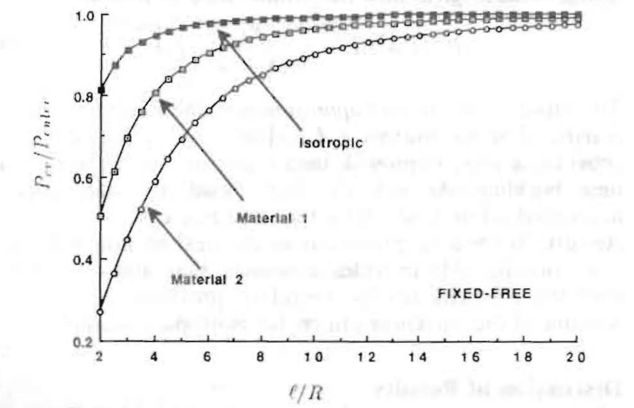

Fig. 1 Eigenvalues P_{cr} / P_{Euler} from the elasticity solution as a function of the length over radius ratio, l / R, for three cases of material data (isotropic plus two transversely isotropic: material 1 approximating glass / epoxy and material 2 approximating graphite / epoxy, both with axial reinforcement). One end is fixed, the other free.

ever, very small for relatively long rods and indeed, it converges to the Euler load for values of length-over-radius ratio greater than about 15. The two examples of transversely isotropic bodies had lower critical loads in comparison with the Euler value based on the axial modulus, and the reduction was larger than that corresponding to isotropic rods with the same length over radius ratio. Especially for material no. 2, which has a markedly reduced shear and radial modulus compared to the (axial) extensional one, reductions of more than ten percent occurred for a length-over-radius ratio less than 10.

The case of a bar with hinged ends is probably an equally important and fundamental case of buckling of a prismatic bar. In this case

$$Z(z) = \sin \lambda z; \quad \lambda = \frac{\pi}{l}, \tag{38a}$$

and the Euler load is

$$P_{\mathrm{Euler}} = \frac{\pi^2 E_3 I}{l^2} = E_3 I \lambda^2. \tag{38b}$$

Table 2 provides a comparison of the critical load for a simply supported bar, as predicted by the present three-dimensional elasticity formulation, with the Euler load. Again, the same three material cases have been considered. The critical load from the elasticity solution, normalized with the Euler load for a simply supported beam, is also given graphically in Fig. 2. It can be concluded that the thickness effect is stronger in this case of end fixity. Again, the critical load is always lower than the Euler value. The two examples of transversely isotropic bodies also had lower critical loads in

Table 2 Both ends pinned

	Critical Loads, $P^{\dagger}_{\mathrm{Euler}}/P_{\mathrm{elast}}$ $[P^{\ddagger}_{T1}/P_{\mathrm{elast}}]$ $(P^{\ddagger}_{T2}/P_{\mathrm{elast}})$		
l/R	Isotropic	Material 1	Material 2
17.0	1.012 [0.988] (0.989)	1.054 [0.962] (0.970)	1.155 [0.912] (0.948)
15.0	1.014 [0.983] (0.984)	1.068 [0.953] (0.962)	1.198 [0.894] (0.944)
13.0	1.019 [0.978] (0.979)	1.093 [0.941] (0.957)	1.266 [0.871] (0.946)
11.0	1.028 [0.970] (0.973)	1.131 [0.922] (0.950)	1.370 [0.838] (0.951)
9.0	1.045 [0.960] (0.967)	1.195 [0.893] (0.943)	1.555 [0.799] (0.975)
7.0	1.074 [0.938] (0.952)	1.321 [0.848] (0.944)	1.908 [0.744] (1.032)
5.0	1.148 [0.893] (0.932)	1.629 [0.777] (0.981)	2.755 [0.678] (1.187)
3.0	1.414 [0.789] (0.930)	2.717 [0.671] (1.174)	5.682 [0.597] (1.642)

$^{\dagger}P_{\mathrm{Euler}}/P_{\mathrm{elast}} = E_3\pi^3R^4/(\sigma_0 4l^2)$, where σ_0 is from the elasticity solution (the determinant of (28)).

‡Timoshenko's first and second transverse shear correction formulas, Eqs. (40a, b).

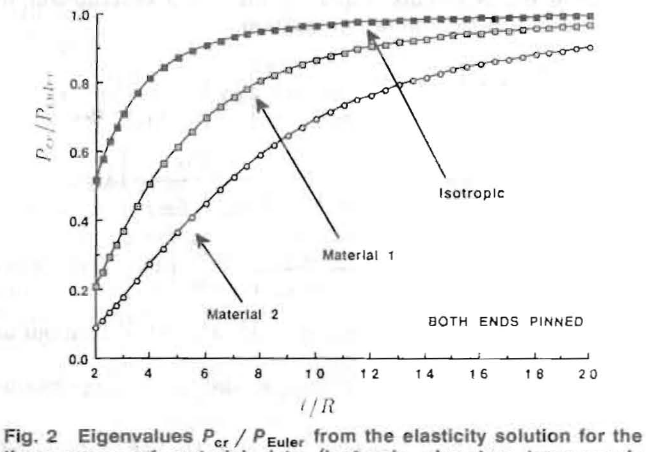

Fig. 2 Eigenvalues P_{cr} / P_{Euler} from the elasticity solution for the three cases of material data (isotropic plus two transversely isotropic) in the case where both ends are pinned

comparison with the Euler values based on the axial modulus, and the reduction was larger than that for isotropic rods with the same length over radius ratio. Furthermore, the reduction was larger than that for the fixed-free case. For this case, material no. 2 has reductions of more than 30 percent for a length over radius ratio less than 10.

If the displacements of the elasticity solution in the fixed-free case are set in a form

$$u_1 = U(r)\cos\theta\cos\frac{\pi z}{2l}, \quad v_1 = V(r)\sin\theta\cos\frac{\pi z}{2l}. \tag{39a}$$

$$w_1 = W(r)\cos\theta\sin\frac{\pi z}{2l}, \tag{39b}$$

we can write these r-dependences as follows:

$$U(r) = C_0\frac{I_1(\lambda q r)}{r} + \sum_{i=1,2} C_i\left[\frac{I_1(\lambda s_i r)}{r} + \lambda s_i I_2(\lambda s_i r)\right], \tag{39c}$$

$$V(r) = -\left\{C_0\left[\frac{I_1(\lambda q r)}{r} + \lambda q I_2(\lambda q r)\right] + \sum_{i=1,2} C_i\frac{I_1(\lambda s_i r)}{r}\right\}, \tag{39d}$$

$$W(r) = -\lambda\sum_{i=1,2} C_i k_i I_1(\lambda s_i r). \tag{39e}$$

As an illustration, for the critical load of the previous example of a transversely isotropic rod made out of Material No. 2 and of length $l/R = 5$ and 8, Figs. 3(a, b, c) show the eigenfunctions $U(r)$, $V(r)$, and $W(r)$. It has been assumed that $U = 1.0$ at the boundary $r = R$. In all cases the shorter

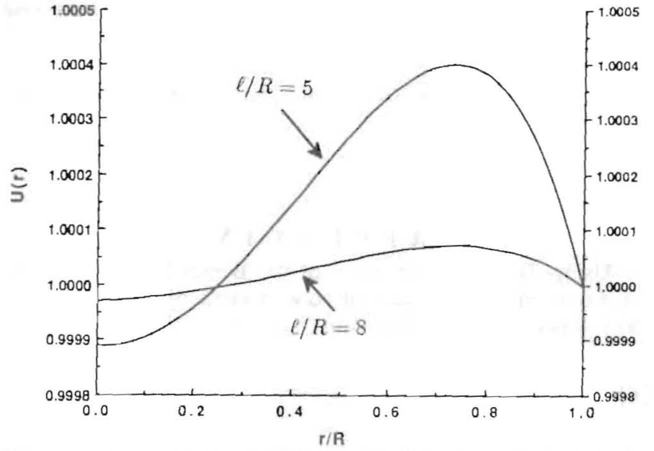

Fig. 3(*a*) "Eigenfunction" $U(r)$ versus normalized radial distance r/R for a transversely isotropic rod made out of Material No. 2 and of length $l/R = 5$ and 8. A unit value for U at the lateral surface $r = R$ has arbitrarily been set (column theory would have a constant value, $U = 1.0$).

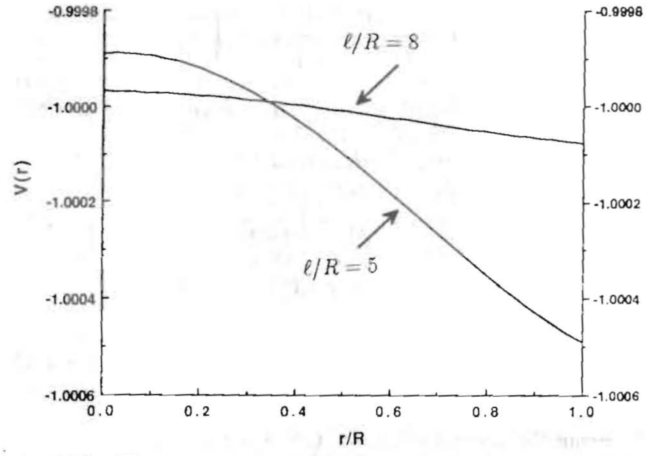

Fig. 3(*b*) "Eigenfunction" $V(r)$ versus normalized radial distance r/R for a transversely isotropic rod made out of Material No. 2 and of length $l/R = 5$ and 8 (column theory would have a linear variation for V)

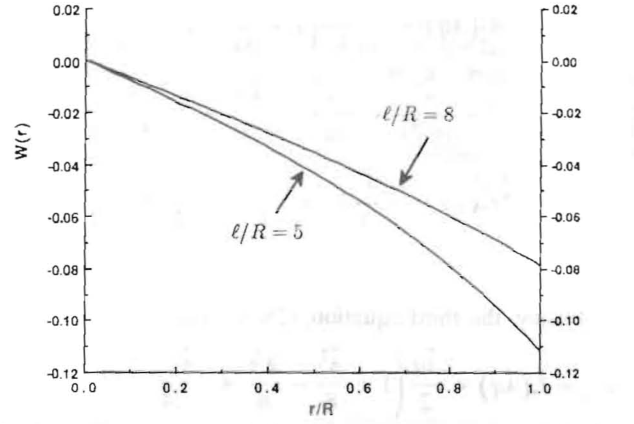

Fig. 3(*c*) "Eigenfunction" $W(r)$ versus normalized radial distance r/R for a transversely isotropic rod made out of Material No. 2 and of length $l/R = 5$ and 8 (column theory would have a linear variation for W)

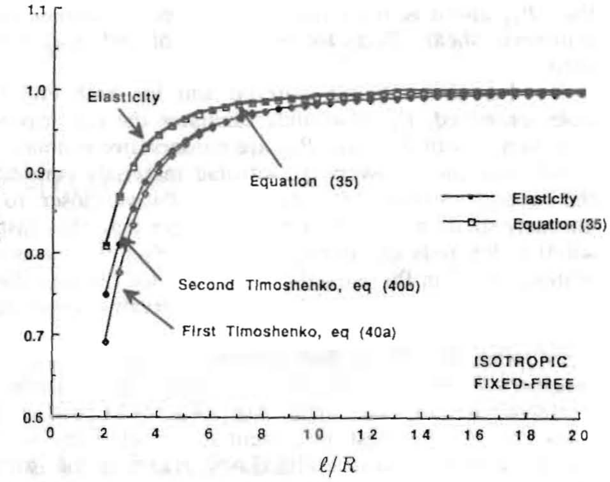

Fig. 4 Comparison of the different direct formulas for isotropic column buckling. Eigenvalues P_{cr}/P_{Euler} from the elasticity solution, Eq. (35), and the two Timoshenko formulas, P_{T1} and P_{T2}, Eqs. (40*a*, *b*) as a function of the length over radius ratio, l/R.

beam, $l/R = 5$, in which transverse normal strains are expected to be of greater importance, shows a higher variation in the displacement fields and a highly nonlinear radial displacement $U(r)$. For the longer beam, $l/R = 8$, the variation is smaller. Notice that classical column theory would have a constant value of $U = 1.0$ and a linear variation for V and W.

A comparison with an improved beam theory will be considered next. Timoshenko suggested two formulas that provide a correction to the Euler load, P_e, due to the influence of transverse shearing forces. The Timoshenko formulas for the critical load, P_{T1} and P_{T2} are (Timoshenko and Gere, 1961):

$$P_{T1} = \frac{P_e}{1 + \alpha P_e/AG}, \qquad (40a)$$

$$P_{T2} = \frac{\sqrt{1 + 4\alpha P_e/AG} - 1}{2\alpha/AG}, \qquad (40b)$$

where α is a numerical factor depending on the shape of the transverse section, A is the cross-sectional area $(= \pi R^2)$, and G is the shear modulus. For a circular cross-section, $\alpha = 1.11$.

Tables 1 and 2 compare results for the Timoshenko formulas with the Euler load and the elasticity solution for the two cases of fixed-free and pinned-pinned rod, respectively, isotropic case and the case of material no. 2 (in the transversely isotropic case we have used $G_{23} = G_{31}$ in place of G in (40); the Euler load P_e is always based on the axial modulus, E_3).

Finally, Fig. 4 compares results from formula (35) with the exact elasticity solution and the two Timoshenko formulas, Eqs. (40*a*, *b*), for a range of l/R values. It is seen that formula (35) almost coincides with the elasticity results.

Conclusions from the comparison of the results in Tables 1 and 2 and from Fig. 4, are summarized next.

Summary and Conclusions

(1) The first Timoshenko estimate, P_{T1}, always underestimates the elasticity solution, for both cases of end fixity considered (i.e., fixed-free and pinned-pinned). It is, therefore, a conservative estimate.

(2) The second Timoshenko estimate, P_{T2}, (which is larger than P_{T1}) is always closer to the elasticity solution

than P_{T1} and it is, therefore, a more precise estimate of the transverse shear effects for both cases of end fixity considered.

(3) For the isotropic material and for both end fixity cases examined, P_{T2} also underestimates the elasticity solution, hence both P_{T1} and P_{T2} are conservative estimates.

(4) For the transversely isotropic materials considered, the second estimate, P_{T2} (which is always closer to the elasticity solution than P_{T1}) may be larger than the elasticity solution for rods of sizable thickness (e.g., this occurs for material no. 2 in the pinned-pinned case for l/R less than 9) and therefore it may provide a nonconservative estimate for rods of sizable thickness.

(5) The new, direct formula (35) that was derived in the present work for isotropic column buckling, gives results that almost coincide with the elasticity solution. Hence, this formula provides the best agreement to the elasticity solution and the best account of the thickness effects in the isotropic case.

In short, the previous results show that the thickness, the end fixity, and the material data have to be jointly considered in assessing the performance of the classical column buckling formulas for cases beyond the traditional long, isotropic rods. If a conservative estimate is desired, then P_{T1} can be used. P_{T2} is more accurate but it may be nonconservative for nonisotropic rods of sizable thickness. If the column is isotropic, the simple direct formula (35) provides an excellent account of the thickness effects.

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APPENDIX

Using the series expansion of the Bessel functions results in a determinant of order three with elements a_{ij}, $i, j = 1, 3$ as follows. From the first equation, (28*a*),

$$a_{11} = (c_{11} - c_{12}) q I_2(\tilde{\lambda} q) = c_{55}\frac{\tilde{\lambda}^2 q}{4}\left(1 + \frac{\tilde{\lambda}^2}{12} - \epsilon - \frac{\tilde{\lambda}^2\epsilon}{6} + \frac{\tilde{\lambda}^4}{16.24} + \frac{\tilde{\lambda}^2\epsilon^2}{12} - \frac{\tilde{\lambda}^4\epsilon}{8.16} + \frac{\tilde{\lambda}^6}{6.15.16.16}\right), \quad \text{(A1)}$$

$$a_{12} = c_{11} q\left[\tilde{\lambda} q I_1(\tilde{\lambda} q) - I_2(\tilde{\lambda} q)\right] + c_{12}\left[q I_2(\tilde{\lambda} q) - k\tilde{\lambda} I_1(\tilde{\lambda} q)\right] = c_{55}\frac{\tilde{\lambda}^2 q}{4}\left\{3 + \frac{5}{12}\tilde{\lambda}^2 - \left(\frac{c_{11} + c_{55}}{c_{11} - c_{55}}\right)\epsilon - \frac{2c_{11}c_{12}}{(c_{11} - c_{55})^2}\epsilon^2 - \left(\frac{7c_{11} - 4c_{55}}{c_{11} - c_{55}}\right)\frac{\tilde{\lambda}^2\epsilon}{12} + \frac{7}{16.24}\tilde{\lambda}^4 + \frac{2c_{11}c_{12}c_{55}}{(c_{11} - c_{55})^3}\epsilon^3 + \left[\frac{3c_{11}c_{55}}{(c_{11} - c_{55})^2} - 1\right]\frac{\tilde{\lambda}^2\epsilon^2}{12} - \left(\frac{17c_{11} - 13c_{55}}{c_{11} - c_{55}}\right)\frac{\tilde{\lambda}^4\epsilon}{16.24} + \frac{\tilde{\lambda}^6}{10.16.16}\right\}, \quad \text{(A2)}$$

$$a_{13} = (c_{11} - c_{12})\left[\tilde{\lambda} I_1(\tilde{\lambda}) - I_2(\tilde{\lambda})\right] = c_{55}\frac{\tilde{\lambda}^2}{4}\left(3 + \frac{5}{12}\tilde{\lambda}^2 + \frac{7}{16.24}\tilde{\lambda}^4 + \frac{1}{10.16.16}\tilde{\lambda}^6\right). \quad \text{(A3)}$$

From the second equation, (28*b*), we obtain

$$a_{21} = q\left[2I_2(\tilde{\lambda} q) - \tilde{\lambda} q I_1(\tilde{\lambda} q)\right] = -\frac{\tilde{\lambda}^2 q}{4}\left(1 + \frac{\tilde{\lambda}^2}{6} - \epsilon - \frac{\tilde{\lambda}^2\epsilon}{3} + \frac{\tilde{\lambda}^4}{8.16} + \frac{\tilde{\lambda}^2\epsilon^2}{6} - \frac{3}{8.16}\tilde{\lambda}^4\epsilon + \frac{\tilde{\lambda}^6}{5.6.12.16}\right), \quad \text{(B1)}$$

$$a_{22} = -2qI_2(\tilde{\lambda} q) = -\frac{\tilde{\lambda}^2 q}{4}\left(1 + \frac{\tilde{\lambda}^2}{12} - \epsilon - \frac{\tilde{\lambda}^2\epsilon}{6} + \frac{\tilde{\lambda}^4}{16.24} + \frac{\tilde{\lambda}^2\epsilon^2}{12} - \frac{\tilde{\lambda}^4\epsilon}{8.16} + \frac{\tilde{\lambda}^6}{6.15.16.16}\right), \quad \text{(B2)}$$

$$a_{23} = -2I_2(\tilde{\lambda}) = -\frac{\tilde{\lambda}^2}{4}\left(1 + \frac{\tilde{\lambda}^2}{12} + \frac{\tilde{\lambda}^4}{16.24} + \frac{\tilde{\lambda}^6}{6.15.16.16}\right). \quad \text{(B3)}$$

Finally, the third equation, (28*c*), gives

$$a_{31} = I_1(\tilde{\lambda} q) = \frac{\tilde{\lambda} q}{2}\left(1 + \frac{\tilde{\lambda}^2}{8} - \frac{\tilde{\lambda}^2\epsilon}{8} + \frac{\tilde{\lambda}^4}{8.24} - \frac{\tilde{\lambda}^4\epsilon}{8.12} + \frac{\tilde{\lambda}^6}{8.24.48}\right), \quad \text{(C1)}$$

$$a_{32} = (k+1)\left[I_1(\tilde{\lambda}q) + \tilde{\lambda}qI_2(\tilde{\lambda}q)\right]$$

$$= \tilde{\lambda}q\left[1 + \frac{3}{8}\tilde{\lambda}^2 - \frac{c_{11}}{2(c_{11}-c_{55})}\epsilon + \frac{c_{11}c_{55}}{2(c_{11}-c_{55})^2}\epsilon^2\right.$$

$$- \frac{3}{16}\left(\frac{3c_{11}-2c_{55}}{c_{11}-c_{55}}\right)\tilde{\lambda}^2\epsilon + \frac{5}{12.16}\tilde{\lambda}^4 - \frac{c_{11}c_{55}^2}{2(c_{11}-c_{55})^3}\epsilon^3$$

$$+ \frac{3}{16}\left(\frac{c_{11}}{c_{11}-c_{55}}\right)^2\tilde{\lambda}^2\epsilon^2 - \frac{5}{16.24}\left(\frac{5c_{11}-4c_{55}}{c_{11}-c_{55}}\right)\tilde{\lambda}^4\epsilon$$

$$\left. + \frac{7}{16.24.24}\tilde{\lambda}^6\right], \quad \text{(C2)}$$

$$a_{33} = 2\left[I_1(\tilde{\lambda}) + \tilde{\lambda}I_2(\tilde{\lambda})\right]$$

$$= \tilde{\lambda}\left(1 + \frac{3}{8}\tilde{\lambda}^2 + \frac{5}{12.16}\tilde{\lambda}^4 + \frac{7}{16.24.24}\tilde{\lambda}^6\right). \quad \text{(C3)}$$
