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Benchmark three-dimensional elasticity solutions for the buckling of thick orthotropic cylindrical shells

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Benchmark solutions to the problem of buckling of orthotropic cylindrical shells, which are based on the three-dimensional theory of elasticity, are presented in this review article. It is assumed that the shell is under external pressure or axial compression or a combination of these loadings. These solutions provide a means of accurately assessing the limitations of the various shell theories in predicting critical loads. A comparison with some classical shell theories shows that the classical shell theories may produce, in general, highly non-conservative results on the critical load of composite shells with thick construction. One noteworthy exception: the Timoshenko shell buckling equations produce conservative results under pure axial compression. Copyright © 1996 Elsevier Science Limited

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INTRODUCTION

A class of important structural applications of fiber-reinforced composite materials involves the configuration of laminated shells. Although thin plate construction has been the thrust of the initial applications, much attention is now being paid to configurations classified as moderately thick shell structures. Such designs can be used in components in the aircraft and automobile industries, as well as in the marine industry. Moreover, composite laminates have been considered in space vehicles in the form of circular cylindrical shells as a primary load carrying structure.

In these light-weight shell structures, loss of stability is of primary concern. This subject has been researched to date through the application of the cylindrical shell theory (e.g. ref. 1). However, previous work[2,3] has shown that considerable care must be exercised in applying thin shell theory formulations to predict the resposne of composite cylinders. Besides the anisotropy, composite shells have one other important distinguishing feature, namely extensional-to-shear modulus ratio much larger than that of their metal counterparts.

In order to more accurately account for the above mentioned effects, various modifications in the classical theory of laminated shells have generally been performed[4–6], (see also ref. 7 for a review of shear deformation theories). These higher order shell theories can be applied to buckling problems with the potential of improved predictions for the critical load. To this extent, Simitses *et al.*[8] used the Galerkin method to produce the critical loads of cylindrical shells under external pressure, as predicted from the first order shear deformation and the higher order shear deformation theories. It was concluded that for moderately thick cylinders, the first order shear deformation theory with a modest shear correction factor provides an adequate correction to the critical load (as compared with the improvements from the higher order theories).

Regarding the classical shell formulation, the critical loads for an isotropic material can be found by solving the eigenvalue problem for the set of cylindrical shell equations that are obtained following the Kirchhoff–Love assumptions. To this group belong the Donnell theory[9] and the Sanders small strain, small rotation about normal and moderate rotation about in-plane theories[10]. Furthermore, in presenting shell theory equilibrium equations for isotropic shells, Timoshenko and Gere[11] included some additional terms (these equations are briefly described in the Appendix). Each of the Donnell, Sanders and Timoshenko equations can be easily extended for the case of an orthotropic material.

The existence of these different shell theories underscores the need for benchmark elasticity solutions, in order to compare the accuracy of the predictions from the classical and the improved shell theories. Classical shell theories with regard to critical loads have been investigated extensively and compared among themselves by Simitses and Aswani[12].

Several benchmark elasticity solutions for composite shell buckling have recently become available. In particular, Kardomateas[13] formulated and solved the problem for the case of uniform external pressure and orthotropic material; a simplified problem definition was used in this study ('ring' assumption), in that the prebuckling stress and displacement field was axisymmetric, and the buckling modes were assumed two-dimensional, i.e. no z component of the displacement field, and no z-dependence of the r and θ displacement components. The ring assumption was relaxed in a further study[14] in which a nonzero axial displacement and a full dependence of the buckling modes on the three coordinates was assumed.

A more thorough investigation of the thickness effects was conducted by Kardomateas[15] for the case of a transversely isotropic thick cylindrical shell under axial compression. This work also included a comprehensive study of the performance of the Donnell[9], Sanders-type[10] (which was also referred to as 'non-simplified Donnell-type' theory), the Flügge[16] and the Danielson and Simmonds[17] theories for isotropic material in the case of axial compression. In a more recent study, Kardomateas[18] considered a generally cylindrically orthotropic material under axial compression. In addition to considering general orthotropy for the material constitutive behavior, the latter work investigated the performance of another classical formulation, i.e. the Timoshenko and Gere formulation[11]. Other three-dimensional elasticity results were provided by Soldatos and Ye[19] based on a successive approximation method. These results were provided for the buckling of complete hollow cylinders subjected to combined axial compression and uniform external pressure and the buckling of open cylindrical panels subjected to axial compression.

Towards this objective, this work summarizes the elasticity solution approach to the problem of buckling of composite cylindrical orthotropic shells subjected to either external pressure or axial compression. Numerical results for fiber reinforced hollow cylinders made out of graphite/epoxy or glass/epoxy are derived and compared with shell theory predictions. These results can be used to assess the accuracy of the classical shell theory and the existing improved shell theories for moderately thick construction. Most of the results in this review article are extracted from these previously cited studies.

Formulation

At the critical load there are two possible infinitely close positions of equilibrium. Denoted by u_0, v_0, w_0 the r, θ and z components of the displacement corresponding to the primary position. A perturbed position is denoted by:

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1, \tag{1}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r,\theta,z)$, $\alpha v_1(r,\theta,z)$, $\alpha w_1(r,\theta,z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r,\theta,z)$, $v_1(r,\theta,z)$, $w_1(r,\theta,z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

The nonlinear strain displacement equations are:

$$\epsilon_{rr} = \frac{\partial u}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right], \tag{2a}$$

$$\epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)^2\right], \tag{2b}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right], \tag{2c}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \left[\frac{\partial u}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + \frac{\partial v}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right], \tag{2d}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} + \left(\frac{\partial u}{\partial r}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\right), \tag{2e}$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} + \left[\frac{\partial u}{\partial z}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) + \frac{\partial v}{\partial z}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) + \frac{1}{r}\frac{\partial w}{\partial \theta}\frac{\partial w}{\partial z}\right]. \tag{2f}$$

Substituting (1) into (2) we find the strain components in the perturbed configuration:

$$\epsilon_{rr} = \epsilon^0_{rr} + \alpha\epsilon'_{rr} + \alpha^2\epsilon''_{rr} \qquad \gamma_{r\theta} = \gamma^0_{r\theta} + \alpha\gamma'_{r\theta} + \alpha^2\gamma''_{r\theta}, \tag{3a}$$

$$\epsilon_{\theta\theta} = \epsilon^0_{\theta\theta} + \alpha\epsilon'_{\theta\theta} + \alpha^2\epsilon''_{\theta\theta} \qquad \gamma_{rz} = \gamma^0_{rz} + \alpha\gamma'_{rz} + \alpha^2\gamma''_{rz}, \tag{3b}$$

$$\epsilon_{zz} = \epsilon^0_{zz} + \alpha\epsilon'_{zz} + \alpha^2\epsilon''_{zz} \qquad \gamma_{\theta z} = \gamma^0_{\theta z} + \alpha\gamma'_{\theta z} + \alpha^2\gamma''_{\theta z}, \tag{3c}$$

where ϵ^0_{rr} are the values of the strain components in the initial position of equilibrium, ϵ'_{rr} are the strain quantities corresponding to the linear terms and ϵ''_{rr} are the ones corresponding to the quadratic terms.

The stress–strain relations for the orthotropic body are:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}, \tag{4}$$

where c_{ij} are the stiffness constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$). Substituting (3) into (4) we get the stresses as:

$$\sigma_{rr} = \sigma^0_{rr} + \alpha\sigma'_{rr} + \alpha^2\sigma''_{rr} \quad \tau_{r\theta} = \tau^0_{r\theta} + \alpha\tau'_{r\theta} + \alpha^2\tau''_{r\theta}, \quad (5a)$$

$$\sigma_{\theta\theta} = \sigma^0_{\theta\theta} + \alpha\sigma'_{\theta\theta} + \alpha^2\sigma''_{\theta\theta} \quad \tau_{rz} = \tau^0_{rz} + \alpha\tau'_{rz} + \alpha^2\tau''_{rz}, \quad (5b)$$

$$\sigma_{zz} = \sigma^0_{zz} + \alpha\sigma'_{zz} + \alpha^2\sigma''_{zz} \quad \tau_{\theta z} = \tau^0_{\theta z} + \alpha\tau'_{\theta z} + \alpha^2\tau''_{\theta z}, \quad (5c)$$

where σ^0_{ij}, σ'_{ij}, σ''_{ij}, are expressed in terms of ϵ^0_{ij}, ϵ'_{ij}, ϵ''_{ij}, respectively, in the same manner as equations (4) for σ_{ij} in terms of ϵ_{ij}.

In the following, we shall keep in (5) and (3) terms up to α, i.e. we neglect the terms which contain α^2.

Governing equations. The equations of equilibrium are taken in terms of the second Piola–Kirchhoff stress tensor Σ in the form (e.g. ref. 20):

$$\operatorname{div}\left(\Sigma \cdot \mathbf{F}^{\mathrm{T}}\right) = 0, \quad (6a)$$

where $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \operatorname{grad}\vec{V}, \quad (6b)$$

where $\vec{V}$ is the displacement vector and $\mathbf{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}). \quad (6c)$$

More specifically, in terms of the linear strains:

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad (7a)$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}, \quad (7b)$$

and the linear rotations:

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \quad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r},$$

$$2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}, \quad (7c)$$

the deformation gradient $\mathbf{F}$ is

$$\mathbf{F} = \begin{bmatrix} 1 + e_{rr} & \frac{1}{2}e_{r\theta} - \omega_z & \frac{1}{2}e_{rz} + \omega_\theta \\ \frac{1}{2}e_{r\theta} + \omega_z & 1 + e_{\theta\theta} & \frac{1}{2}e_{\theta z} - \omega_r \\ \frac{1}{2}e_{rz} - \omega_\theta & \frac{1}{2}e_{\theta z} + \omega_r & 1 + e_{zz} \end{bmatrix} \quad (8)$$

and the equilibrium equation (6a) gives:

$$\begin{aligned} &\frac{\partial}{\partial r}[\sigma_{rr}(1 + e_{rr}) + \tau_{r\theta}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \tau_{rz}(\tfrac{1}{2}e_{rz} + \omega_\theta)] \\ &+ \frac{1}{r}\frac{\partial}{\partial \theta}[\tau_{r\theta}(1 + e_{rr}) + \sigma_{\theta\theta}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \tau_{\theta z}(\tfrac{1}{2}e_{rz} + \omega_\theta)] \\ &+ \frac{\partial}{\partial z}[\tau_{rz}(1 + e_{rr}) + \tau_{\theta z}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \sigma_{zz}(\tfrac{1}{2}e_{rz} + \omega_\theta)] \\ &+ \frac{1}{r}[\sigma_{rr}(1 + e_{rr}) - \sigma_{\theta\theta}(1 + e_{\theta\theta}) + \tau_{rz}(\tfrac{1}{2}e_{rz} + \omega_\theta) \\ &- \tau_{\theta z}(\tfrac{1}{2}e_{\theta z} - \omega_r) - 2\tau_{r\theta}\omega_z] = 0, \end{aligned} \quad (9a)$$

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial \theta}[\tau_{r\theta}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \sigma_{\theta\theta}(1 + e_{\theta\theta}) + \tau_{\theta z}(\tfrac{1}{2}e_{\theta z} - \omega_r)] \\ &+ \frac{\partial}{\partial z}[\tau_{rz}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \tau_{\theta z}(1 + e_{\theta\theta}) + \sigma_{zz}(\tfrac{1}{2}e_{\theta z} - \omega_r)] \\ &+ \frac{\partial}{\partial r}[\sigma_{rr}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \tau_{r\theta}(1 + e_{\theta\theta}) + \tau_{rz}(\tfrac{1}{2}e_{\theta z} - \omega_r)] \\ &+ \frac{1}{r}[\sigma_{rr}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \sigma_{\theta\theta}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \tau_{rz}(\tfrac{1}{2}e_{\theta z} - \omega_r) \\ &+ \tau_{\theta z}(\tfrac{1}{2}e_{rz} + \omega_\theta) + \tau_{r\theta}(2 + e_{rr} + e_{\theta\theta})] = 0, \end{aligned} \quad (9b)$$

$$\begin{aligned} &\frac{\partial}{\partial z}[\tau_{rz}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \tau_{\theta z}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \sigma_{zz}(1 + e_{zz})] \\ &+ \frac{\partial}{\partial r}[\sigma_{rr}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \tau_{r\theta}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \tau_{rz}(1 + e_{zz})] \\ &+ \frac{1}{r}\frac{\partial}{\partial \theta}[\tau_{r\theta}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \sigma_{\theta\theta}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \tau_{\theta z}(1 + e_{zz})] \\ &+ \frac{1}{r}[\sigma_{rr}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \tau_{r\theta}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \tau_{rz}(1 + e_{zz})] = 0. \end{aligned} \quad (9c)$$

Introducing the linear strains and rotations in equation (3), e.g. $e_{rr} = e^0_{rr} + \alpha e'_{rr}$, $\omega_z = \omega^0_z + \alpha\omega'_z$, as well as the stresses from equation (5) and keeping up to α^1 terms, we obtain a set of equations for the perturbed state in terms of the e^0_{ij}, ω^0_j and e'_{ij}, ω'_j. Notice that in addition to the notations we adopted earlier, e^0_{ij} and ω^0_j are the values of e_{ij} and ω_j for $u = u_0$, $v = v_0$ and $w = w_0$, and e'_{ij} and ω'_j are the values for $u = u_1$, $v = v_1$ and $w = w_1$.

Since the displacements u_0, v_0, w_0, correspond to positions of equilibrium, there must also exist equations of the form of equations (9) with the zero superscript, which are obtained by referring equation (6a) to the initial position of equilibrium.

Thus, after subtracting the equilibrium equations at the perturbed and initial positions, we arrive at a system of homogeneous differential equations which are linear in the derivatives of u_1, v_1 and w_1 with respect to r, θ, z. This follows from the fact that σ'_{ij}, e'_{ij}, ω'_j appear linearly in the equation, and are themselves, in virtue of equations (7), linear functions of these derivatives. The system of equations, corresponding to equation (9), at the initial position of equilibrium, is, on the other hand, nonlinear in the derivatives of u_0, v_0, w_0. However, if we make the additional assumption to neglect the terms that have e^0_{ij} and ω^0_j as coefficients, i.e. terms $e^0_{ij}\sigma^0_{ij}$ and $\omega^0_j\sigma^0_{ij}$, we can use the linear classical equilibrium equations to solve for the initial position of equilibrium.

Moreover, if we make the assumption to neglect the terms that have e^0_{ij} and ω^0_j as coefficients, i.e. terms $e^0_{ij}\sigma'_{ij}$ and $\omega^0_j\sigma'_{ij}$, and furthermore, since a characteristic feature of stability problems is the shift from positions with small rotations to positions with rotations substantially exceeding the strains, if we neglect the terms $e'_{ij}\sigma^0_{ij}$, thus keeping only the $\omega'_j\sigma^0_{ij}$ terms, we obtain the following buckling equations:

$$\frac{\partial}{\partial r}(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)$$

$$+\frac{\partial}{\partial z}(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)$$
$$+\frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z) = 0, \quad (10a)$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)$$
$$+\frac{\partial}{\partial z}(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)$$
$$+\frac{1}{r}(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r) = 0, \quad (10b)$$

$$\frac{\partial}{\partial r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r)$$
$$+\frac{\partial}{\partial z}(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r)$$
$$+\frac{1}{r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) = 0. \quad (10c)$$

Boundary conditions. The boundary conditions associated with equation (6a) can be expressed as[20]:

$$(\mathbf{F}\cdot\Sigma^{\mathrm{T}})\cdot\hat{n} = \vec{t}(\vec{V}), \quad (11)$$

where $\vec{t}$ is the traction vector on the surface which has outward unit normal $\hat{n} = (l, m, n)$ before any deformation. The traction vector $\vec{t}$ depends on the displacement field $\vec{V} = (u, v, w)$. Indeed, because of the hydrostatic pressure loading, the magnitude of the surface load remains invariant under deformation, but its direction changes (since hydrostatis pressure is always directed along the normal to the surface on which it acts).

This gives

$$[\sigma_{rr}(1 + e_{rr}) + \tau_{r\theta}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \tau_{rz}(\tfrac{1}{2}e_{rz} + \omega_\theta)]l$$
$$+ [\tau_{r\theta}(1 + e_{rr}) + \sigma_{\theta\theta}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \tau_{\theta z}(\tfrac{1}{2}e_{rz} + \omega_\theta)]m$$
$$+ [\tau_{rz}(1 + e_{rr}) + \tau_{\theta z}(\tfrac{1}{2}e_{r\theta} - \omega_z) + \sigma_{zz}(\tfrac{1}{2}e_{rz} + \omega_\theta)]n = t_r, \quad (12a)$$

$$[\sigma_{rr}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \tau_{r\theta}(1 + e_{\theta\theta}) + \tau_{rz}(\tfrac{1}{2}e_{\theta z} - \omega_r)]l$$
$$+ [\tau_{r\theta}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \sigma_{\theta\theta}(1 + e_{\theta\theta}) + \tau_{\theta z}(\tfrac{1}{2}e_{\theta z} - \omega_r)]m$$
$$+ [\tau_{rz}(\tfrac{1}{2}e_{r\theta} + \omega_z) + \tau_{\theta z}(1 + e_{\theta\theta}) + \sigma_{zz}(\tfrac{1}{2}e_{\theta z} - \omega_r)]n = t_\theta, \quad (12b)$$

$$[\sigma_{rr}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \tau_{r\theta}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \tau_{rz}(1 + e_{zz})]l$$
$$+ [\tau_{r\theta}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \sigma_{\theta\theta}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \tau_{\theta z}(1 + e_{zz})]m$$
$$+ [\tau_{rz}(\tfrac{1}{2}e_{rz} - \omega_\theta) + \tau_{\theta z}(\tfrac{1}{2}e_{\theta z} + \omega_r) + \sigma_{zz}(1 + e_{zz})]n = t_z. \quad (12c)$$

If we write these equations for the initial and the perturbed equilibrium position and then subtract them and use the previous arguments on the relative magnitudes of the rotations ω'_j we obtain:

$$(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta)l + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)m$$
$$+ (\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)n$$
$$= \lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_r(\vec{V}_0 + \alpha\vec{V}_1) - t_r(\vec{V}_0)]\right\}, \quad (13a)$$

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r)l + (\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)m$$
$$+ (\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)n$$
$$= \lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_\theta(\vec{V}_0 + \alpha\vec{V}_1) - t_\theta(\vec{V}_0)]\right\}, \quad (13b)$$

$$(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta)l + (\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta)m$$
$$+ (\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta)n$$
$$= \lim_{\alpha\to 0}\left\{\frac{1}{\alpha}[t_z(\vec{V}_0 + \alpha\vec{V}_1) - t_z(\vec{V}_0)]\right\}. \quad (13c)$$

Let $\hat{n}^0$ and $\hat{n}^1$ denote the normal unit vectors to the bounding surface at the initial and perturbed positions of equilibrium, respectively. Before any deformation, this vector is $\hat{n} = (l, m, n)$. For external pressure p loading at the initial position

$$t_r(\vec{V}_0) = -p\cos(\hat{n}^0, \hat{r}); \quad t_\theta(\vec{V}_0) = -p\cos(\hat{n}^0, \hat{\theta});$$
$$t_z(\vec{V}_0) = -p\cos(\hat{n}^0, \hat{z}), \quad (14a)$$

and at the perturbed position

$$t_r(\vec{V}_0 + \alpha\vec{V}_1) = -p\cos(\hat{n}^1, \hat{r});$$
$$t_\theta(\vec{V}_0 + \alpha\vec{V}_1) = -p\cos(\hat{n}^1, \hat{\theta}); \quad (14b)$$
$$t_z(\vec{V}_0 + \alpha\vec{V}_1) = -p\cos(\hat{n}^1, \hat{z}).$$

But in terms of the deformation gradient

$$\mathbf{F}^{0,1}\cdot\hat{n} = (1 + E_n^{0,1})\hat{n}^{0,1}, \quad (15)$$

where E_n^0, E_n^1 is the relative elongation normal to the bounding surface at the initial and perturbed equilibrium positions, respectively. More explicitly,

$$\cos(\hat{n}^0, \hat{r}) = \frac{1}{1 + E_n^0}[(1 + e^0_{rr})l + (\tfrac{1}{2}e^0_{r\theta} - \omega^0_z)m$$
$$+ (\tfrac{1}{2}e^0_{rz} + \omega^0_\theta)n], \quad (16a)$$

$$\cos(\hat{n}^0, \hat{\theta}) = \frac{1}{1 + E_n^0}[(\tfrac{1}{2}e^0_{r\theta} + \omega^0_z)l + (1 + e^0_{\theta\theta})m$$
$$+ (\tfrac{1}{2}e^0_{\theta z} - \omega^0_r)n], \quad (16b)$$

$$\cos(\hat{n}^0, \hat{z}) = \frac{1}{1 + E_n^0}[(\tfrac{1}{2}e^0_{rz} - \omega^0_\theta)l + (\tfrac{1}{2}e^0_{\theta z} + \omega^0_r)m$$
$$+ (1 + e^0_{zz})n]. \quad (16c)$$

Similar expressions hold true for the perturbed state. For example,

$$\cos(\hat{n}^1, \hat{r}) = \frac{1}{1 + E_n^1}\Big\{(1 + e^0_{rr} + \alpha e'_{rr})l$$
$$+ [(\tfrac{1}{2}e^0_{r\theta} - \omega^0_z) + \alpha(\tfrac{1}{2}e'_{r\theta} - \omega'_z)]m$$
$$+ [(\tfrac{1}{2}e^0_{rz} + \omega^0_\theta) + \alpha(\tfrac{1}{2}e'_{rz} + \omega'_\theta)]n\Big\}. \quad (17)$$

The assumption of small strains allows neglecting E_n^0 and E_n^1 in comparison with unity. Substituting into the expressions (14) for the tractions in terms of the pressure and subtracting the initial and perturbed state and using the same arguments on the magnitude of rotations to neglect e'_{ij} in comparison with ω', we arrive at the following expressions:

$$t_r(\vec{V}_0 + \alpha\vec{V}_1) - t_r(\vec{V}_0) = p\alpha(\omega'_z m - \omega'_\theta n), \tag{18a}$$

$$t_\theta(\vec{V}_0 + \alpha\vec{V}_1) - t_\theta(\vec{V}_0) = -p\alpha(\omega'_z l - \omega'_r n), \tag{18b}$$

$$t_z(\vec{V}_0 + \alpha\vec{V}_1) - t_z(\vec{V}_0) = p\alpha(\omega'_\theta l - \omega'_r m), \tag{18c}$$

and in lieu of equation (13) for the lateral surfaces, i.e. for $m = n = 0$ and $l = 1$,

$$\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta = 0, \tag{19a}$$

$$\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r = -p\omega'_z, \tag{19b}$$

$$\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta = p\omega'_\theta, \tag{19c}$$

and for the end surfaces, i.e. $\ell = m = 0$ and $n = 1$,

$$\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta = -p\omega'_\theta, \tag{19d}$$

$$\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r = p\omega'_r, \tag{19e}$$

$$\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta = 0. \tag{19f}$$

Pre-buckling state. The problem under consideration is that of an orthotropic cylindrical shell subjected to a uniform external pressure, p, and an axial compression, P (*Figure 1*). The constitutive elasticity relations for the orthotropic body are:

$$\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix}. \tag{20}$$

where a_{ij} are the compliance constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

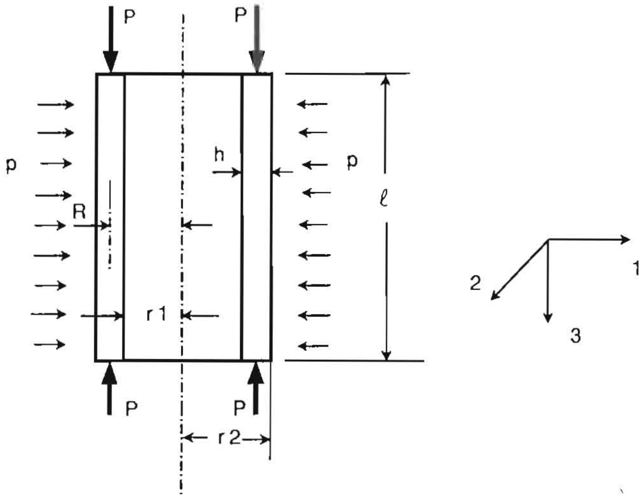

Figure 1 Definition of the geometry and the loading

In terms of the elastic constants:

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}} \qquad (i,j = 1,2,4,5,6), \tag{21a}$$

set[21]:

$$k = \sqrt{\frac{\beta_{11}}{\beta_{22}}}; \qquad x_1 = \frac{(a_{13} - a_{23})}{\beta_{22} - \beta_{11}}. \tag{21b}$$

Also, for convenience, set:

$$f_k = -\frac{r_2^{k+1}}{r_2^{2k} - r_1^{2k}}; \qquad f_{-k} = \frac{r_2^{k+1}r_1^{2k}}{r_2^{2k} - r_1^{2k}}, \tag{21c}$$

$$h_k = x_1\frac{r_2^{k+1} - r_1^{k+1}}{r_2^{2k} - r_1^{2k}}; \quad h_{-k} = x_1\frac{r_2^{k-1} - r_1^{k-1}}{r_2^{2k} - r_1^{2k}}r_1^{k+1}r_2^{k+1}. \tag{21d}$$

Then the normal stresses are given as follows:

$$\sigma_{rr} = p(f_k r^{k-1} + f_{-k}r^{-k-1}) + C(x_1 - h_k r^{k-1} - h_{-k}r^{-k-1}), \tag{22a}$$

$$\sigma_{\theta\theta} = p(f_k k r^{k-1} - f_{-k}kr^{-k-1}) + C(x_1 - h_k k r^{k-1} + h_{-k}kr^{-k-1}), \tag{22b}$$

and the shear stresses are:

$$\tau_{\theta z} = \tau_{rz} = \tau_{r\theta} = 0. \tag{22c}$$

The axial stress σ_{zz} is found from Lekhnitskii[21]:

$$\sigma_{zz} = C - \frac{1}{a_{33}}(a_{13}\sigma_{rr} + a_{23}\sigma_{\theta\theta}). \tag{23a}$$

For convenience, set:

$$\alpha_k = \frac{a_{13} + ka_{23}}{a_{33}}; \qquad \alpha_{-k} = \frac{a_{13} - ka_{23}}{a_{23}};$$

$$\gamma_1 = 1 - \frac{(a_{13} + a_{23})x_1}{a_{33}}. \tag{23b}$$

Then the axial stress is in the form:

$$\sigma_{zz} = -p(f_k\alpha_k r^{k-1} + f_{-k}\alpha_{-k}r^{-k-1}) + C(\gamma_1 + h_k\alpha_k r^{k-1} + h_{-k}\alpha_{-k}r^{-k-1}). \tag{23c}$$

Now the constant C is found from the condition of axial load:

$$\int_{r_1}^{r_2}\sigma_{zz}2\pi r \mathrm{d}r = -P, \tag{24a}$$

which gives

$$C\alpha_{11} = p\beta_1 - \frac{P}{2\pi}, \tag{24b}$$

where

$$\alpha_{11} = \gamma_1\frac{r_2^2 - r_1^2}{2} + h_k\alpha_k\frac{r_2^{k+1} - r_1^{r+1}}{k+1} + h_{-k}\alpha_{-k}\frac{r_2^{-k+1} - r_1^{-k+1}}{1-k}, \tag{24c}$$

and

$$\beta_1 = f_k\alpha_k\frac{r_2^{k+1} - r_1^{k+1}}{k+1} + f_{-k}\alpha_{-k}\frac{r_2^{-k+1} - r_1^{-k+1}}{1-k}. \tag{24d}$$

Equation (24b) may be changed to a single parameter equation by setting

$$\frac{P}{2\pi} = Spr_2^2, \tag{25}$$

where S is a nondimensional constant, which we shall call 'load interaction parameter'. The problem may then be solved for a series of selected values of S. A case of particular interest is represented by the ratio $S = 0.5$. For that value, $P = p\pi r_2^2$, and the cylindrical shell is seen to be subjected to a uniform pressure, p, applied to both its lateral surface and its ends, which are assumed to be capped. This case of pure hydrostatic-pressure loading has been treated in detail in ref. 14.

Introduction into (24b) gives:

$$C = p\tilde{C}; \qquad \tilde{C} = (\beta_1 - Sr_2^2)/\alpha_{11}. \tag{26}$$

Hence, we can write the stresses as follows:

$$\sigma_{rr} = p(\zeta_1 + \zeta_k r^{k-1} + \zeta_{-k} r^{-k-1}), \tag{27a}$$

$$\sigma_{\theta\theta} = p(\zeta_1 + \zeta_k k r^{k-1} - \zeta_{-k} k r^{-k-1}), \tag{27b}$$

$$\sigma_{zz} = p(\zeta_3 - \zeta_k \alpha_k r^{k-1} - \zeta_{-k}\alpha_{-k} r^{-k-1}), \tag{27c}$$

where

$$\zeta_k = -\tilde{C}h_k + f_k; \qquad \zeta_{-k} = -\tilde{C}h_{-k} + f_{-k}, \tag{27d}$$

$$\zeta_1 = \tilde{C}x_1; \qquad \zeta_3 = \tilde{C}\gamma_1. \tag{27e}$$

Therefore, it turns out that for a given load interaction parameter, S, the pre-buckling shear stresses are zero and the pre-buckling normal stresses are linearly dependent on the external pressure, p, in the form:

$$\sigma_{ij}^0 = p(C_{ij,0} + C_{ij,1} r^{k-1} + C_{ij,2} r^{-k-1}). \tag{28}$$

This observation allows a direct implementation of a standard solution scheme, since, as will be seen, the derivatives of the stresses with respect to p will be needed, and these are directly found from equations (28).

Perturbed state. Using the constitutive relations in equations (4) for the stresses σ'_{ij} in terms of the strains e'_{ij}, the strain–displacement relations in equation (7) for the strains e'_{ij} and the rotations ω'_j in terms of the displacements u_1, v_1, w_1, the buckling equation (10a) for the problem at hands is written in terms of the displacements at the perturbed state as follows:

$$c_{11}\left(u_{1,rr} + \frac{u_{1,r}}{r}\right) - c_{22}\frac{u_1}{r^2} + \left(c_{66} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta\theta}}{r^2}$$
$$+ \left(c_{55} + \frac{\sigma_{zz}^0}{2}\right)u_{1,zz} + \left(c_{12} + c_{66} - \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,r\theta}}{r}$$
$$- \left(c_{22} + c_{66} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta}}{r^2}$$
$$+ \left(c_{13} + c_{55} - \frac{\sigma_{zz}^0}{2}\right)w_{1,rz} + (c_{13} - c_{23})\frac{w_{1,z}}{r} = 0. \tag{29a}$$

The second buckling equation (10b) gives:

$$\left(c_{66} + \frac{\sigma_{rr}^0}{2}\right)\left(v_{1,rr} + \frac{v_{1,r}}{r} - \frac{v_1}{r^2}\right) + \left(\frac{\sigma_{rr}^0 - \sigma_{\theta\theta}^0}{2}\right)\left(\frac{v_{1,r}}{r} + \frac{v_1}{r^2}\right)$$
$$+ c_{22}\frac{v_{1,\theta\theta}}{r^2} + \left(c_{44} + \frac{\sigma_{zz}^0}{2}\right)v_{1,zz}$$
$$+ \left(c_{66} + c_{12} - \frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,r\theta}}{r} + \left(c_{66} + c_{22} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta}}{r^2}$$
$$+ \left(c_{23} + c_{44} - \frac{\sigma_{zz}^0}{2}\right)\frac{w_{1,\theta z}}{r}$$
$$+ \frac{1}{2}\frac{d\sigma_{rr}^0}{dr}\left(v_{1,r} + \frac{v_1}{r} - \frac{u_{1,\theta}}{r}\right) = 0. \tag{29b}$$

In a similar fashion, the third buckling equation (10c) gives:

$$\left(c_{55} + \frac{\sigma_{rr}^0}{2}\right)\left(w_{1,rr} + \frac{w_{1,r}}{r}\right) + \left(c_{44} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{w_{1,\theta\theta}}{r^2}$$
$$+ c_{33}w_{1,zz} + \left(c_{13} + c_{55} - \frac{\sigma_{rr}^0}{2}\right)u_{1,rz}$$
$$+ \left(c_{23} + c_{55} - \frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,z}}{r} + \left(c_{23} + c_{44} - \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta z}}{r}$$
$$+ \frac{1}{2}\frac{d\sigma_{rr}^0}{dr}(w_{1,r} - u_{1,z}) = 0. \tag{29c}$$

In the perturbed position, we seek equilibrium modes in the form:

$$u_1(r,\theta,z) = U(r)\cos n\theta \sin \lambda z;$$
$$v_1(r,\theta,z) = V(r)\sin n\theta \sin \lambda z, \tag{30}$$
$$w_1(r,\theta,z) = W(r)\cos n\theta \cos \lambda z,$$

where the functions $U(r)$, $V(r)$, $W(r)$ are uniquely determined for a particular choice of n and λ.

Substituting in (29a), we obtain the following linear homogeneous ordinary differential equation for $r_1 \le r \le r_2$:

$$U(r)''c_{11} + U(r)'\frac{c_{11}}{r}$$
$$+ U(r)\left[-c_{55}\lambda^2 - \frac{c_{22} + c_{66}n^2}{r^2} - \sigma_{zz}^0\frac{\lambda^2}{2} - \sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]$$
$$+ V(r)'\left[\frac{(c_{12} + c_{66})n}{r} - \sigma_{\theta\theta}^0\frac{n}{2r}\right]$$
$$+ V(r)\left[\frac{-(c_{22} + c_{66})n}{r^2} - \sigma_{\theta\theta}^0\frac{n}{2r^2}\right]$$
$$+ W(r)'\left[-(c_{13} + c_{55})\lambda + \sigma_{zz}^0\frac{\lambda}{2}\right]$$
$$+ W(r)\frac{(c_{23} - c_{13})\lambda}{r} = 0. \tag{31a}$$

The second differential equation (29b) gives for $r_1 \leq r \leq r_2$:

$$V(r)''\left(c_{66} + \frac{\sigma_{rr}^0}{2}\right) + V(r)'\left[\frac{c_{66}}{r} + \frac{1}{r}\left(\sigma_{rr}^0 - \frac{\sigma_{\theta\theta}^0}{2}\right) + \frac{\sigma_{rr}^{0\prime}}{2}\right]$$
$$+ V(r)\left[-c_{44}\lambda^2 - \frac{c_{66} + c_{22}n^2}{r^2} - \sigma_{zz}^0\frac{\lambda^2}{2} - \frac{\sigma_{\theta\theta}^0}{2r^2} + \frac{\sigma_{rr}^{0\prime}}{2r}\right]$$
$$+ U(r)'\left[\frac{-(c_{12} + c_{66})n}{r} + \sigma_{rr}^0\frac{n}{2r}\right]$$
$$+ U(r)\left[\frac{-(c_{22} + c_{66})n}{r^2} - \sigma_{\theta\theta}^0\frac{n}{2r^2} + \sigma_{rr}^{0\prime}\frac{n}{2r}\right]$$
$$+ W(r)\left[(c_{23} + c_{44})\frac{n\lambda}{r} - \sigma_{zz}^0\frac{n\lambda}{2r}\right] = 0. \qquad (31b)$$

In a similar fashion, (29c) gives for $r_1 \leq r \leq r_2$:

$$W(r)''\left(c_{55} + \frac{\sigma_{rr}^0}{2}\right) + W(r)'\left[\frac{c_{55}}{r} + \frac{\sigma_{rr}^0}{2r} + \frac{\sigma_{rr}^{0\prime}}{2}\right]$$
$$+ W(r)\left[-c_{33}\lambda^2 - c_{44}\frac{n^2}{r^2} - \sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]$$
$$+ U(r)'\left[(c_{13} + c_{55})\lambda - \sigma_{rr}^0\frac{\lambda}{2}\right]$$
$$+ U(r)\left[\frac{(c_{23} + c_{55})\lambda}{r} - \sigma_{rr}^0\frac{\lambda}{2r} - \sigma_{rr}^{0\prime}\frac{\lambda}{2}\right]$$
$$+ V(r)\left[(c_{23} + c_{44})\frac{n\lambda}{r} - \sigma_{\theta\theta}^0\frac{n\lambda}{2r}\right] = 0. \qquad (31c)$$

All the previous three equations (31) are linear, homogeneous, ordinary differential equations of the second order for $U(r)$, $V(r)$ and $W(r)$. In these equations $\sigma_{rr}^0(r)$, $\sigma_{\theta\theta}^0(r)$, $\sigma_{zz}^0(r)$ and $\sigma_{rr}^{0\prime}(r)$ depend linearly on the external pressure p through expressions in the form of equation (28).

Now we proceed to the boundary conditions on the lateral surfaces $r = r_1$, r_2. These will complete the formulation of the eigenvalue problem for the critical load.

From equation (19), we obtain for $\hat{l} = \pm 1$, $\hat{m} = \hat{n} = 0$:

$$\sigma_{rr}' = 0; \qquad \tau_{r\theta}' + (\sigma_{rr}^0 + p_j)\omega_z' = 0;$$
$$\tau_{rz}' - (\sigma_{rr}^0 + p_j)\omega_\theta' = 0, \quad \text{at } r = r_1, r_2. \qquad (32)$$

where $p_j = p$ for $r = r_2$ (outside boundary) and $p_j = 0$ for $r = r_1$ (inside boundary).

Substituting in equations (20), (7) and (30), the boundary condition $\sigma_{rr}' = 0$ at $r = r_j, j = 1, 2$ gives:

$$U'(r_j)c_{11} + [U(r_j) + nV(r_j)]\frac{c_{12}}{r_j} - c_{13}\lambda W(r_j) = 0,$$
$$j = 1, 2. \qquad (33a)$$

The boundary condition $\tau_{r\theta}' + (\sigma_{rr}^0 + p_j)\omega_z' = 0$ gives:

$$V'(r_j)\left[c_{66} + (\sigma_{rr}^0 + p_j)\frac{1}{2}\right]$$
$$+ [V(r_j) + nU(r_j)]\left[-c_{66} + (\sigma_{rr}^0 + p_j)\frac{1}{2}\right]\frac{1}{r_j} = 0,$$
$$j = 1, 2. \qquad (33b)$$

In a similar fashion, the condition $\tau_{rz}' - (\sigma_{rr}^0 + p_j)\omega_\theta' = 0$ at $r = r_j, j = 1, 2$ gives:

$$\lambda U(r_j)\left[c_{55} - (\sigma_{rr}^0 + p_j)\tfrac{1}{2}\right]$$
$$+ W'(r_j)\left[c_{55} + (\sigma_{rr}^0 + p_j)\tfrac{1}{2}\right] = 0, \quad j = 1, 2. \qquad (33c)$$

Therefore, for a given load interaction, S, equations (31) and (33) constitute an eigenvalue problem for differential equations, with the applied external pressure, p, the parameter, which can be solved by standard numerical methods (two point boundary value problem).

Before discussing the numerical procedure used for solving this eigenvalue problem, one final point will be addressed. To completely satisfy all the elasticity requirements, we should discuss the boundary conditions at the ends. From equations (19), the boundary conditions on the ends $\hat{l} = \hat{m} = 0$, $\hat{n} = \pm 1$, are:

$$\tau_{rz}' + (\sigma_{zz}^0 + p)\omega_\theta' = 0; \qquad \tau_{\theta z}' - (\sigma_{zz}^0 + p)\omega_r' = 0;$$
$$\sigma_{zz}' = 0, \qquad \text{at } z = 0, \ell. \qquad (34)$$

Since σ_{zz}' varies as $\sin\lambda z$, the condition $\sigma_{zz}' = 0$ on both the lower end $z = 0$, and the upper end $z = \ell$, is satisfied if

$$\lambda = \frac{m\pi}{\ell}. \qquad (35)$$

It will be proved now that these remaining two conditions are satisfied on the average. To show this we write each of the first two expressions in equations (34) in the form: $S_{rz} = \tau_{rz}' + (\sigma_{zz}^0 + p)\omega_\theta'$ and $S_{\theta z} = \tau_{\theta z}' - (\sigma_{zz}^0 + p)\omega_r'$, and integrate their resultants in the Cartesian coordinate system (x, y, z), e.g. the x-resultant of S_{rz} is: $\int_{r_1}^{r_2}\int_0^{2\pi} S_{rz}(\cos\theta)(rd\theta)dr$.

Since τ_{rz}' and ω_θ' have the form of $F(r)\cos n\theta\cos\lambda z$, i.e. they have a $\cos n\theta$ variation, the x-component of S_{rz} has a $\cos n\theta\cos\theta$ variation, which, when integrated over the entire angle range from zero to 2π, will result in zero. The y-component has a $\cos n\theta\sin\theta$ variation, which, again, when integrated over the entire angle range, will result in zero. Similar arguments hold for $S_{\theta z}$, which has the form of $F(r)\sin n\theta\cos\lambda z$.

Moreover, it can also be proved that the system of resultant stresses, equations (34) would produce no torsional moment. Indeed, this moment would be given by $\int_{r_1}^{r_2}\int_0^{2\pi} S_{\theta z}(rd\theta)rdr$. Since $\tau_{\theta z}'$ and ω_r' and hence $S_{\theta z}$ have a $\sin n\theta$ variation, the previous integral will be in the form $\int_{r_1}^{r_2}\int_0^{2\pi} r^2F(r)\sin n\theta\cos\lambda z dr d\theta$, which, when integrated over the entire θ-range from zero to 2π, will result in zero.

An alternative method of proving these conditions by using the equilibrium equations in a Cartesian coordinate system and the divergence theorem for transformation of an area integral into a contour integral as well as the lateral boundary conditions in the Cartesian coordinate system (analogous to equations (19)) was outlined in ref. 18.

Returning to the discussion of the eigenvalue problem, as has already been stated, equations (31) and (33) constitute an eigenvalue problem for ordinary second order linear differential equations in the r variable, with the applied external pressure, p, the parameter. This is

Table 1 Comparison with shell theories—axial compression

Orthotropic with circumferential reinforcement, $\ell/r_2 = 5$

Critical loads, $\tilde{P} = \dfrac{P}{\pi(r_2^2 - r_1^2)}\dfrac{r_2}{E_3 h}$

Moduli in GN/m^2: $E_2 = 57$; $E_1 = E_3 = 14$; $G_{31} = 5.0$; $G_{12} = G_{23} = 5.7$

Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$

r_2/r_1	Elasticity (n, m)	Sanders-type* (n, m) (% increase)	Timoshenko* (n, m) (% increase)
1.05	0.6764 (2,1)	0.7904 (4,9) (16.9%)	0.6735 (2,1) (−0.4%)
1.10	0.6641 (2,2)	0.7883 (3,6) (18.7%)	0.6461 (2,2) (−2.7%)
1.15	0.6284 (2,2)	0.7716 (2,3) (22.8%)	0.6218 (2,3) (−1.1%)
1.20	0.6134 (2,3)	0.7505 (2,3) (22.4%)	0.5559 (1,1) (−9.4%)
1.25	0.5186 (1,1)	0.7560 (2,4) (45.8%)	0.4549 (1,1) (−12.3%)
1.30	0.4429 (1,1)	0.7771 (1,1) (75.5%)	0.3876 (1,1) (−12.5%)

* See Appendix with $p = 0$

Table 2 Comparison with shell theories—axial compression

Isotropic, $E = 14$ GN/m^2, $\nu = 0.3$, $\ell/r_2 = 5$

Critical loads, $\tilde{P} = \dfrac{P}{\pi(r_2^2 - r_1^2)}\dfrac{r_2}{E_3 h}$

r_2/r_1	Elasticity (n, m)	Sanders-type (n, m) (% increase)	Timoshenko (n, m) (% increase)	Flügge (n, m) (% increase)	Danielson & Simmonds (n, m) (% increase)
1.05	0.4426 (2,1)	0.5474 (2,1) (23.7%)	0.4348 (2,1) (−1.8%)	0.4525 (2,1) (2.2%)	0.4559 (2,1) (3.0%)
1.10	0.3910 (2,1)	0.4871 (2,1) (24,6%)	0.3865 (2,1) (−1.2%)	0.4019 (2,1) (2.8%)	0.4088 (2,1) (4.6%)
1.15	0.4547 (2,1)	0.5488 (2,2) (20.7%)	0.4373 (2,2) (−3.8%)	0.4710 (2,1) (3.6%)	0.4814 (2,1) (5.9%)
1.20	0.4371 (2,2)	0.5272 (2,2) (20.6%)	0.4184 (2,2) (−4.3%)	0.4620 (2,2) (5.7%)	0.4705 (2,2) (7.6%)
1.25	0.4426 (2,2)	0.5403 (2,2) (22.0%)	0.4269 (2,2) (−3.5%)	0.4728 (2,2) (6.8%)	0.4837 (2,2) (9.3%)
1.30	0.4487 (1,1)	0.5709 (2,2) (27.2%)	0.3895 (1,1) (−13.2%)	0.4915 (1,1) (9.5%)	0.4987 (1,1) (11.1%)

essentially a standard two point boundary value problem. The relaxation method was used[22], which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the shell. For this purpose an equally spaced mesh of 241 points was employed and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the shell theory solution was used. An investigation of the convergence showed that essentially the same results were produced with even three times as many mesh points. The procedure employs the derivatives of the equations with respect to the functions U, V, W, U', V', W' and the pressure p; hence, because of the linear nature of the equations and the linear dependence of σ_{ij}^0 on p through equations (28), it can be directly implemented. Finally, it should be noted that finding the critical load involves a minimization step in the sense that the eigenvalue is obtained for different combinations of n, m, and the critical load is the minimum. The specific results are presented in the following.

DISCUSSION OF RESULTS

The critical loads for pure axial compression, P, are given in *Tables 1* and *2*. The critical loads for pure external pressure, p (corresponding to a load interaction parameter $S = 0.5$), are given in *Tables 3* and *4*. Results for combined external pressure and axial compression are presented in *Tables 5* and *6*; the critical condition is defined by the external pressure and the axial load (p, P), normalized as:

$$\tilde{p} = \frac{pr_2^3}{E_2 h^3}; \qquad \tilde{P} = \frac{P}{\pi(r_2^2 - r_1^2)}\frac{r_2}{E_3 h}. \tag{36}$$

The results were produced for two composites; a typical glass/epoxy material with moduli in GN/m^2 and Poisson's ratios listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$, and a typical graphite/epoxy material, with moduli: $E_2 = 140$, $E_1 = 9.9$, $E_3 = 9.1$, $G_{31} = 5.9$, $G_{12} = 4.7$, $G_{23} = 4.3$ and Poisson's ratios: $\nu_{12} = 0.020$, $\nu_{23} = 0.300$, $\nu_{31} = 0.490$. It has been assumed that the reinforcing direction is along the circumferential direction. In all these studies, an external radius $r_2 = 1$ m and a length ratio $\ell/r_2 = 5$ or 10 have been assumed. A range of outside *versus* inside radius, r_2/r_1 from somewhat thin, 1.03, to thick, 1.30, is examined.

In the shell theory solutions, the radial displacement is constant through the thickness and the axial and

Table 3 Comparison with shell theories—external pressure

Glass/epoxy (orthotropic) with circumferential reinforcement, $\ell/r_2 = 10$

Critical pressure, $\tilde{p} = pr_2^3/(E_2h^3)$

Moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$

Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$

$n = 2$, $m = 1$

r_2/r_1	Elasticity	Sanders-type* (% increase)	Timoshenko* (% increase)
1.05	0.2813	0.2926 (4.0%)	0.2914 (3.6%)
1.10	0.2744	0.2973 (8.3%)	0.2962 (7.9%)
1.15	0.2758	0.3133 (13.6%)	0.3122 (13.2%)
1.20	0.2764	0.3308 (19.7%)	0.3296 (19.2%)
1.25	0.2755	0.3485 (26.5%)	0.3473 (26.1%)
1.30	0.2733	0.3662 (34.0%)	0.3649 (33.5%)

* See Appendix with $P = 0$

circumferential ones have a linear variation, i.e. they are in the form:

$$u_1(r,\theta,z) = U_0 \cos n\theta \sin \lambda z,$$

$$v_1(r,\theta,z) = \left[V_0 + \frac{r-R}{R}(V_0 + nU_0)\right] \sin n\theta \sin \lambda z, \quad (37a)$$

$$w_1(r,\theta,z) = [W_0 - (r-R)\lambda U_0] \cos n\theta \cos \lambda z, \quad (37b)$$

where U_0, V_0, W_0, are constants (these displacement field variations would satisfy the classical assumptions of $e_{rr} = e_{r\theta} = e_{rz} = 0$).

A distinct eigenvalue corresponds to each pair of the positive integers m and n. The pair corresponding to the smallest eigenvalue can be determined by trial. As noted in the Introduction, one of the classical theories that will be used for comparison purposes is the non-shallow shell theory based on Sanders-type[10] kinematic relations (e.g. ref. 23). The other benchmark shell buckling equations used in this paper are the ones described in

Table 4 Comparison with shell theories—external pressure

Graphite/epoxy (orthotropic) with circumferential reinforcement, $\ell/r_2 = 10$

Critical pressure, $\tilde{p} = pr_2^3/(E_2h^3)$

Moduli in GN/m^2: $E_2 = 140$, $E_1 = 9.9$, $E_3 = 9.1$, $G_{31} = 5.9$, $G_{12} = 4.7$, $G_{23} = 4.3$

Poisson's ratios: $\nu_{12} = 0.020$, $\nu_{23} = 0.300$, $\nu_{31} = 0.490$

$n = 2$, $m = 1$

r_2/r_1	Elasticity (n,m)	Sanders-type (n,m) (% increase)	Timoshenko (n,m) (% increase)
1.05	0.2576 (2,1)	0.2723 (2,1) (5.7%)	0.2713 (2,1) (5.3%)
1.10	0.2513 (2,1)	0.2871 (2,1) (14.2%)	0.2861 (2,1) (13,8%)
1.15	0.2347 (2,2)	0.3037 (2,2) (29.4%)	0.2995 (2,2) (27.6%)
1.20	0.2166 (2,3)	0.3183 (2,2) (47.0%)	0.3111 (2,3) (43.6%)
1.25	0.1978 (2,3)	0.3310 (2,3) (67.3%)	0.3198 (2,4) (61.7%)
1.30	0.1808 (2,4)	0.3429 (2,4) (89.7%)	0.3261 (2,5) (80.4%)

Table 5 Comparison with shell theories—combined external pressure and axial compression

Glass/epoxy (orthotropic) with circumferential reinforcement, $\ell/r_2 = 5$

Load interaction parameter (equation (25)), $S = 5.0$

Critical loads (equation (36)), $(\tilde{p}, \tilde{P})$ (n,m)

r_2/r_1	Elasticity	Sanders-type* (% increase)	Timoshenko* (% increase)
1.03	(0.5561, 0.3346) (2,1)	(0.6209, 0.3736) (2,1) (11.7%)	(0.5653, 0.3401) (2,1) (1.7%)
1.05	(0.3014, 0.2993) (2,1)	(0.3435, 0.3411) (2,1) (14.0%)	(0.3130, 0.3108) (2,1) (3.8%)
1.10	(0.1971, 0.3822) (2,1)	(0.2371, 0.4597) (2,1) (20.3%)	(0.2165, 0.4198) (2,1) (9.8%)
1.15	(0.1665, 0.4730) (2,2)	(0.2218, 0.6300) (2,2) (33.2%)	(0.1886, 0.5356) (2,2) (13.3%)
1.20	(0.1335, 0.4940) (2,2)	(0.1909, 0.7067) (2,2) (43.0%)	(0.1624, 0.6009) (2,2) (21.6%)
1.25	(0.1167, 0.5278) (1,1)	(0.1753, 0.7932) (2,3) (50.2%)	(0.1241, 0.5615) (1,1) (6.3%)

* See Appendix

Table 6 Comparison with shell theories—combined external pressure and axial compression

Glass/epoxy (orthotropic) with circumferential reinforcement, $\ell/r_2 = 5$

Load interaction parameter (equation (25)), $S = 1.0$

Critical loads (equation (36)), $(\tilde{p}, \tilde{P})$ (n, m)

r_2/r_1	Elasticity	Sanders-type (% increase)	Timoshenko (% increase)
1.03	(0.7311, 0.0880) (3,1)	(0.7518, 0.0905) (3,1) (2.8%)	(0.7480, 0.0900) (3,1) (2.3%)
1.05	(0.4666, 0.0927) (2,1)	(0.4965, 0.0986) (2,1) (6.4%)	(0.4829, 0.0959) (2,1) (3.5%)
1.10	(0.3038, 0.1178) (2,1)	(0.3386, 0.1313) (2,1) (11.4%)	(0.3297, 0.1278) (2,1) (8.5%)
1.15	(0.2758, 0.1567) (2,1)	(0.3235, 0.1838) (2,1) (17.3%)	(0.3152, 0.1791) (2,1) (14.3%)
1.20	(0.2659, 0.1968) (2,1)	(0.3297, 0.2440) (2,1) (24.0%)	(0.3214, 0.2379) (2,1) (20.9%)
1.25	(0.2600, 0.2353) (2,1)	(0.3418, 0.3093) (2,1) (31.5%)	(0.3334, 0.3017) (2,1) (28.2%)

ref. 11. In these equations, an additional term in the first equation, namely $-N_\theta^0(v_{,\theta z}+u_{,z})$, and an additional term in the second equation, namely $RN_z^0 v_{,zz}$, exist.

In the comparison studies we have used an extension of the original, isotropic Donnell[9], Sanders[10] and Timoshenko and Gere[11] formulations for the case of orthotropy. The linear algebraic equations for the eigenvalues of both the Sanders-type and Timoshenko-type formulations are given in more detail in the Appendix.

Concerning the present elasticity formulation, the critical load is obtained for a given load interaction parameter, S, by finding the solution for p for a range of n and m, and keeping the minimum value. The following observations can be made:

- For *pure axial compression*:

1. The bifurcation points from the Timoshenko formulation are always closer to the elasticity predictions than the ones from the Sanders-type formulation.
2. For both the orthotropic and the isotropic cases, the bifurcation point for the Sanders-type shell theory, is always higher than the elasticity solution, which means that the Sanders-type formulation is non-conservative. Moreover, this Sanders-type shell theory becomes, in general, more non-conservative with thicker construction.
3. On the contrary, the Timoshenko bifurcation point is lower than the elasticity one in all cases considered, i.e. the Timoshenko formulation is actually conservative in predicting stability loss under pure axial compression. The degree of conservatism of the Timoshenko formulation generally increases for thicker shells.

- For *pure external pressure*:

1. For both the orthotropic material cases, the bifurcation points from both the Sanders-type and the Timoshenko-type formulations are always higher than the elasticity solution, which means that both these shell formulations are non-conservative. Moreover, they become more non-conservative with thicker construction.
2. The bifurcation points from the Timoshenko formulation are always slightly closer to the elasticity predictions than the ones from the Sanders-type formulation.
3. The degree of non-conservatism is strongly dependent on the material; the shell theories predict much higher deviations from the elasticity solution for the graphite/epoxy (which is also noted to have a much higher extensional-to-shear modulus ratio).

- For the *combined axial compression and external pressure*:

  It is seen that for the relatively high axial load case, $S = 5$, the Timoshenko equations perform remarkably well, approaching closely the elasticity results, especially for thick construction; the degree of non-conservatism for these theories is dependent not only on the material, but also on the load interaction, S.

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APPENDIX

Eigenvalues from non-shallow Sanders-type and Timoshenko shell formulations

In the shell theory formulation, the mid-thickness ($r = R$) displacements are in the form:

$$u_1 = U_0 \cos n\theta \sin \lambda z, \quad v_1 = V_0 \sin n\theta \sin \lambda z,$$

$$w_1 = W_0 \cos n\theta \cos \lambda z,$$

where U_0, V_0, W_0 are constants.

The equations for the non-shallow shell theory based on the Sanders[10] kinematic relations are (Brush and Almroth[23]):

$$RN_{z,z} + N_{z\theta,\theta} = 0$$

$$RN_{z\theta,z} + N_{\theta,\theta} + \frac{M_{\theta,\theta}}{R} + M_{z\theta,z} = 0$$

$$N_\theta - RN_z^0 u_{,zz} - RM_{z,zz} - \frac{M_{\theta,\theta\theta}}{R} - 2M_{z\theta,z\theta} + N_\theta^0 \beta_{\theta,\theta} + p(v_{,\theta} + u) = 0,$$

where $R\beta_\theta = v - u_{,\theta}$. The Timoshenko shell buckling equations[11] have the additional term $-N_\theta^0(v_{,\theta z} + u_{,z})$ in the first equation, and the additional term $RN_z^0 v_{,zz}$ in the second equation. We have denoted by R the mean shell radius and by p the absolute value of the external pressure. Notice that the external pressure p would give, $N_{z\theta}^0 = 0$ and $N_\theta^0 = -pR$ and the axial compression, P, would give $N_z^0 = -P/(2\pi R) = -pSR$ (where S is the load interaction parameter, defined in equation (25)). It should be pointed out that in previous work (Kardomateas[18]), these Sanders-type equations have been referred to as the 'non-simplified' Donnell equations (because the Donnell equations can be directly derived from these if some simplifying assumptions are made in the kinematic relations).

In terms of the 'equivalent property' constants:

$$C_{22} = E_2 h/(1 - \nu_{23}\nu_{32}); \quad C_{33} = E_3 h/(1 - \nu_{23}, \nu_{32}),$$

$$C_{23} = \frac{E_3 \nu_{23} h}{1 - \nu_{23}\nu_{32}}; \quad C_{44} = G_{23} h, \quad D_{ij} = C_{ij}\frac{h^2}{12},$$

the coefficient terms in the homogeneous equations system that gives the eigenvalues are:

$$\alpha_{11} = C_{23}\lambda; \quad \alpha_{12} = (C_{23} + C_{44})n\lambda;$$

$$\alpha_{13} = -(C_{33}R\lambda^2 + C_{44}n^2/R),$$

$$\alpha_{21} = -\left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 2\frac{D_{44}\lambda^2}{R}\right)n,$$

$$\alpha_{22} = -\left(\frac{C_{22}n^2}{R} + C_{44}R\lambda^2 + \frac{D_{22}n^2}{R^3} + 2\frac{D_{44}\lambda^2}{R}\right),$$

$$\alpha_{23} = (C_{23} + C_{44})n\lambda,$$

$$\alpha_{31} = \frac{C_{22}}{R} + \frac{D_{22}n^4}{R^3} + 2\frac{D_{23}\lambda^2 n^2}{R} + D_{33}\lambda^4 R + 4\frac{D_{44}\lambda^2 n^2}{R},$$

$$\alpha_{32} = \left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 4\frac{D_{44}\lambda^2}{R}\right)n,$$

$$\alpha_{33} = -C_{23}\lambda.$$

Notice that in the above formulas we have used the curvature expression $\kappa_{z\theta} = (v_{,z} - u_{,z\theta})/R$ for both theories.

Then the linear homogeneous equations system that gives the eigenvalues for the Timoshenko formulation for the case of combined axial compression, P, and external pressure, p, is:

$$(\alpha_{11} + pR\lambda)U_0 + (\alpha_{12} + pRn\lambda)V_0 + \alpha_{13}W_0 = 0, \quad \text{(A1)}$$

$$\alpha_{21}U_0 + \left(\alpha_{22} + P\frac{\lambda^2}{2\pi}\right)V_0 + \alpha_{23}W_0 = 0, \quad \text{(A2)}$$

$$\left[\alpha_{31} - P\frac{\lambda^2}{2\pi} - p(n^2 - 1)\right]U_0 + \alpha_{32}V_0 + \alpha_{33}W_0 = 0. \quad \text{(A3)}$$

For the Sanders shell formulation, the additional term in the coefficient of V_0 in (A2) is ommitted, i.e. the coefficient of V_0 is only α_{22} and the additional terms in the coefficients of U_0 and V_0 in equation (A1) are also ommitted, i.e. the coefficient of U_0 is only α_{11} and the coefficient of V_0 is only α_{12}. The eigenvalues (for a given load interaction, S, in general) are naturally found by equating to zero the determinant of the coefficients of U_0, V_0 and W_0. Notice that in the case of combined axial compression and external pressure, the axial load, P, is expressed in terms of the external pressure, p, through the load interaction parameter, S, defined in equation (25).