Pergamon

Int. J. Solids Structures Vol. 35, No. 13, pp. 1355–1362, 1998
© 1998 Elsevier Science Ltd
All rights reserved. Printed in Great Britain
0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00110–8

BUCKLING OF ORTHOTROPIC BEAM-PLATES WITH MULTIPLE CENTRAL DELAMINATIONS

HAIYING HUANG and GEORGE A. KARDOMATEAS
School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, Georgia 30332-0150, U.S.A.

(*Received* 15 *September* 1996; *in revised form* 24 *March* 1997)

Abstract—A closed form solution is developed for predicting the critical load of a composite beam-plate with multiple delaminations. The characteristic equation is derived by using non-linear beam theory, performing proper linearization and by imposing the appropriate kinematical continuity and equilibrium conditions. The effects of the dimensions and locations of the delaminations on the critical load are investigated and the results are compared with previously published data. © 1998 Elsevier Science Ltd.

INTRODUCTION

Composite materials have many advantages over conventional materials such as metals and alloys, especially high strength to weight ratios as well as stiffness to weight ratios. The increasing usage of composite materials in industry requires better understanding of their structural behavior and failure conditions. Delamination (interlayer cracking) is one of the most common failure modes of laminated composite materials, and can be caused by manufacturing defects or impact loading. Under compression, a delaminated composite plate may buckle and possibly undergo propagation of the interlayer crack.

Most studies in the field of delamination buckling are concentrated on the critical load prediction of an idealized single delamination in isotropic or orthotropic materials. Chai *et al.* (1981) investigated the growth of a general single delamination by determining the energy release rate through a numerical differentiation of the total energy with respect to the delamination length. Simitses *et al.* (1985) and Yin *et al.* (1986) developed a simple model for predicting the critical load and the ultimate load capacity of a beam-plate with a single delamination. Davidson and Ferrie (1994) discussed the effect of stretching-shear coupling on delamination growth. Kardomateas (1989, 1993) and Kardomateas and Pelegri (1994) analyzed the post-buckling and growth behavior of an internal delamination in a composite plate under compression by using perturbation methods.

Because of its complexity, the multiple delamination problem has not yet been as extensively studied as the single delamination problem. Most of the research conducted for the case of multiple delaminations was based on numerical methods. Lim and Parsons (1993) employed energy methods to derive a finite element solution for equal-length delaminations. A finite element analytical and an experimental investigation was performed by Kutlu and Chang (1992) to study the compression response of a multiply delaminated composite plate. Contact between delaminated layers was studied by Suemasu (1993) and Larsson (1991). Sheinman and Soffer (1991) and Adan *et al.* (1993) developed an analytical model for the buckling of multiply delaminated composite beams and studied the interactive effect between delaminations.

In this paper, the nonlinear differential equation for a beam is linearized and combined with appropriate kinematical continuity conditions, equilibrium equations and boundary conditions. A closed form expression of the characteristic equation of a plate with multiple central delaminations is derived. In this manner, the critical load of a plate with delaminations of different sizes and locations can be quickly calculated. The method can be easily extended to nonsymmetric delaminations, and also to a plate with more than two delaminations.

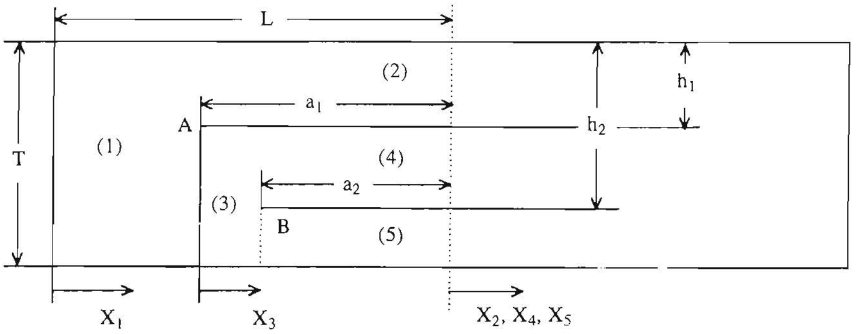

Fig. 1. Definition of the geometry of the multiple delamination problem.

ANALYTICAL FORMULATION

Problem definition

The geometry of the delaminated plates employed in the study is shown in Fig. 1. A plate of half length L, thickness T and width W has two pre-existing central delaminations. Both ends of the plate are clamped and the external load is symmetrically applied. Because of the symmetry of the structure, only the left half needs to be considered. The half plate is divided into five subplates, each with a size parameter pair (L_i, t_i), L_i and t_i being the length and thickness of the ith subplate, respectively. The coordinate systems for the subplates are also shown in Fig. 1. Point A and B are the crack tips. Denote the delamination closest to the surface as delamination I and the one further inside as delamination II. The sizes and locations of these two delaminations are defined by their length a_i and depth h_i. Notice that $t_1 = T$, $t_2 = h_1$, $t_3 = T-h_1$, $t_4 = h_2-h_1$, $t_5 = T-h_2$, and $L_1 = L-a_1$, $L_2 = a_1$, $L_3 = a_1-a_2$, $L_4 = L_5 = a_2$.

Basic equations

According to the nonlinear beam theory, the differential equation for a beam is:

$$D\frac{d^2\theta}{dx^2}+P\sin\theta = 0,$$

where D is the bending stiffness of the plate, $D = EI/(1-\nu^2)$ for isotropic materials and $D = E_1I/(1-\nu_{12}\nu_{21})$ for orthotropic materials. To calculate the critical load, the nonlinear equation is linearized as:

$$D\frac{d^2\theta}{dx^2}+P\theta = 0. \tag{1}$$

The general solution for eqn (1) is:

$$\theta = A\sin\lambda x+B\cos\lambda x$$

with $\lambda = \sqrt{P/D}$. Thus, for subplates (1)–(5):

$$\theta_i = A_i\sin\lambda_i x_i+B_i\cos\lambda_i x_i$$

where $\lambda_i = \sqrt{P_i^0/D}$, $P_i^0 = \bar{P}t_i/T$ and $\bar{P}$ is the unknown external load. Here, uniform distribution of the compressive load at the pre-buckling state was assumed.

According to the boundary condition of clamped ends, $\theta_1(0) = 0$, we have:

$$\theta_1 = A_1\sin\lambda_1 x_1. \tag{2a}$$

Because the structure is symmetric, $\theta_2(0) = \theta_4(0) = \theta_5(0) = 0$, so:

$$\theta_2 = A_2 \sin \lambda_2 x_2, \tag{2b}$$

$$\theta_4 = A_4 \sin \lambda_4 x_4, \tag{2c}$$

$$\theta_5 = A_5 \sin \lambda_5 x_5, \tag{2d}$$

and

$$\theta_3 = A_3 \sin \lambda_3 x_3 + B_3 \cos \lambda_3 x_3. \tag{2e}$$

We have six unknown coefficients and one unknown force $\bar{P}$ in eqn (2). To calculate the critical load, the continuity conditions and equilibrium equations at the crack tips A and B must be used.

(1) Continuity conditions:

At point A:

$$\theta_1(L_1) = \theta_2(-L_2);\quad \theta_1(L_1) = \theta_3(0);\quad u_2^A = u_3^A. \tag{3a}$$

At point B:

$$\theta_3(L_3) = \theta_4(-L_4);\quad \theta_4(-L_4) = \theta_5(-L_5);\quad u_4^B = u_5^B. \tag{3b}$$

u is the displacement along the x direction. The displacements at point A and B are due to the shortening of the mid-plane under compression loading and the rotation of the cross section about the midplane. So:

$$u_2^A = \frac{P_2 L_2}{WEt_2} + \frac{t_2}{2}\theta^A, \tag{4a}$$

$$u_3^A = u_5^c + \frac{P_3 L_3}{WEt_3} - \frac{t_3}{2}\theta^A = \frac{P_5 L_5}{WEt_5} - \frac{t_4}{2}\theta^B + \frac{P_3 L_3}{WEt_3} - \frac{t_3}{2}\theta^A, \tag{4b}$$

$$u_4^B = \frac{P_4 L_4}{WEt_4} + \frac{t_4}{2}\theta^B, \tag{4c}$$

$$u_5^B = \frac{P_5 L_5}{WEt_5} - \frac{t_5}{2}\theta^B. \tag{4d}$$

In eqn (4b), c refers to the mid-thickness point of subplate 3 at the section of the delamination tip B (Fig. 1).

(2) Equilibrium equations:

At point A:

$$P_1 = P_2 + P_3, \tag{5a}$$

$$M_1 - M_2 - M_3 - \frac{P_2 t_3}{2} + \frac{P_3 t_2}{2} = 0. \tag{5b}$$

At point B:

$$P_3 = P_4 + P_5, \tag{5c}$$

$$M_3 - M_4 - M_5 - \frac{P_4 t_5}{2} + \frac{P_5 t_4}{2} = 0. \tag{5d}$$

We denote $P_i = P_i^0 + P_i^a$, where P_i^a is the additional force due to bending of the subplates. Notice that at primary state (i.e., before buckling):

$$P_1^0 = P_2^0 + P_3^0, \tag{6a}$$

$$P_3^0 = P_4^0 + P_5^0, \tag{6b}$$

$$\frac{P_2^0 t_3}{2} - \frac{P_3^0 t_2}{2} = 0, \tag{6c}$$

$$\frac{P_4^0 t_5}{2} - \frac{P_5^0 t_4}{2} = 0, \tag{6d}$$

$$\frac{P_2^0 L_2}{WEt_2} = \frac{P_3^0 L_3}{WEt_3} + \frac{P_4^0 L_4}{WEt_4}, \tag{7a}$$

$$\frac{P_5^0 L_5}{WEt_5} = \frac{P_4^0 L_4}{WEt_4}. \tag{7b}$$

Substituting (6) into the equilibrium equations (5) gives:

$$P_2^a - P_3^a = 0, \tag{8a}$$

$$M_1 - M_2 - M_3 - \frac{P_2^a t_3}{2} + \frac{P_3^a t_2}{2} = 0, \tag{8b}$$

$$P_3^a = P_4^a + P_5^a, \tag{8c}$$

$$M_3 - M_4 - M_5 - \frac{P_4^a t_5}{2} + \frac{P_5^a t_4}{2} = 0, \tag{8d}$$

where $M_i = D_i \mathrm{d}\theta_i/\mathrm{d}x_i$.

Substituting now (4), (7), (8a), (8c) into the third eqn in (3a) and the third eqn in (3b), we have:

$$\frac{P_2^a L_2}{WEt_2} + \frac{t_2}{2}\theta^A = \frac{P_5^a L_5}{WEt_5} - \frac{t_4}{2}\theta^B + \frac{P_3^a L_3}{WEt_3} - \frac{t_3}{2}\theta^A, \tag{9a}$$

$$\frac{P_4^a L_4}{WEt_4} + \frac{t_4}{2}\theta^B = \frac{P_5^a L_5}{WEt_5} - \frac{t_5}{2}\theta^B. \tag{9b}$$

Introducing eqn (2) into eqns (3), (8), (9) and eliminating the force terms P_2^a, P_3^a, P_4^a, P_5^a, and B_3, we get five linear homogeneous algebraic equations. Express them in matrix form:

$$[F]\{A\} = 0, \tag{10}$$

where $\{A\} = \{A_1, A_2, A_3, A_4, A_5\}^{\mathrm{T}}$. The characteristic equation is:

$$\det\{F(\bar{P})\} = 0. \tag{11}$$

The lowest external load $\bar{P}$ satisfying eqn (11) is the critical load.

Table 1. Critical load, $\bar{P}_{cr}$, for a single delamination of varying length and location

$\bar{a}$	$\bar{h}$ 0.10	0.20	0.30	0.40	0.50
0.1	0.9799	0.9997	0.9998	0.9999	0.9999
0.2	0.2495	0.9264	0.9924	0.9950	0.9956
0.3	0.1103	0.4371	0.8582	0.9543	0.9638
0.4	0.0624	0.2470	0.5314	0.7883	0.8561
0.5	0.0400	0.1585	0.3469	0.5675	0.6896
0.6	0.0278	0.1103	0.2435	0.4124	0.5411
0.7	0.0204	0.0812	0.1804	0.3111	0.4310
0.8	0.0156	0.0623	0.1390	0.2428	0.3514
0.9	0.0123	0.0493	0.1105	0.1949	0.2933

DISCUSSION OF RESULTS

The roots of the five by five determinant as a function of the external load $\bar{P}$ were found by a numerical solution. Delaminations of different sizes and locations were investigated. The calculation results are presented both in tables and graphs. The parameters are nondimensionalized as: $\bar{h}_i = h_i/T$; $\bar{a}_i = a_i/L$; $\bar{P}_{cr} = P_{cr}/P^0_{cr}$, where P^0_{cr} is the critical load of the plate without delamination.

Single delamination

To verify the accuracy of the method employed in this study, the critical loads of a plate with a single delamination were computed first. The results for a plate with a symmetric delamination of different lengths and locations are shown in Table 1, which are in good agreement with the data presented by Simitses *et al.* (1985).

Two delaminations

The advantage of this method is its simplicity in the calculation of critical loads for composite plates with multiple delaminations. Because of the complexity of the problem, most of the methods published had to turn to numerical techniques, mainly finite element models. As mentioned before, the number of linear homogeneous equations in the study is reduced to five, which makes the calculation of the critical load simple and fast.

To further verify the proposed method, an example by Lim and Parsons (1993) is reconsidered here. This example considers a plate with two delaminations symmetrically located about the midplane and with the same length $\bar{a}$. Ignoring the buckling pattern involving contact between the laminae, the results obtained by the present method are listed in Table 2, as well as those published by Lim and Parsons (1993). Again there is an excellent agreement between these two methods.

It is rather difficult to consider all the possible combinations of lengths and depths of the two delaminations. Attention here is concentrated on the effect of delamination II on the critical load of the plate with delamination I only. It was found that the effect of delamination II depends not only on the length and location of delamination II itself, but it also depends on the size of delamination I. First we investigated the critical load of a

Table 2. Comparison of results (critical load, $\bar{P}_{cr}$) from the present approach with the data in Lim and Parsons (1993) for two delaminations

$\bar{a}$	0.10	0.40	0.50	0.60	0.70	0.80	0.90
Present	0.9996	0.5057	0.3350	0.2374	0.1771	0.1374	0.1099
Lim and Parsons (1993)	1.000	0.505	0.335	0.237	0.177	0.137	0.110

Table 3. Critical load, $\bar{P}_{cr}$, of two delaminations with fixed location $\bar{h}_1 = 0.125$, $\bar{h}_2 = 0.25$, and varying length

	$\bar{a}_1$										
$\bar{a}_2$	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.10	0.9998	0.9982	0.6903	0.3890	0.1731	0.0974	0.0624	0.0433	0.0318	0.0244	0.0193
0.20	0.9868	0.9628	0.6002	0.3791	0.1731	0.0974	0.0624	0.0433	0.0318	0.0244	0.0193
0.30	0.6616	0.6279	0.4723	0.3197	0.1726	0.0974	0.0624	0.0433	0.0318	0.0244	0.0193
0.40	0.3802	0.3754	0.3418	0.2668	0.1508	0.0972	0.0624	0.0433	0.0318	0.0244	0.0193
0.50	0.2452	0.2442	0.2368	0.2114	0.1346	0.0875	0.0623	0.0433	0.0318	0.0244	0.0193
0.60	0.1711	0.1709	0.1688	0.1610	0.1191	0.0802	0.0571	0.0433	0.0318	0.0244	0.0193
0.70	0.1263	0.1262	0.1255	0.1228	0.1028	0.0737	0.0532	0.0402	0.0318	0.0244	0.0193
0.80	0.0970	0.0970	0.0967	0.0957	0.0867	0.0672	0.0497	0.0378	0.0299	0.0244	0.0193
0.90	0.0769	0.0769	0.0768	0.0763	0.0722	0.0604	0.0464	0.0358	0.0283	0.0230	0.0193
1.00	0.0625	0.0625	0.0624	0.0622	0.0602	0.0535	0.0431	0.0338	0.0269	0.0221	0.0188

plate by fixing the depths of the delaminations and changing their lengths. The results are shown in Table 3.

Figure 2 shows the change of ratio between critical load P_{cr} and P_{crs}, which is the critical load of the plate with delamination I only. It is clear that the critical load is the same as that of the plate with delamination I only when the delamination II is very short. However, the critical load will decrease as a_2 increases. The effect of delamination II on critical load depends on the length of delamination I, too. If delamination I is short, the critical load drops drastically as a_2 increases. If delamination I is long, the effect of delamination II is not significant until delamination II is longer than delamination I.

The critical loads with varying $\bar{h}_2$, $\bar{a}_2$ and fixed $\bar{h}_1$, $\bar{a}_1$ were also calculated. Figure 3 is the plot of P_{cr}/P_{crs} vs $\bar{a}_2$ with $\bar{h}_2$ as a parameter. From the results, we can conclude that: delamination II has a significant effect on the critical load when it is close to delamination I. The effect can be ignored when it is far away and shorter than delamination I.

Secondly, consider the situation where the lengths of the delaminations are fixed and the locations of the delaminations are varying. The calculated critical loads are shown in Table 4. It can be seen that the presence of delamination II does not affect the critical load when delamination I is very shallow, even though these two delaminations are very close to each other. As $\bar{h}_1$ increases, the effect becomes more significant. There are three ways that the presence of delamination II may make the critical load decrease: (1) Subplate 2 still buckles first, but the presence of delamination II relaxes the constraints at the ends of

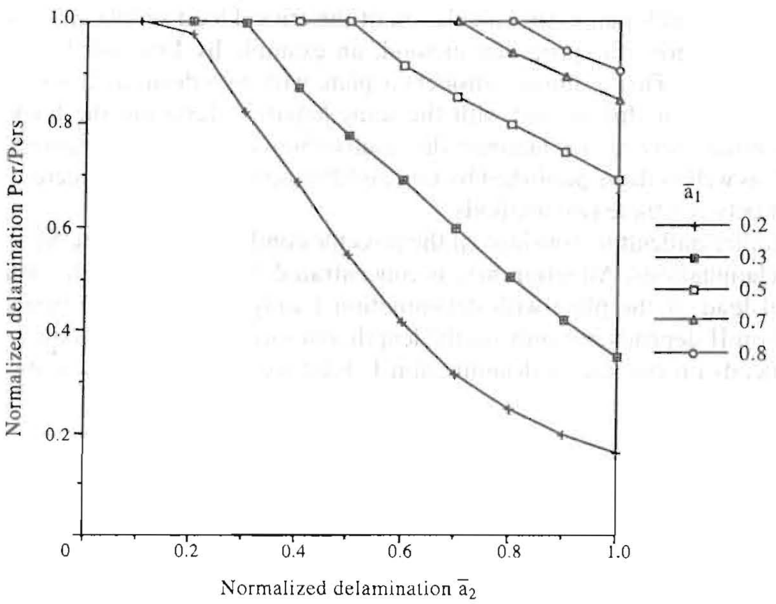

Fig. 2. The effect of delamination II on the critical load, P_{cr}, for a plate with $\bar{h}_1 = 0.125$ and $\bar{h}_2 = 0.25$. P_{crs} is the critical load for a plate with delamination I only.

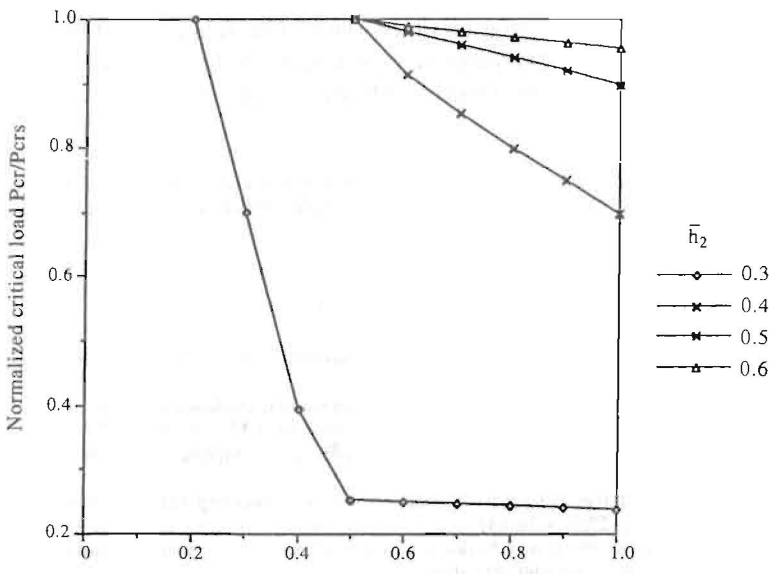

Fig. 3. Normalized critical load, P_{cr}/P_{crs}, for a plate with $\bar{a}_1 = 0.5$, $\bar{h}_1 = 0.2$ and varying $\bar{h}_2$ and $\bar{a}_2$.

Table 4. Critical load, $\bar{P}_{cr}$, of a plate with $\bar{a}_1 = 0.5$, $\bar{a}_2 = 0.3$ and varying $\bar{h}_1$ and $\bar{h}_2$

	$\bar{h}_1$				
$\bar{h}_2$	0.10	0.20	0.30	0.40	0.50
0.20	0.04				
0.30	0.04	0.1109			
0.40	0.04	0.1585	0.1108		
0.45	0.04	0.1585	0.2442	0.2455	
0.50	0.04	0.1585	0.2452	0.1106	
0.55	0.04	0.1585	0.2452	0.2455	0.0278
0.60	0.04	0.1585	0.2452	0.4209	0.1102
0.65	0.04	0.1585	0.2452	0.5501	0.2425
0.70	0.04	0.1585	0.2452	0.5588	0.4042
0.75	0.04	0.1585	0.2452	0.5501	0.5060
0.80	0.04	0.1585	0.2452	0.4209	0.4042
0.90	0.04	0.1109	0.2442	0.1106	0.1102

subplate 2, thus making the critical load decrease. (2) Subplate 4 will buckle first because the two delaminations are too close to each other. In this case, the buckling of subplate 4 will be constrained by subplate 2 or subplate 5. Contact analysis is called for to predict the critical load more precisely. (3) Subplate 5 buckles first because delamination II is too close to the boundary. This situation is the same as situation (1) if the subplates and delaminations are re-numbered.

CONCLUSIONS

An analytical method is developed for predicting the critical loads of composite plates with multiple delaminations. Linearized beam theory, end fixity conditions, kinematic continuity conditions and equilibrium equations at the delamination tip sections are used to obtain the closed form characteristic equation. This method allows determining the critical load and buckling mode for an arbitrary plate with delaminations of different magnitudes and locations.

Special attention is paid to the interactive effect of the two delaminations on the critical load. It was found that the deeply located delamination affects the critical load significantly when it is close to the nearest-to-the-surface delamination or when it is longer. When the two delaminations are too close to each other, the middle subplate will buckle first, thus a contact analysis is called for. The proposed method is the first step for a nonlinear post-buckling analysis, which would be more complicated but necessary for studying the growth characteristics.

Acknowledgement—The financial support of the Office of Naval Research, Ship Structures and Systems, S&T Division, Grant N00014-91-J-1892, and the interest and encouragement of the Grant Monitor, Dr Y.D.S. Rajapakse, are both gratefully acknowledged.

REFERENCES

Adan, M., Sheinman, I. and Atlus, E. (1993) Buckling of multiple delamination beams. *International Journal of Solids and Structures* **30**(10), 1289–1300.

Chai, H., Babcock, C. D. and Knauss, W. G. (1981) One dimensional modeling of failure in laminated plates by delamination buckling. *International Journal of Solids and Structures* **17**(11), 1069–1083.

Davidson, B. D. and Ferrie, C. H. (1994) Effect of stretching-shearing coupling on the instability-related delamination growth. *Composite Materials* **29**, 624–631.

Kardomateas, G. A. (1989) Large deformation effects in the post-buckling behavior of composites with thin delamination. *AIAA Journal* **27**(5), 624–631.

Kardomateas, G. A. (1993) The initial post-buckling and growth behavior of internal delaminations in composite plates. *Journal of Applied Mechanics* **60**, 903–910.

Kardomateas, G. A. and Pelegri, A. A. (1994) The stability of delamination growth in compressively loaded composite plates. *International Journal of Fracture* **65**, 262–276.

Kutlu, Z. and Chang, F. K. (1992) Modeling compressing failure of delaminated composites containing multiple through-the-width delaminations. *Journal of Composite Materials* **26**(3), 350–387.

Larsson, P. L. (1991) On multiple delamination buckling and growth in composite plates. *International Journal of Solids and Structures* **27**, 1623–1637.

Lim, Y. B. and Parsons, I. D. (1993) The linearized buckling analysis of a composite beam with multiple delaminations. *International Journal of Solids and Structures* **30**(22), 3085–3099.

Sheinman, I. and Soffer, M. (1991) Post buckling analysis of composite delaminated beams. *International Journal of Solids and Structures* **27**(5), 639–646.

Simitses, G. J., Sallam, S. and Yin, W. L. (1985) Effect of delamination of axially loaded homogeneous laminated plates. *AIAA Journal* **23**(9), 1437–1444.

Suemasu, H. (1993) Post buckling behavior of composite panels with multiple delaminations. *Journal of Composite Materials* **27**(11), 1077–1097.

Yin, W. L., Sallam, S. and Simitses, G. J. (1986) Ultimate axial load capacity of a delaminated beam-plate. *AIAA Journal* **24**(1), 123–128.