Effect of Normal Strains in Buckling of Thick Orthotropic Shells

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Abstract: An improved elasticity solution to the problem of buckling of orthotropic cylindrical shells subjected to external pressure is presented. The 2D axisymmetric cylindrical shell is studied (ring approximation). Specifically, in the development of the governing equations and boundary conditions for the buckling state, the solution includes the terms with the prebuckling normal strains and stresses as coefficients (i.e., the terms $e^0_{kk}\sigma'_{ij}$ and $\sigma^0_{kk}e'_{ij}$, which were neglected in the earlier work as being too small compared to the terms σ'_{ij} and $\sigma^0_{kk}\omega'_j$, respectively). The formulation results in a two-point boundary eigenvalue problem for ordinary differential equations in r, with the external pressure p as the parameter. The results show that the effect of including the normal strains and stresses is to further decrease the critical load. This decrease (versus the earlier elasticity solution without these terms) depends on the shell thickness and is generally moderate, and in no event comparable with the (quite large) decrease of the elasticity versus the shell theory prediction. This decrease depends also on the degree of orthotropy, and it is smaller for the isotropic case. Finally, a formula is derived for the critical pressure based on a first-order shear deformation formulation, and the comparison shows an improvement versus the classical shell for thick shells, but still the elasticity solution is noticeably lower than the first-order shear deformation prediction.

INTRODUCTION

Shell buckling has traditionally been studied by use of classical shell theories [e.g., Simitses and Aswani (1974) and Simitses et al. (1985)]. However, the application of fiber-reinforced composite materials in configurations of laminated shells necessitates the use of improved methods of buckling analysis. This is a result of the anisotropy and because of one other important distinguishing feature, namely an extensional-to-shear modulus ratio much larger than that of their metal counterparts. In addition, composite shells are envisioned in applications involving relatively thick construction (submersibles, support columns, etc.).

Therefore, improved shell theories have been formulated by Whitney and Sun (1974), Librescu (1975), Reddy and Liu (1985), and others. These higher-order shell theories can be applied to buckling problems with the potential of improved predictions for the critical load. To this extent, Simitses et al. (1993) used the Galerkin method to produce the critical loads of cylindrical shells under external pressure, as predicted from the first-order shear deformation (FOSD) and the higher-order shear deformation theories. It was concluded that for moderately thick cylinders (with a ratio of outside-over-inside radius of about 1.03), the FOSD theory with a modest shear correction factor provides an adequate correction to the critical load (as compared with the improvements from the higher-order theories).

The existence of these different shell theories underscores the need for benchmark elasticity solutions to compare the accuracy of the predictions from the classical and the improved shell theories. Several elasticity solutions for composite shell buckling have become available. In particular, Kardomateas (1993a) formulated and solved the problem for the case of uniform external pressure and orthotropic material; a simplified problem definition was used in this study ("ring" assumption), in that the prebuckling stress and displacement field was axisymmetric and the buckling modes were assumed 2D (i.e., no z component of the displacement field and no z-dependence of the r and θ displacement components). The ring assumption was relaxed in a further study (Kardomateas and Chung 1994), in which a nonzero axial displacement and a full dependence of the buckling modes on the three coordinates was assumed.

A more thorough investigation of the thickness effects was conducted by Kardomateas (1993b) for the case of a transversely isotropic thick cylindrical shell under axial compression. This work also included a comprehensive study of the performance of the Donnell (1933), Sanders-type (1963) (which was also referred to as the "nonsimplified Donnell-type" theory), Flügge (1960), and Danielson and Simmonds (1969) theories for isotropic material in the case of axial compression. In a later study, Kardomateas (1995) considered a generally cylindrical orthotropic material under axial compression. In addition to considering general orthotropy for the material constitutive behavior, the latter work investigated the performance of another classical formulation [i.e., the Timoshenko and Gere (1961) one]. Other 3D elasticity results were provided by Soldatos and Ye (1994) based on a successive approximation method. These results were provided for the buckling of complete hollow cylinders subjected to combined axial compression and uniform external pressure and the buckling of open cylindrical panels subjected to axial compression.

The elasticity solutions presented by Kardomateas (1993a,b, 1994, 1995) were based on certain orders of magnitude arguments for the terms involved in the 3D nonlinear elasticity equations for the perturbed (buckling) versus the prebuckling state quantities. For example, use was made of the fact that a characteristic feature of stability problems is the shift from positions with small rotations to positions with rotations substantially exceeding the strains. If we denote the linear strains e_{ij} and the linear rotations ω_j and use the superscript 0 to denote the prebuckling state and the prime (′) to denote the perturbed state, then in these elasticity solutions, the terms $e'_{ij}\sigma^0_{ij}$ were neglected, thus keeping only the $\omega'_j\sigma^0_{ij}$ terms. In addition, the terms that have e^0_{ij} and ω^0_j as coefficients were neglected (i.e., terms $e^0_{ij}\sigma'_{ij}$ and $\omega^0_j\sigma'_{ij}$). Also, in the prebuckling equations, the terms that have e^0_{ij} and ω^0_j as coefficients (i.e., terms $e^0_{ij}\sigma^0_{ij}$ and $\omega^0_j\sigma^0_{ij}$) were neglected, thus only using the linear classical equilibrium equations for the initial position of equilibrium.

In this work, we revisit the issue of neglecting these normal strain and normal stress terms to investigate the degree of influence these terms may have on the critical load. The simplest configuration is studied, namely the 2D axisymmetric cylindrical shell (ring approximation) that was studied in Kardom-

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ateas (1993a). Numerical results for a standard glass/epoxy and an isotropic material are derived and compared with the predictions of the shell theory and the elasticity solution without the normal stress and strain terms of Kardomateas (1993a). It will be shown that the effect of these terms is to render the critical load even lower, and that this effect is more pronounced with increased thickness. However, this effect is very small with moderate thickness, and in no way compares with the (quite large) decrease of the elasticity versus the shell theory prediction.

FORMULATION

We shall refer to specific equations in the formulation of Kardomateas (1993a) and point out the improvements introduced in the present analysis. At the critical load, there are two possible infinitely close positions of equilibrium. The r, θ, and z components of the displacement corresponding to the primary position are denoted by u_0, v_0, and w_0, respectively. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \quad v = v_0 + \alpha v_1; \quad w = w_0 + \alpha w_1 \tag{1}$$

where α = infinitesimally small quantity. Here, $\alpha u_1(r, \theta, z)$, $\alpha v_1(r, \theta, z)$, and $\alpha w_1(r, \theta, z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r, \theta, z)$, $v_1(r, \theta, z)$, and $w_1(r, \theta, z)$ are assumed finite, and α is an infinitesimally small quantity independent of r, θ, and z.

The stress-strain relations for the orthotropic body are as follows:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \tag{2}$$

where c_{ij} = stiffness constants (we have used the notation $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

In the following, we shall use e_{ij} to denote the linear strains and ω_j to denote the linear rotations. The equilibrium equations in terms of the stresses and the linear strains and rotations are given by Kardomateas (1993a, equation 9). For completeness, we repeat the first of these equations here

$$\begin{aligned} &\frac{\partial}{\partial r}\left[\sigma_{rr}(1 + e_{rr}) + \tau_{r\theta}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \tau_{rz}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right] \\ &+ \frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{r\theta}(1 + e_{rr}) + \sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \tau_{\theta z}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right] \\ &+ \frac{\partial}{\partial z}\left[\tau_{rz}(1 + e_{rr}) + \tau_{\theta z}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \sigma_{zz}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right] \\ &+ \frac{1}{r}\left[\sigma_{rr}(1 + e_{rr}) - \sigma_{\theta\theta}(1 + e_{\theta\theta}) + \tau_{rz}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right. \\ &\left.- \tau_{\theta z}\left(\frac{1}{2}e_{\theta z} - \omega_r\right) - 2\tau_{r\theta}\omega_z\right] = 0 \end{aligned}$$

Introducing the linear strains and rotations in the form $e_{ij} = e_{ij}^0 + \alpha e_{ij}'$, $\omega_j = \omega_j^0 + \alpha\omega_j'$, as well as the stresses from $\sigma_{ij} = \sigma_{ij}^0 + \alpha\sigma_{ij}'$ and keeping up to the α^1 terms, we obtain a set of equations for the perturbed state in terms of the e_{ij}^0, ω_j^0 and e_{ij}', ω_j'. Notice that e_{ij}^0 and ω_j^0 are the values of e_{ij} and ω_j for $u = u_0$, $v = v_0$, and $w = w_0$, and e_{ij}' and ω_j' are the values for $u = u_1$, $v = v_1$, and $w = w_1$.

Although the displacements u_0, v_0, and w_0 correspond to positions of equilibrium, there must also exist equations of the same form with the zero superscript, which are obtained by referring to the initial position of equilibrium. Thus, after subtracting the equilibrium equations at the perturbed and initial positions, we arrive at a system of homogeneous differential equations that are linear in the derivatives of u_1, v_1, and w_1 with respect to r, θ, and z. This follows from σ_{ij}', e_{ij}', and ω_j' appearing linearly in the equation, and they are linear functions of these derivatives. The system of equations, corresponding to Kardomateas (1993a, equation 9) at the initial position of equilibrium, is, on the other hand, nonlinear in the derivatives of u_0, v_0, and w_0.

However, if we make the additional assumption to neglect the terms that have e_{ij}^0 and ω_j^0 as coefficients (i.e., terms $e_{ij}^0\sigma_{ij}^0$ and $\omega_j^0\sigma_{ij}^0$), we can use the linear classical equilibrium equations to solve for the initial position of equilibrium. Moreover, if we make the assumption to neglect the terms that have e_{ij}^0 and ω_j^0 as coefficients (i.e., terms $e_{ij}^0\sigma_{ij}'$ and $\omega_j^0\sigma_{ij}'$) and furthermore, because a characteristic feature of stability problems is the shift from positions with small rotations to positions with rotations substantially exceeding the strains, if we neglect the terms $e_{ij}'\sigma_{ij}^0$, thus keeping only the $\omega_j'\sigma_{ij}^0$ terms, we obtain the buckling equations as shown in Kardomateas (1993a, equation 10). For completeness, we repeat the first of these equations here

$$\begin{aligned} &\frac{\partial}{\partial r}(\sigma_{rr}' - \tau_{r\theta}^0\omega_z' + \tau_{rz}^0\omega_\theta') + \frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{r\theta}' - \sigma_{\theta\theta}^0\omega_z' + \tau_{\theta z}^0\omega_\theta') \\ &+ \frac{\partial}{\partial z}(\tau_{rz}' - \tau_{\theta z}^0\omega_z' + \sigma_{zz}^0\omega_\theta') \\ &+ \frac{1}{r}(\sigma_{rr}' - \sigma_{\theta\theta}' + \tau_{rz}^0\omega_\theta' + \tau_{\theta z}^0\omega_r' - 2\tau_{r\theta}^0\omega_z') = 0 \end{aligned}$$

Now, let us relax the assumption to neglect the terms that have e_{ij}^0 as coefficients (i.e., let us keep the terms $e_{kk}^0\sigma_{ij}'$), and let us also relax the assumption to neglect $e_{ij}'\sigma_{ij}^0$ in comparison to $\omega_j'\sigma_{ij}^0$ (i.e., let us keep the terms $e_{kl}'\sigma_{ij}^0$).

Because we are dealing with a case in which the prebuckling shear stresses and strains are zero (i.e., $\tau_{r\theta}^0 = \tau_{rz}^0 = \tau_{\theta z}^0 = 0$ and $e_{r\theta}^0 = e_{rz}^0 = e_{\theta z}^0 = 0$), the prebuckling rotations are zero (i.e., $\omega_r^0 = \omega_\theta^0 = \omega_z^0 = 0$), and the prebuckling normal stresses and strains depend only on r, the first equilibrium equation with normal stress and strain terms is written

$$\begin{aligned} &\frac{\partial}{\partial r}[\sigma_{rr}'(1 + e_{rr}^0) + \sigma_{rr}^0 e_{rr}'] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{r\theta}'(1 + e_{rr}^0) + \sigma_{\theta\theta}^0\left(\frac{1}{2}e_{r\theta}' - \omega_z'\right)\right] \\ &+ \frac{1}{r}[\sigma_{rr}'(1 + e_{rr}^0) + \sigma_{rr}^0 e_{rr}' - \sigma_{\theta\theta}'(1 + e_{\theta\theta}^0) - \sigma_{\theta\theta}^0 e_{\theta\theta}'] = 0 \end{aligned} \tag{3a}$$

By comparison, the corresponding equation to the earlier elasticity formulation of Kardomateas (1993a) is repeated here

$$\frac{\partial}{\partial r}\sigma_{rr}' + \frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{r\theta}' - \sigma_{\theta\theta}^0\omega_z') + \frac{1}{r}(\sigma_{rr}' - \sigma_{\theta\theta}') = 0 \tag{3b}$$

The second equilibrium equation with normal strain terms is written

$$\begin{aligned} &\frac{\partial}{\partial r}\left[\tau_{r\theta}'(1 + e_{\theta\theta}^0) + \sigma_{rr}^0\left(\frac{1}{2}e_{r\theta}' + \omega_z'\right)\right] \\ &+ \frac{1}{r}\frac{\partial}{\partial\theta}[\sigma_{\theta\theta}'(1 + e_{\theta\theta}^0) + \sigma_{\theta\theta}^0 e_{\theta\theta}'] + \frac{1}{r}\left[\sigma_{rr}^0\left(\frac{1}{2}e_{r\theta}' + \omega_z'\right)\right. \\ &\left.+ \sigma_{\theta\theta}^0\left(\frac{1}{2}e_{r\theta}' - \omega_z'\right) + \tau_{r\theta}'(2 + e_{rr}^0 + e_{\theta\theta}^0)\right] = 0 \end{aligned} \tag{4a}$$

and by comparison, the corresponding second equilibrium equation in Kardomateas (1993a) is repeated here

$$\frac{\partial}{\partial r}(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z) + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma'_{\theta\theta} + \frac{1}{r}(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z) = 0 \tag{4b}$$

Boundary Conditions

Boundary conditions for hydrostatic loading in the context of shell theory can be found in Brush and Almroth (1975). In the context of the elasticity formulation, the complete set of boundary conditions is given in Kardomateas (1993a, equation 12). For completeness, we repeat the first of these equations here

$$\left[\sigma_{rr}(1 + e_{rr}) + \tau_{r\theta}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \tau_{rz}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right]l$$
$$+ \left[\tau_{r\theta}(1 + e_{rr}) + \sigma_{\theta\theta}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \tau_{\theta z}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right]m$$
$$+ \left[\tau_{rz}(1 + e_{rr}) + \tau_{\theta z}\left(\frac{1}{2}e_{r\theta} - \omega_z\right) + \sigma_{zz}\left(\frac{1}{2}e_{rz} + \omega_\theta\right)\right]n = t_r$$

where t_r = r-component of the traction vector on the surface that has outward unit normal $\hat{n} = (l, m, n)$ before any deformation. The traction vector $\mathbf{t}$ depends on the displacement field $\mathbf{V} = (u, v, w)$. Indeed, because of the hydrostatic pressure loading, the magnitude of the surface load remains invariant under deformation, but its direction changes (because hydrostatic pressure is always directed along the normal to the surface on which it acts).

Now, if we write these equations for the initial and the perturbed equilibrium position, then subtract them and take into account the aforementioned characteristic of the hydrostatic pressure loading, and then use the previous arguments on the relative magnitudes of the rotations ω'_j, we obtain equations (13) and (18) as found in Kardomateas (1993a).

Now, if we relax these assumptions and include the normal strain and stress terms, we obtain the following first boundary condition:

$$[\sigma'_{rr}(1 + e^0_{rr}) + \sigma^0_{rr}e'_{rr}]l + \left[\tau'_{r\theta}(1 + e^0_{rr}) + \sigma^0_{\theta\theta}\left(\frac{1}{2}e'_{r\theta} - \omega'_z\right)\right]m$$
$$+ \left[\tau'_{rz}(1 + e^0_{rr}) + \sigma^0_{zz}\left(\frac{1}{2}e'_{rz} + \omega'_\theta\right)\right]n$$
$$= -p\left[e'_{rr}l + \left(\frac{1}{2}e'_{r\theta} - \omega'_z\right)m + \left(\frac{1}{2}e'_{rz} + \omega'_\theta\right)n\right] \tag{5a}$$

By comparison, the corresponding condition to the earlier elasticity formulation of Kardomateas (1993a) is repeated here

$$\sigma'_{rr}l + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z)m + (\tau'_{rz} + \sigma^0_{zz}\omega'_\theta)n = p(\omega'_z m - \omega'_\theta n) \tag{5b}$$

In a similar fashion, the second boundary condition with normal strain and stress terms is written

$$\left[\tau'_{r\theta}(1 + e^0_{\theta\theta}) + \sigma^0_{rr}\left(\frac{1}{2}e'_{r\theta} + \omega'_z\right)\right]l + [\sigma'_{\theta\theta}(1 + e^0_{\theta\theta}) + \sigma^0_{\theta\theta}e'_{\theta\theta}]m$$
$$+ \left[\tau'_{\theta z}(1 + e^0_{\theta\theta}) + \sigma^0_{zz}\left(\frac{1}{2}e'_{\theta z} - \omega'_r\right)\right]n = -p\left(\frac{1}{2}e'_{r\theta} + \omega'_z\right)l$$
$$+ e'_{\theta\theta}m + \left(\frac{1}{2}e'_{\theta z} - \omega'_r\right)n\Bigg] \tag{6a}$$

and by comparison, the corresponding second boundary condition in Kardomateas (1993a) is repeated here

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z)l + \sigma'_{\theta\theta}m + (\tau'_{\theta z} - \sigma^0_{zz}\omega'_r)n = -p(\omega'_z l - \omega'_r n) \tag{6b}$$

The main difference between (6*a*) and (6*b*) is the added normal strain $\varepsilon^0_{\theta\theta}$ and normal stress $\sigma^0_{\theta\theta}$ terms. Another minor difference is that the strain terms e'_{ij} are no longer neglected in comparison with the rotations ω'_k.

Prebuckling State

The problem under investigation is of a very long hollow cylinder rigidly fixed at its ends and deformed by uniformly distributed external pressure p. The axially symmetric distribution of external forces produces stresses identical at all cross sections and dependent only on the radial coordinate r (generalized plane assumption). In this manner, the forces at the ends are distributed identically over both surfaces and reduce to equal and opposite resultant forces and moments. Let R_1 be the internal radius and R_2 the external radius and set $c = R_1/R_2$. Lekhnitskii (1963) gave the stress field as follows:

$$\sigma^0_{rr} = -\frac{p}{1 - c^{2k}}\left(\frac{r}{R_2}\right)^{k-1} + \frac{pc^{k-1}}{1 - c^{2k}}c^{k+1}\left(\frac{R_2}{r}\right)^{k+1} \tag{7a}$$

$$\sigma^0_{\theta\theta} = -\frac{p}{1 - c^{2k}}k\left(\frac{r}{R_2}\right)^{k-1} - \frac{pc^{k-1}}{1 - c^{2k}}kc^{k+1}\left(\frac{R_2}{r}\right)^{k+1} \tag{7b}$$

$$\sigma^0_{zz} = \frac{p}{(1 - c^{2k})a_{33}}(a_{13} + a_{23}k)\left(\frac{r}{R_2}\right)^{k-1}$$
$$- \frac{pc^{k-1}}{(1 - c^{2k})a_{33}}(a_{13} - a_{23}k)c^{k+1}\left(\frac{R_2}{r}\right)^{k+1} \tag{7c}$$

$$\tau^0_{r\theta} = \tau^0_{rz} = \tau^0_{\theta z} = 0 \tag{7d}$$

Eq. (2) for the orthotropic constitutive behavior (where c_{ij} are the stiffness constants) and the inverse relationship (where a_{ij} are the compliance constants) are used, i.e.

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} \tag{7e}$$

Integration of the strain field, resulting by substituting (7*a*)–(7*d*) into the strain-stress relationships (7*e*) through the linear strain-displacement relations, gives

$$u_0(r) = D_1pr^k + D_2pk^{-k}; \quad v_0 = w_0 = 0 \tag{8a}$$

where

$$k = \sqrt{\frac{a_{11}a_{33} - a^2_{13}}{a_{22}a_{33} - a^2_{23}}} = \sqrt{\frac{c_{22}}{c_{11}}} \tag{8b}$$

$$D_1 = -\frac{1}{(1 - c^{2k})kR_2^{k-1}}\left[a_{11} + a_{12}k - \frac{a_{13}}{a_{33}}(a_{13} + a_{23}k)\right]$$
$$= -\frac{1}{(c_{11}k + c_{12})(1 - c^{2k})R_2^{k-1}} \tag{8c}$$

$$D_2 = -\frac{c^{k-1}R_1^{k+1}}{(1 - c^{2k})k}\left[a_{11} - a_{12}k - \frac{a_{13}}{a_{33}}(a_{13} - a_{23}k)\right]$$
$$= \frac{c^{k-1}R_1^{k+1}}{(-c_{11}k + c_{12})(1 - c^{2k})} \tag{8d}$$

with D_1 and D_2 being found by integrating

$$\frac{\partial u_0}{\partial r} = a_{11}\sigma^0_{rr} + a_{12}\sigma^0_{\theta\theta} + a_{13}\sigma^0_{zz}$$

Accordingly, the prebuckling strains are written

$$e^0_{rr} = pk(D_1r^{k-1} - D_2r^{-k-1}) \tag{9a}$$

$$e^0_{\theta\theta} = p(D_1 r^{k-1} + D_2 r^{-k-1}) \quad (9b)$$

$$e^0_{zz} = e^0_{r\theta} = e^0_{rz} = e^0_{\theta z} = \omega^0_r = \omega^0_\theta = \omega^0_z = 0 \quad (9c)$$

Perturbed State

In the perturbed position, we seek plane equilibrium modes as follows:

$$u_1(r, \theta) = A_n(r)\cos n\theta; \quad v_1(r, \theta) = B_n(r)\sin n\theta; \quad w_1(r, \theta) = 0 \quad (10)$$

The first-order strains are accordingly

$$e'_{rr} = \frac{\partial u_1}{\partial r} = A'_n(r)\cos n\theta;$$

$$e'_{\theta\theta} = \frac{1}{r}\frac{\partial v_1}{\partial \theta} + \frac{u_1}{r} = \frac{A_n(r) + nB_n(r)}{r}\cos n\theta \quad (11a)$$

$$e'_{r\theta} = \frac{1}{r}\frac{\partial u_1}{\partial \theta} + \frac{\partial v_1}{\partial r} - \frac{v_1}{r} = \left[B'_n(r) - \frac{B_n(r) + nA_n(r)}{r}\right]\sin n\theta \quad (11b)$$

$$e'_{zz} = \frac{\partial w_1}{\partial z} = 0; \quad e'_{\theta z} = \frac{\partial v_1}{\partial z} + \frac{1}{r}\frac{\partial w_1}{\partial \theta} = 0; \quad e'_{rz} = \frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial r} = 0 \quad (11c)$$

and the first order rotations are written

$$2\omega'_z = \frac{\partial v_1}{\partial r} + \frac{v_1}{r} - \frac{1}{r}\frac{\partial u_1}{\partial \theta} = \left[B'_n(r) + \frac{B_n(r) + nA_n(r)}{r}\right]\sin n\theta \quad (11d)$$

$$2\omega'_\theta = \frac{\partial u_1}{\partial z} - \frac{\partial w_1}{\partial r} = 0; \quad 2\omega'_r = \frac{1}{r}\frac{\partial w_1}{\partial \theta} - \frac{\partial v_1}{\partial z} = 0 \quad (11e)$$

Substituting into (3a), we obtain the first equilibrium equation as follows:

$$\begin{aligned}
&[c_{11}(1 + e^0_{rr}) + \sigma^0_{rr}]A_n(r)'' \\
&+ \left[\frac{c_{11}}{r}(1 + e^0_{rr}) + \frac{c_{12}}{r}(e^0_{rr} - e^0_{\theta\theta}) + c_{11}e^{0'}_{rr} + \frac{\sigma^0_{rr}}{r} + \sigma^{0'}_{rr}\right]A_n(r)' \\
&+ \left[\frac{c_{12}}{r}e^{0'}_{rr} - \frac{c_{22}}{r^2}(1 + e^0_{\theta\theta}) - \frac{\sigma^0_{\theta\theta}(n^2 + 1)}{r^2} - \frac{c_{66}n^2}{r^2}(1 + e^0_{rr})\right]A_n(r) + \frac{(c_{12} + c_{66})n}{r}(1 + e^0_{rr})B_n(r)' \\
&+ \left[\frac{c_{12}}{r}e^{0'}_{rr} - \frac{c_{22}}{r^2}(1 + e^0_{\theta\theta}) - \frac{c_{66}}{r^2}(1 + e^0_{rr}) - 2\frac{\sigma^0_{\theta\theta}}{r^2}\right]nB_n(r) = 0
\end{aligned} \quad (12a)$$

In a similar fashion, the second equilibrium equation [(4a)] becomes

$$\begin{aligned}
&[\sigma^0_{rr} + c_{66}(1 + e^0_{\theta\theta})]B''_n(r) \\
&+ \left[\sigma^{0'}_{rr} + \frac{\sigma^0_{rr}}{r} + \frac{c_{66}}{r}(1 + e^0_{rr}) + c_{66}e^{0'}_{\theta\theta}\right]B'_n(r) \\
&- \left[\frac{c_{66}}{r^2}(1 + e^0_{rr}) + \frac{\sigma^0_{\theta\theta}}{r^2} + \frac{c_{66}}{r}e^{0'}_{\theta\theta} + \frac{c_{22}n^2}{r^2}(1 + e^0_{\theta\theta}) + \frac{\sigma^0_{\theta\theta}n^2}{r^2}\right]B_n(r) \\
&- \frac{n(c_{12} + c_{66})}{r}(1 + e^0_{\theta\theta})A'_n(r) - n\left[\frac{c_{22}}{r^2}(1 + e^0_{\theta\theta}) + \frac{c_{66}}{r^2}(1 + e^0_{rr}) + \frac{c_{66}}{r}e^{0'}_{\theta\theta} + 2\frac{\sigma^0_{\theta\theta}}{r^2}\right]A_n(r) = 0
\end{aligned} \quad (12b)$$

The first boundary condition [(5a)] on the lateral surfaces $m = n = 0$, $l = \pm 1$ becomes

$$[c_{11}(1 + e^0_{rr}) + \sigma^0_{rr} + p_j]A_n(R_j)' + \frac{c_{12}}{r}(1 + e^0_{rr})A_n(R_j) + \frac{c_{12}n}{r}(1 + e^0_{rr})B_n(R_j) = 0 \quad (13a)$$

and the second boundary condition [(6a)] becomes

$$[c_{66}(1 + e^0_{\theta\theta}) + \sigma^0_{rr} + p_j]B_n(R_j)' - \frac{c_{66}}{r}(1 + e^0_{\theta\theta})B_n(R_j) - \frac{c_{66}n}{r}(1 + e^0_{\theta\theta})A_n(R_j) = 0 \quad (13b)$$

where $p_j = p$ for $j = 2$ (i.e., at the outer surface R_2); and $p_j = 0$ for $j = 1$ (i.e., at the inner surface R_1). Therefore, we obtain two linear, homogeneous, ordinary differential equations of the second order for $A_n(r)$ and $B_n(r)$. Eqs. (12) and (13) constitute an eigenvalue problem for differential equations with parameter p, which can be solved by standard numerical methods (two point boundary value problem). The relaxation method (Press et al. 1989) was used to obtain results that are discussed next. The minimum eigenvalue is obtained for $n = 2$. An equally spaced mesh of 241 points was used to derive the results as in the case without the normal terms, and the method is highly efficient with rapid convergence.

RESULTS AND DISCUSSION

Results were produced for the same configuration as the one in Kardomateas (1993a), namely, a circular cylinder of inner radius $R_1 = 1$ m with moduli in GN/m^2 and Poisson's ratios used typically for a glass/epoxy material and listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 is the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$.

Fig. 1 shows the critical pressure as a function of the ratio of outside versus inside radius R_2/R_1. The elasticity solution with normal stress and strain terms included is compared with the elasticity solution in the earlier work of Kardomateas (1993a) and with the predictions of the classical shell theory. Table 1 shows the tabulated data. It is seen that the normal stress and strain terms reduce the critical load by about 1.3% for $R_2/R_1 = 1.05$ and by 7% for $R_2/R_1 = 1.20$ versus the earlier elasticity solution and by about 10% for $R_2/R_1 = 1.30$. However, compared with the elasticity solution in Kardomateas (1993a), the classical shell theory showed a 20% increase in critical load for $R_2/R_1 = 1.20$ and a 34% increase for $R_2/R_1 = 1.30$; therefore, this effect is far more important.

Table 2 shows the same data for the isotropic case with $\nu = 0.3$. It is seen that the normal stress and strain terms reduce the critical load by about 1% for $R_2/R_1 = 1.05$ and by about 5% for $R_2/R_1 = 1.20$ versus the earlier elasticity solution and by about 7% for $R_2/R_1 = 1.30$ (i.e., the effect is less than in the orthotropic case). Again, compared with the elasticity solution in Kardomateas (1993a), the classical shell theory showed a 12% increase in critical load for $R_2/R_1 = 1.20$ and an increase of about 18% for $R_2/R_1 = 1.30$; therefore, this effect is far more important than the normal terms effect, but all of these effects are less than in the orthotropic case.

The direct expression for the critical pressure from classical shell theory is written

$$p_{cr,sh} = \frac{E_2}{(1 - \nu_{23}\nu_{32})}(n^2 - 1)\frac{h^3}{12R^3} \quad (14a)$$

where $R = (R_1 + R_2)/2$ = midsurface radius; and $h = R_2 - R_1$ = shell thickness. The previous value can be found by using the Donnell nonlinear shell theory equations (Brush and Almroth 1975) and seeking the buckled shapes in the form (10)

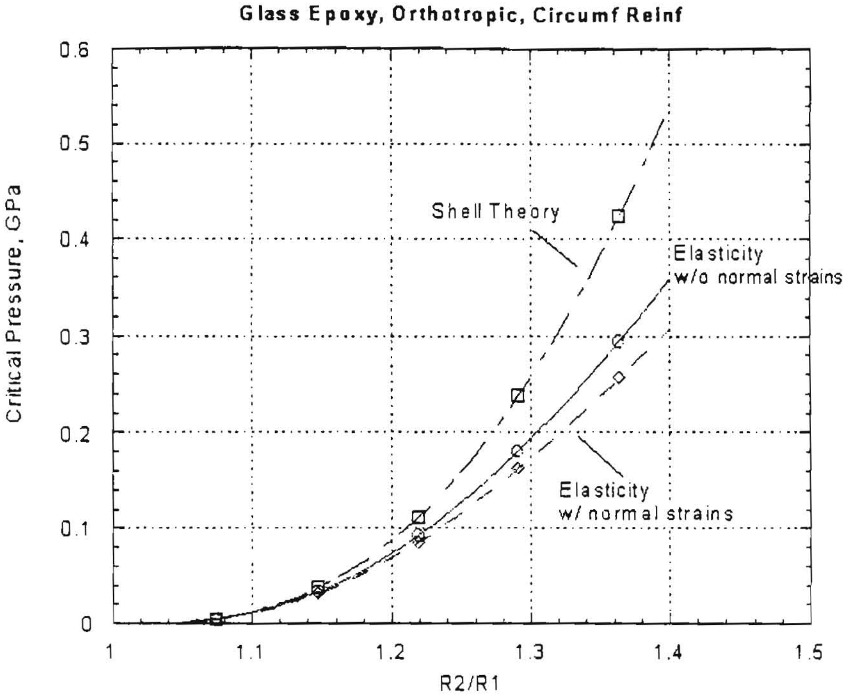

FIG. 1. Critical Pressure p_{cr} versus Ratio of Outside/Inside Radius R_2/R_1; Comparison of Elasticity Solution with and without Normal Terms and Shell Theory Predictions

TABLE 1. Elasticity with and without Normal Stress and Strain Terms—Orthotropic Critical Pressure (GPa)

R_2/R_1 (1)	Elasticity without normal terms (2)	Classical shell theory[a] (3)	Elasticity with normal terms (4)	With normal versus without normal terms (%) (5)
1.05	0.001635	0.001686	0.001614	−1.28
1.11	0.01532	0.01661	0.01482	−3.26
1.159	0.04055	0.0461	0.03847	−5.13
1.207	0.07996	0.09574	0.07434	−7.03
1.255	0.1331	0.1683	0.1212	−8.94
1.303	0.1985	0.2656	0.1772	−10.73
1.352	0.2743	0.3887	0.2402	−12.43
1.4	0.3585	0.5379	0.3083	−14.002

[a]From Eq. (14*a*).

TABLE 2. Elasticity with and without Normal Stress and Strain Terms—Isotropic Critical Pressure (GPa)

R_2/R_1 (1)	Elasticity with normal terms (2)	Classical shell theory[a] (3)	Elasticity without normal terms (4)	With normal versus without normal terms (%) (5)
1.05	0.001752	0.001818	0.001772	−1.13
1.11	0.01647	0.01791	0.01691	−2.60
1.159	0.044	0.04971	0.04574	−3.80
1.207	0.08790	0.1032	0.09254	−5.01
1.255	0.1487	0.1815	0.1585	−6.18
1.303	0.2257	0.2864	0.2438	−7.42
1.352	0.3179	0.4191	0.3476	−8.54
1.4	0.4237	0.58	0.4691	−9.68

[a]From Eq. (14*a*).

where $A_n(r) = A_n$ (i.e., it is now a constant instead of function f r) and $B_n(r) = B_n + (r - R)\beta$ with B_n being a constant (i.e., it admits a linear variation through the thickness). Because $\beta = (v_1 - u_{1,\theta})/R$, the latter can also be written in the form $B_n(r) = B_n + (r - R)(B_n + n)/R$. Consequently, we obtain the following shell theory buckling equations:

$$u_{1,\theta} + v_{1,\theta\theta} - \frac{h^2}{12R^2}(u_{1,\theta\theta\theta} - v_{1,\theta\theta}) = 0 \qquad (14b)$$

$$\frac{h^2}{12R^2}(u_{1,\theta\theta\theta\theta} - v_{1,\theta\theta\theta}) + (u_1 + v_{1,\theta}) - \frac{pR(1 - \nu_{23}\nu_{32})}{E_2 h}(v_{1,\theta} - u_{1,\theta\theta}) = 0 \qquad (14c)$$

Substituting the displacements from (10) and using the previous expressions for $A_n(r)$ and $B_n(r)$ arguments results in the eigenvalue (14*a*) and the "eigenvectors" given by

$$A_n = 1; \quad B_n = -\left(1 + \frac{h^2}{12R^2}n^2\right)\Bigg/\left[n\left(1 + \frac{h^2}{12R^2}\right)\right] \qquad (14d)$$

The comparison of our elasticity solution was performed with the Donnell shell theory. It has been shown (Danielson and Simmonds 1969) that the Donnell shell theory can produce, in some instances, inaccurate results (such as for long tube behavior) as opposed to the more elaborate Flügge theory that provides more accurate predictions. However, for the problem under consideration, due to the assumed 2D buckling modes (i.e., no z component of the displacement field and no z-dependence of the r and θ displacement components), the Flügge and Donnell equations would give the same critical load. Indeed, the buckling equations for the Flügge shell theory [see e.g., Simmonds (1966)] would give (14*b*) without the term $(h^2/12R^2)(u_{1,\theta\theta\theta} - v_{1,\theta\theta})$ and (14*c*) with the first term being $(h^2/12R^2)(u_{1,\theta\theta\theta\theta} + 2u_{1,\theta\theta} + u_1)$ instead of $(h^2/12R^2)(u_{1,\theta\theta\theta\theta} - v_{1,\theta\theta\theta})$. Substitution of the buckling modes [(10)] gives the same critical load [(14*a*)] as the Donnell shell theory.

Next, we are going to compare with the critical pressure from an FOSD theory because this is expected to improve the shell theory results. To do so, let us first derive the formula

The critical pressure based on an FOSD shell formulation.

We are referring to a coordinate system z, θ, and r, in which z and θ are in the axial and circumferential directions and r is in the (radial) direction of the outward normal to the middle surface. The corresponding displacements at the middle surface are denoted by w, v, and u, and the rotation of the normal to the middle surface in the $r\theta$-plane is denoted by ψ. Then the nonlinear strain displacement relations are written

$$\varepsilon_\theta = \frac{v_{,\theta} + u}{R} + \frac{1}{2R^2} u_{,\theta}^2; \quad \gamma_{r\theta} = \psi + \frac{u_{,\theta}}{R} \tag{15a}$$

and the bending strain (or curvature) relationship is written

$$\kappa_\theta = \frac{\psi_{,\theta}}{R} \tag{15b}$$

All other strains and curvatures are zero, i.e.

$$\varepsilon_z = \varepsilon_r = \gamma_{rz} = \gamma_{\theta z} = \kappa_z = \kappa_{\theta z} = 0 \tag{15c}$$

Notice that based on the Timoshenko-Mindlin kinematic hypothesis, the displacement field $\bar{u}$, $\bar{v}$, and $\bar{w}$ at an arbitrary point is represented by

$$\bar{u}(r, \theta) = u(\theta); \quad \bar{v}(r, \theta) = v(\theta) + (r - R)\psi(\theta) \tag{15d,e}$$

In these equations, a comma denotes differentiation with respect to the corresponding coordinate; ε_θ = inplane strain; and $\gamma_{r\theta}$ = transverse shear strain. Notice that for the classical shell theory, $\psi = -u_{,\theta}/R$, and therefore $\gamma_{r\theta} = 0$.

In the FOSD theory considered, the relationship between force resultants and moment resultants and membrane strains and bending strains is written

$$N_\theta = C_{22}\varepsilon_\theta; \quad Q_{r\theta} = k_s^2 C_{66}\gamma_{r\theta}; \quad M_\theta = D_{22}\kappa_\theta \tag{16a}$$

$$N_z = c_{12}\varepsilon_\theta; \quad M_z = D_{12}\kappa_\theta; \quad Q_{rz} = N_{\theta z} = M_{\theta z} = 0 \tag{16b}$$

where

$$C_{22} = \frac{hE_\theta}{1 - \nu_{23}\nu_{32}}; \quad C_{66} = hG_{r\theta}; \quad D_{ij} = \frac{h^2}{12} c_{ij} \tag{16c}$$

For the shear correction factor, results will be presented for the usual value of $k_s^2 = 5/6$. A discussion of various methods for determining these factors can be found in Dong and Nelson (1972) and Whitney (1973).

The governing equations of equilibrium for the shear deformable ring can be derived from the principle of virtual work and can be directly extracted from these for the shell [e.g., see Kardomateas (1997)]. These are written as follows:

$$N_{\theta,\theta} - p(v - u_{,\theta}) = 0 \tag{17a}$$

$$M_{\theta,\theta} - RQ_{r\theta} = 0 \tag{17b}$$

$$Q_{r\theta,\theta} - N_\theta + \frac{1}{R}(N_\theta u_{,\theta})_{,\theta} - p(v_{,\theta} + u) - pR = 0 \tag{17c}$$

In the prebuckling state, the axially symmetric distribution of external forces produces stresses identical at all cross sections. For external pressure

$$u_0 = -pR^2/C_{22}; \quad v_0 = 0; \quad \psi_0 = 0; \quad N_{\theta 0} = -pR = 0 \tag{18}$$

We shall also use the superscript c to refer to the critical state (i.e., $N_{\theta 0}^c = -p_c R$).

Substituting the displacement field in the perturbed form [(1)] into the equilibrium equations, retaining the first-order terms, then using the relations that express the first-order resultant forces and moments in terms of the first-order strains, and subsequently using the strain-displacement relations gives the buckling equations in the form

$$\frac{C_{22}}{R} v_{1,\theta\theta} - p_c v_1 + \left(\frac{C_{22}}{R} + p_c\right) u_{1,\theta} = 0 \tag{19a}$$

$$D_{22}\frac{\psi_{1,\theta\theta}}{R} - k_s^2 C_{66}(R\psi_1 + u_{1,\theta}) = 0 \tag{19b}$$

$$-\left(\frac{C_{22}}{R} + p_c\right)(v_{1,\theta} + u_1) + (N_{\theta 0}^c + k_s^2 C_{66})\frac{u_{1,\theta\theta}}{R} + k_s^2 C_{66}\psi_{1,\theta} = 0 \tag{19c}$$

The first-order displacement field is set in the form

$$u_1(\theta) = A_n \cos n\theta; \quad v_1(\theta) = B_n \sin n\theta; \quad \psi_1(\theta) = C_n \sin n\theta \tag{20}$$

Substitution results in the following three equations for A_n, B_n, and C_n:

$$A_n n\left(\frac{C_{22}}{R} + p_c\right) + B_n\left(\frac{C_{22}n^2}{R} + p_c\right) = 0 \tag{21a}$$

$$A_n n k_s^2 C_{66} - C_n(Rk_s^2 C_{66} + D_{22}n^2) = 0 \tag{21b}$$

$$-A_n\left[\frac{C_{22}}{R} + p_c + \frac{n^2}{R}(k_s^2 C_{66} - p_c R)\right] - B_n n\left(\frac{C_{22}}{R} + p_c\right) + C_n n k_s^2 C_{66} = 0 \tag{21c}$$

Eqs. (21a) and (21b) give

$$B_n = -A_n n\left(\frac{C_{22}}{R} + p_c\right) \bigg/ \left(\frac{C_{22}n^2}{R} + p_c\right) \tag{22a}$$

$$C_n = A_n n k_s^2 C_{66}/(Rk_s^2 C_{66} + D_{22}n^2) \tag{22b}$$

Substituting into (21c) gives the following quadratic characteristic equation for p_c:

$$\alpha p_c^2 + \beta p_c + \gamma = 0 \tag{23a}$$

where

$$\alpha = 2n^2 - 1 \tag{23b}$$

$$\beta = \frac{C_{22}}{R}(n^4 + n^2 - 1) + \frac{n^2 k_s^2 C_{66}}{R}\left(\frac{R^2 k_s^2 C_{66}}{R^2 k_s^2 C_{66} + D_{22}n^2} - 1\right) \tag{23c}$$

$$\gamma = \frac{C_{22}}{R^2} n^4 k_s^2 C_{66}\left(\frac{R^2 k_s^2 C_{66}}{R^2 k_s^2 C_{66} + D_{22}n^2} - 1\right) \tag{23d}$$

Table 3 gives the results from a comparison of the improved elasticity solution with the FOSD and with the classical shell formulations for thick construction. It is seen that the FOSD provides an improvement versus the classical shell theory but not to the extent of covering the overestimation versus the elasticity prediction. However, when the more complete problem of cylindrical shells of finite length under external pressure, rather than the simplified ring approximation, was analyzed in Kardomateas (1996), the FOSD results of Simitses et al. (1993) were found to be adequate for the moderately thick (R_2/R_1 no more than 1.10) boron/epoxy shells considered.

TABLE 3. First-Order Shear (k_s^2 = 5/6)—Orthotropic Critical Pressure (GPa) for Thick Shells

R_2/R_1 (1)	Elasticity with normal terms (2)	Classical shell theory[a] (3)	FOSD shell[b] (4)	Elasticity with normal terms versus shell (%) (5)	Elasticity with normal terms versus FOSD (%) (6)
1.207	0.07434	0.09574	0.09392	−22.35	−20.85
1.255	0.1212	0.1683	0.1561	−27.99	−22.36
1.303	0.1772	0.2656	0.2321	−33.28	−23.65
1.352	0.2402	0.3887	0.3191	−38.20	−24.73
1.4	0.3083	0.5379	0.4146	−42.68	−25.64

[a]From Eq. (14a).
[b]From Eq. (23).

From the results presented previously, it can be concluded that the effect of including the normal strain and normal stress terms, and thus having an improved but more complicated elasticity formulation, is to render the critical load further lower. However, this reduction (versus the elasticity solution without these terms) is small compared to the large reduction in critical load between the elasticity and the classical shell theory predictions for thick construction.

CONCLUSIONS

This paper presented an improved elasticity solution to the problem of buckling of orthotropic cylindrical shells subjected to external pressure. The generalized plane deformation case is studied (equivalent to a ring approximation). The main improvement versus the earlier elasticity solution is that the present solution includes the terms with the prebuckling normal strains and stresses as coefficients, which were neglected in the buckling equations of the earlier work. The results show that the effect of including these normal strains and stresses further decreases the critical load, albeit modestly. Moreover, a closed-form formula is derived in this paper for the critical pressure based on an FOSD theory. The elasticity critical load is still found to be lower than the FOSD prediction.

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