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Stress intensity factors for a mixed mode center crack in an anisotropic strip

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Abstract. An approach based on the continuous dislocation technique is formulated and used to obtain the Mode I and II stress intensity factors in a fully anisotropic infinite strip with a central crack. First, the elastic solution of a single dislocation in an anisotropic infinite strip is obtained from that of a dislocation in an anisotropic half plane, by applying an array of dislocations along the boundary of the infinite strip, which is supposed to be traction-free. The dislocation densities of the dislocation array are determined in such a way that the traction forces generated by the dislocation array cancel the residual tractions along the boundary due to the single dislocation in the half plane. The stress field of a single dislocation in the infinite strip is thus a superposition of that of the single dislocation and the dislocation array in the half plane. Subsequently, the elastic solution is applied to calculate the stress intensity factors for a center crack in an anisotropic strip. Crack length and material anisotropy effects are discussed in detail.

Key words: Anisotropic, center crack, dislocation, mode mixity, stress intensity factors, strip.

1. Introduction

The problem of determining the stress intensity factors of cracks in anisotropic materials is of considerable importance in the design of safe structures. This is due to the ever increasing usage of anisotropic materials in modern technology, especially in the aerospace industry and electronic industry, in which composite materials and bimaterials are commonly used.

Many approaches have been developed to calculate mixed-mode stress intensity factors in the past decades. A detailed review of stress intensity factor calculation can be found in Cartwright and Rooke (1975). One of the most effective methods is the distributed (also called continuous) dislocation technique, which is a semi-analytical technique. The basic idea of the continuous dislocation method is to model the crack as an array of dislocations along the crack lines in otherwise perfect bodies and determine the dislocation densities by satisfying the crack surface traction-free conditions. The crack tip stress intensity factors can be then calculated from the dislocation densities.

In order to apply the continuous dislocation technique to a solid body, the proper dislocation elastic field in the crack domain must be available. This is why the continuous dislocation approach is most suitable to determining stress intensity factors for cracks of relatively small dimensions, because the number of fundamental solutions available for the various kinds of dislocations is limited to simple geometries such as infinite space, half plane, near a circular inclusion, etc. The elastic solution of a dislocation in isotropic infinite plane or half plane can be easily found in many publications such as in Hill et al. (1996). As far as anisotropy, Lee (1990) proposed an analytic method to calculate the elastic fields of a dislocation in an anisotropic infinite plane and half plane.

However, the analytic solution of a dislocation in either isotropic or anisotropic infinite strips (i.e., one dimension of the domain being finite) is not available. Civelek (1985) presented a numerical method by superposing the infinite plane field with an additional elastic field, expressed via an Airy stress function with Fourier transformation and the Fourier series determined from the residual stresses along the boundaries. He also applied this method to several typical cases of practical importance involving a single or two cracks. Suo (1990) and Suo and Hutchinson (1990) extended this method to orthotropic materials and calculated the mixed-mode stress intensity factors for an infinite strip with semi-infinite cracks subjected to edge bending. In an earlier study, Delale et al. (1979) studied the problem of an inclined crack in an orthotropic strip, in which the plane of the crack must be a plane of material orthotropy. Georgiadis and Papadopoulos (1987) determined the stress intensity factor for an orthotropic infinite strip with a semi-infinite crack located mid-distance of the strip faces, by using Fourier transforms in combination with the Wiener-Hopf technique. A solution procedure that is based on a synthesis of Stroh's formalism for anisotropic elasticity and the Fourier transformation was presented by Wu and Chiu (1995) to analyze the elastic fields of a dislocation in an anisotropic strip. In their paper, the elastic field of interest is divided into an unbounded medium and a regular part. Interactive forces between two like dislocations in the central plane of the strip in these materials are also calculated to study the influence of the boundaries. A surface dislocation model was proposed by Jagannagham and Marcinkowski (1979) to calculate the stress fields of a finite body subjected to either an applied stress or an internal stress. However, neither of these two papers went further to explore the calculation of stress intensity factors. In a more recent study, Qian and Sun (1997) obtained stress intensity factors for interface cracks between two monoclinic media, by either calculating the finite-extension strain energy release rates or utilizing the relationships between the crack surface displacements and the stress intensity factors, both carried out with a finite element analysis.

In this paper, the elastic solution of a dislocation in an anisotropic infinite strip is derived by assigning an array of dislocations along a line in the half plane which is supposed to be the boundary of the infinite strip. The dislocation densities of these added dislocations are determined by satisfying the traction-free boundary conditions. The stress fields of a single dislocation in the anisotropic strip is thus the combination of that of the single dislocation and those of the dislocations distributed along the boundary. Thereafter, the elastic solution is employed to calculate the mixed-mode stress intensity factors of a center crack in an anisotropic infinite strip. Effects of both the geometry and the material anisotropy on the stress intensity factors are investigated.

2. Formulation

2.1. Elastic solution of a dislocation in an anisotropic infinite strip

As shown in Fig. 1, the geometry of a dislocation in an infinite strip can be decomposed into two geometries. The first one is a half plane with a single dislocation located at point (x_0, y_0), for which the elastic solution of the dislocation can be found in Lee (1990) and is summarized in Appendix A. In short, the stress components at point (x, y) due to a dislocation $B = B_x + iB_y$ located at (x_0, y_0) can be expressed as:

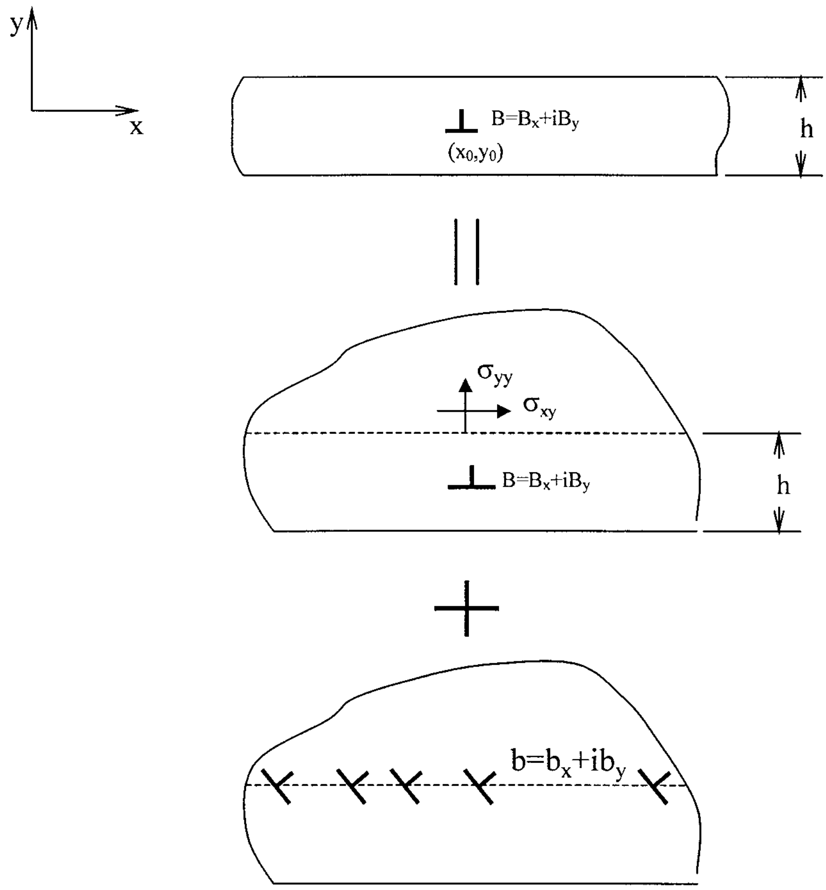

Figure 1. Single dislocation in an infinite strip as a superposition of a single dislocation in a half plane and an array of dislocations at the boundary of the strip.

$$\sigma_{ij}(x, y) = B_x(x_0, y_0)G_{xij}(x, y, x_0, y_0) + B_y(x_0, y_0)G_{yij}(x, y, x_0, y_0), \tag{1}$$

where $ij = xx, yy, xy$ and $G_{xij}(x, y, x_0, y_0)$ are the stress components at (x, y) due to a unit dislocation $B_x = 1$ at (x_0, y_0) and $G_{yij}(x, y, x_0, y_0)$ are the stress components at (x, y) due to a unit dislocation $B_y = 1$ at (x_0, y_0).

Accordingly, the traction forces along the dashed line, which is supposed to be the boundary of the infinite strip, due to the single dislocation $B(x_0, y_0)$ in the half plane are:

$$\sigma_{yy}^{(s)}(x, h) = B_x(x_0, y_0)G_{xyy}(x, h, x_0, y_0) + B_y(x_0, y_0)G_{yyy}(x, h, x_0, y_0), \tag{2a}$$

and

$$\sigma_{xy}^{(s)}(x, h) = B_x(x_0, y_0)G_{xxy}(x, h, x_0, y_0) + B_y(x_0, y_0)G_{yxy}(x, h, x_0, y_0). \tag{2b}$$

The second geometry is a half plane with an array of dislocations along the dashed line. The dislocation densities of the dislocations $b(x, h)$ along the boundary are determined in such a way that the traction forces generated by these dislocations along the dashed line $\sigma_{yy}^{(a)}(x, h)$ and $\sigma_{xy}^{(a)}(x, h)$ are the opposite of $\sigma_{yy}^{(s)}(x, h)$ and $\sigma_{xy}^{(s)}(x, h)$. Thus the traction-free boundary conditions of the infinite strip are satisfied after superposing these two geometries together.

Suppose that the dislocation array $b(t,h) = b_x(t,h) + i b_y(t,h)$ are distributed from $-\infty$ to ∞ along $y = h$, then the traction forces along the dashed line due to the dislocation array are:

$$\sigma_{yy}^{(a)}(x,h) = \int_{-\infty}^{\infty} [b_x(t,h) G_{xyy}(x,h,t,h) + \\ + b_y(t,h) G_{yyy}(x,h,t,h)] \, \mathrm{d}t = -\sigma_{yy}^{(s)}(x,h), \tag{3a}$$

$$\sigma_{xy}^{(a)}(x,h) = \int_{-\infty}^{\infty} [b_x(t,h) G_{xxy}(x,h,t,h) + b_y(t,h) G_{yxy}(x,h,t,h)] \, \mathrm{d}t = -\sigma_{xy}^{(s)}(x,h). \tag{3b}$$

where $\sigma_{ij}^{(s)}$ are defined in Equation (2).

The functions $G_{xyy}(x,h,t,h)$, $G_{yyy}(x,h,t,h)$, $G_{xxy}(x,h,t,h)$ and $G_{yxy}(x,h,t,h)$ are singular at $x = t$. Since the single dislocation is in self- equilibrium, the traction forces $\sigma_{yy}^{(s)}(x,h)$ and $\sigma_{xy}^{(s)}(x,h)$ vanish as $t \to -\infty, +\infty$. As a result, the dislocation densities $b_x(t,h)$ and $b_y(t,h)$ go to zero as $t \to -\infty, +\infty$. Therefore, for calculation purposes, in the singular integral Equations (3) we can ignore the dislocations located at $x > d$ and $x < -d$, where d is a value large enough compared to h. A value of $d = 100h$ was found to be more than adequate for this purpose. Thus, Equations (3) become:

$$\sigma_{yy}^{(a)}(x,h) = \int_{-d}^{d} [b_x(t,h) G_{xyy}(x,h,t,h) + b_y(t,h) G_{yyy}(x,h,t,h)] \, \mathrm{d}t = -\sigma_{yy}^{(s)}(x,h), \tag{4a}$$

$$\sigma_{xy}^{(a)}(x,h) = \int_{-d}^{d} [b_x(t,h) G_{xxy}(x,h,t,h) + b_y(t,h) G_{yxy}(x,h,t,h)] \, \mathrm{d}t = -\sigma_{xy}^{(s)}(x,h). \tag{4b}$$

Now, normalize Equation (4) as following:

$$\tilde{t} = \frac{t}{d} \quad \text{and} \quad \tilde{x} = \frac{x}{d},$$

so that these equations can be written in the form:

$$\sigma_{ij}^{(a)} = \pi d \left[\frac{1}{\pi} \int_{-1}^{1} b_x(\tilde{t},h) G_{xij}(x,h,t,h) + b_y(\tilde{t},h) G_{yij}(x,h,t,h) \right] \mathrm{d}\tilde{t} = -\sigma_{ij}^{(s)}(x,h),$$

$$ij = yy, xy. \tag{5}$$

We can actually enforce that $b(\tilde{t},h) = b_x(\tilde{t},h) + i b_y(\tilde{t},h)$ be zero at $\tilde{t} = -1$ and $\tilde{t} = 1$ ($t = -d$ and $t = d$) assuming that d is large enough. This can be built into the solution by expressing $b(\tilde{t},h)$ as the product of a fundamental function $W(\tilde{t})$ and an unknown function $\tilde{b}(\tilde{t},h)$ (Hills et al., 1996):

$$b(\tilde{t}) = W(\tilde{t}) \tilde{b}(\tilde{t},h); \qquad W(\tilde{t}) = \sqrt{1 - \tilde{t}^2}. \tag{6}$$

Substituting Equation (6) into Equation (5), the numerical form of the singular integral equations can be expressed as:

$$\pi d\left[\sum_{m=1}^{N} W_m\tilde{b}_x(\tilde{t}_m,h)G_{xij}(x_k,h,t_m,h)+\sum_{m=1}^{N} W_m\tilde{b}_y(\tilde{t}_m,h)G_{yij}(x_k,h,t_m,h)\right]= \\ -\sigma_{ij}^{(s)}(x_k,h),\quad ij=yy,\ xy,\quad k=1,\ldots,N+1, \tag{7}$$

where $\tilde{t}_m$ are the N discrete integral points and $\tilde{x}_k$ are the collocation points and W_m are the weight coefficients:

$$\tilde{t}_m=\cos\left(\frac{\pi m}{N+1}\right);\quad \tilde{x}_k=\cos\left[\frac{\pi(2k-1)}{2(N+1)}\right];\qquad W_m=\frac{1-\tilde{t}_m^2}{N+1}. \tag{8}$$

Equation (7) allows us to determine the dislocation densities $b(\tilde{t}_m,h)$ of the dislocation array along the dashed line, which cancel out the residual traction forces due to a single dislocation $B(x_0,y_0)$ in the first geometry. After the dislocation densities $b(\tilde{t}_m,h)$ are known, the stress components at every point (x,y) in the second geometry can be calculated as follows:

$$\sigma_{ij}^{(a)}(x,y)=\pi d\left[\sum_{m=1}^{N} W_m\tilde{b}_x(\tilde{t}_m,h)G_{xij}(x,y,t_m,h)+\sum_{m=1}^{N} W_m\tilde{b}_y(\tilde{t}_m,h)G_{yij}(x,y,t_m,h)\right], \tag{9}$$

where $ij=xx,yy$ and xy.

Obviously, the dislocation densities $b(t_m,h)$ along the dashed line are related to the single dislocation $B(x_0,y_0)$ in the half plane. Denoting the dislocation densities $b(\tilde{t}_m,h)$ as $b^{(x)}(\tilde{t}_m,h)=b_x^{(x)}(\tilde{t}_m,h)+ib_y^{(x)}(\tilde{t}_m,h)$ for a single dislocation $B(x_0,y_0)=1$ and as $b^{(y)}(\tilde{t}_m,h)=b_x^{(y)}(\tilde{t}_m,h)+ib_y^{(y)}(\tilde{t}_m,h)$ for a dislocation $B(x_0,y_0)=i$, and superposing the elastic fields of the single dislocation and the dislocation array along $y=h$, we have:

$$\tilde{G}_{xij}(x,y,x_0,y_0)=G_{xij}(x,y,x_0,y_0)+ \\ +\pi d\left[\sum_{m=1}^{N} W_m\tilde{b}_x^{(x)}(\tilde{t}_m,h)G_{xij}(x,y,t_m,h)+\sum_{m=1}^{N} W_m\tilde{b}_y^{(x)}(\tilde{t}_m,h)G_{yij}(x,y,t_m,h)\right], \tag{10a}$$

and

$$\tilde{G}_{yij}(x,y,x_0,y_0)=G_{yij}(x,y,x_0,y_0)+ \\ +\pi d\left[\sum_{m=1}^{N} W_m\tilde{b}_x^{(y)}(\tilde{t}_m,h)G_{xij}(x,y,t_m,h)+\sum_{m=1}^{N} W_m\tilde{b}_y^{(y)}(\tilde{t}_m,h)G_{yij}(x,y,t_m,h)\right], \tag{10b}$$

where $ij=xx,yy$ and xy.

Physically, $\tilde{G}_{xij}(x,y)$ in Equation (10a) represent the stresses at (x,y) due to a dislocation $B_x=1$ at (x_0,y_0) in the infinite strip; Similarly, $\tilde{G}_{yij}(x,y)$ in Equation (10b) represent the stresses at (x,y) due to a dislocation $B_y=1$ at (x_0,y_0) in the infinite strip. Because the elastic fields of the dislocation are linear, the stress components at (x,y) due to a dislocation $B(x_0,y_0)=B_x(x_0,y_0)+iB_y(x_0,y_0)$ in the infinite strip are:

$$\sigma_{ij}(x,y)=B_x(x_0,y_0)\tilde{G}_{xij}(x,y,x_0,y_0)+B_y(x_0,y_0)\tilde{G}_{yij}(x,y,x_0,y_0). \tag{11}$$

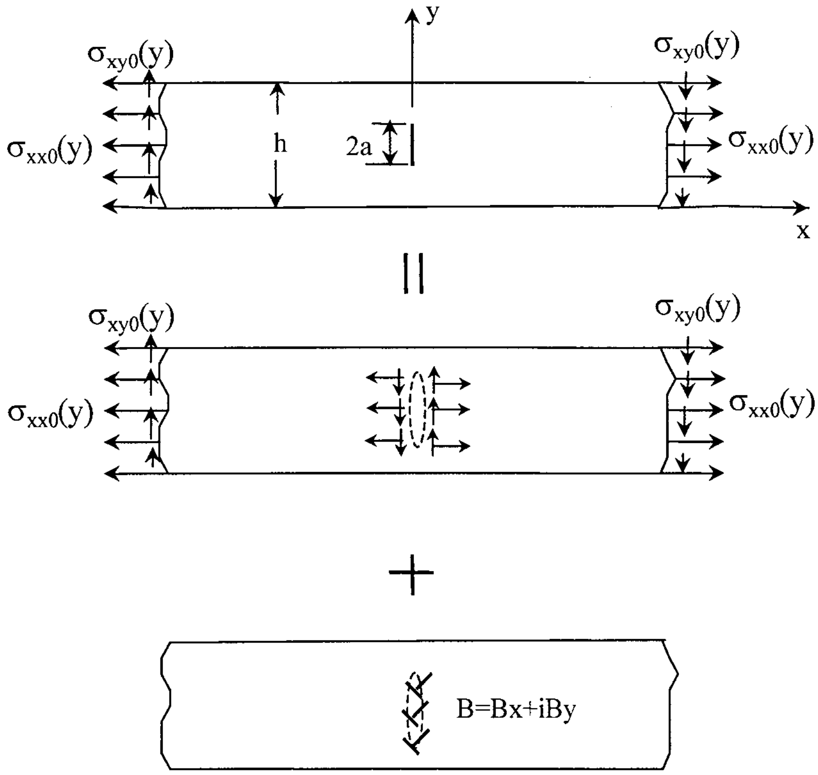

Figure 2. The continuous dislocation approach in an infinite strip. The array of dislocations $B = B_x + iB_y$ is determined so that the traction-free conditions at the crack site are satisfied.

Next, the elastic solution of the dislocation in an anistropic infinite strip is applied to an anisotropic strip with a central crack perpendicular to the boundary of the thin strip.

2.2. Application to a central crack in an infinite anisotropic strip

Now, let's consider an infinite strip with a central crack subjected to uniform tension. The crack of length $2a$ is aligned with the y axis and the middle point of the crack is located at $y = h/2$. Thus, the crack tips are located at $y = h/2 + a$ and $y = h/2 - a$, respectively. Applying the routine continuous dislocation method, i.e., replacing the crack in the infinite strip with a series of dislocations (Fig. 2), the tractions at a point $(0, y)$ along the crack surface due to the dislocations are:

$$\sigma_{xx}^{(d)}(0, y) = \int_{h/2-a}^{h/2+a} \left[B_x(0, t)\tilde{G}_{xxx}(0, y, 0, t) + B_y(0, t)\tilde{G}_{yxx}(0, y, 0, t) \right] \mathrm{d}t, \tag{12a}$$

and

$$\sigma_{xy}^{(d)}(0, y) = \int_{h/2-a}^{h/2+a} \left[B_x(0, t)\tilde{G}_{xxy}(0, y, 0, t) + B_y(0, t)\tilde{G}_{yxy}(0, y, 0, t) \right] \mathrm{d}t. \tag{12b}$$

To satisfy the crack surface traction-free condition, the tractions $\sigma_{xx}^{(d)}$ and $\sigma_{xy}^{(d)}$ in Equations (12) should cancel out the tractions along the crack face due to the external force, $\sigma_{xx0}(y)$ and $\sigma_{xy0}(y)$. In our case, the thin strip is subjected to uniform tensile load σ, i.e. $\sigma_{xx0}(y) = \sigma$ and $\sigma_{xy0}(y) = 0$. To satisfy the crack surface tranction free boundary conditions, we have:

$$\sigma_{xx}^{(d)}(0, y) = -\sigma \quad \text{and} \quad \sigma_{xy}^{(d)}(0, y) = 0. \tag{13}$$

Substituting Equation (12) into Equation (13) gives two singular integral equations, which can be solved by Gaussian quadrature. First, Equations (12) are normalized through the substitution:

$$\bar{y} = \frac{y - h/2}{a}, \qquad \bar{t} = \frac{t - h/2}{a},$$

so that Equation (13) can be written in the form:

$$\begin{aligned}\sigma_{ij}^{(d)}(0, y) = \ & \pi a \left[\frac{1}{\pi} \int_{-1}^{1} B_x(0, \bar{t}) \tilde{G}_{xij}(0, y, 0, t) \, \mathrm{d}\bar{t} + \right. \\ & \left. + \frac{1}{\pi} \int_{-1}^{1} B_y(0, \bar{t}) \tilde{G}_{yij}(0, y, 0, t) \, \mathrm{d}\bar{t} \right] = -\sigma_{ij0}(y) \quad ij = xx, xy.\end{aligned} \tag{14}$$

Now, the form of $B_x(\bar{t})$ and $B_y(\bar{t})$ must be singular at the crack tips $\bar{t} = -1, 1$ ($t = h/2 - a, h/2 + a$) in order to preserve the correct asymptotic form of stresses and displacements. Note that it has been proven by Stroh (1958) that the crack tip stresses have a singularity of the $r^{-1/2}$ type in anisotropic materials, just as in the isotropic case. This may be built into the solution by expressing $B(\bar{t})$ as the product of a fundamental function $W(\bar{t})$ and an unknown regular function $\tilde{B}(\bar{t})$ (Hills et al., 1996):

$$B(\bar{t}) = W(\bar{t}) \tilde{B}(\bar{t}); \qquad W(\bar{t}) = \frac{1}{\sqrt{1 - \bar{t}^2}}. \tag{15}$$

Substituting Equation (15) into Equation (14), the numerical form of the singular integral equations can be expressed as:

$$\pi a \left\{ \sum_{m=1}^{N} W_m \tilde{B}_x(0, \bar{t}_m) \tilde{G}_{xxx}(0, y_k, 0, t_m) + \sum_{m=1}^{N} W_m \tilde{B}_y(0, \bar{t}_m) \tilde{G}_{yxx}(0, y_k, 0, t_m) \right\} = -\sigma,$$

$$k = 1 \ldots N - 1, \tag{16a}$$

where $\bar{t}_m$ are the N discrete integration points and $\bar{y}_k$ are the $N - 1$ collocation points and W_m are weight coefficients:

$$\bar{t}_m = \cos \frac{\pi(2m - 1)}{2N}; \qquad \bar{y}_k = \cos \frac{\pi k}{N}; \qquad W_m = \frac{1}{N}.$$

Also, $y_k = a\bar{y}_k + h/2$ and $t_m = a\bar{t}_m + h/2$. Similarly:

$$\pi a \left\{ \sum_{m=1}^{N} W_m \tilde{B}_x(0, \bar{t}_m) \tilde{G}_{xxy}(0, y_k, 0, t_m) + \sum_{m=1}^{N} W_m \tilde{B}_y(0, \bar{t}_m) \tilde{G}_{yxy}(0, y_k, 0, t_m) \right\} = 0,$$

$$k = 1 \ldots N - 1. \tag{16b}$$

Although only the structure subjected to uniform tension is considered in this paper, it should be mentioned that the external load can be arbitrary as long as the elastic solution of the structure subjected to such external loading is known.

In addition, the requirement that the crack surfaces physically come together at both ends imposes two side conditions:

$$\int_{h/2-a}^{h/2+a} B_x(0,t)\,\mathrm{d}t = a\int_{-1}^{1} B_x(0,\bar{t})\,\mathrm{d}\bar{t} = 0, \tag{17a}$$

$$\int_{h/2-a}^{h/2+a} B_y(0,t)\,\mathrm{d}t = a\int_{-1}^{1} B_y(0,\bar{t})\,\mathrm{d}\bar{t} = 0. \tag{17b}$$

The conditions (17) arise from the fact that the dislocation density is related to the displacements as:

$$B(0,t)\,\mathrm{d}t = \left[B_x(0,t)\,\mathrm{d}s + iB_y(0,t)\right]\,\mathrm{d}t = d\left[u^+(t) - u^-(t)\right] + id\left[v^+(t) - v^-(t)\right],$$

where $+$ denotes the upper surface and $-$ the lower surface of the crack

The discrete forms of equation (17a) and (17b) are:

$$\sum_{m=1}^{N} W_m\tilde{B}_x(0,\bar{t}_m) = 0; \qquad \sum_{m=1}^{N} W_m\tilde{B}_y(0,\bar{t}_m) = 0. \tag{18}$$

Now the values of $\tilde{B}(0,\bar{t}_m) = \tilde{B}_x(0,\bar{t}_m) + i\tilde{B}_y(0,\bar{t}_m)$ can be solved at the discrete set of points $\bar{t}_m$. The system of $2N$ linear equations for the determination of $\tilde{B}_x(0,\bar{t}_m)$ and $\tilde{B}_y(0,\bar{t}_m)$ are the Equations (16) and (18).

Of major significance are the values of the dislocation densities at the crack tips, $\tilde{B}(+1)$ and $\tilde{B}(-1)$, as they are directly related to the stress intensity factors. The value of $\tilde{B}(+1)$ and $\tilde{B}(-1)$ can be obtained from Krenk's interpolation formulae (Hills et al., 1996):

$$\tilde{B}(+1) = M_E\sum_{m=1}^{N} B_{Em}\tilde{B}(\bar{t}_m), \tag{19a}$$

$$\tilde{B}(-1) = M_E\sum_{m=1}^{N} B_{Em}\tilde{B}(\bar{t}_{N+1-m}), \tag{19b}$$

where

$$M_E = \frac{1}{N}, \qquad \text{and} \qquad B_{Em} = \sin\left[\frac{\pi(2m-1)(2N-1)}{4N}\right] \Big/ \sin\left[\frac{\pi(2m-1)}{4N}\right].$$

The stress intensity factors at the crack tip $y = h/2 + a$, are defined as:

$$(K_I + iK_{II})\,|_{h/2+a} = \lim_{y\to h/2+a}\{\sqrt{2\pi(h/2+a-y)}[\sigma_{xx}(y) + i\tau_{xy}(y)]_{x=0}\}. \tag{20a}$$

Using relations for the stresses in terms of the complex potentials as in (A3), gives

$$\sigma_{xx}(y) + i\sigma_{xy}(y)\,|_{x=0} = \sum_{j=1,2}\int_{h/2-a}^{h/2+a}\left[(\mu_j^2 - i\mu_j)\phi_j'(y,\xi) + (\bar{\mu}_j^2 - i\bar{\mu}_j)\bar{\phi}_j'(y,\xi)\right]\,\mathrm{d}\xi. \tag{20b}$$

Only the singular parts of the stress potential contribute, Therefore, the first term in (20b) gives

$$(\mu_1 - i)\lim_{y\to h/2+a}\sqrt{2\pi(h/2+a-y)}\int_{h/2-a}^{h/2+a}\left[A_{11}\frac{b(\xi)}{y-\xi} + A_{12}\frac{\bar{b}(\xi)}{y-\xi}\right]\,\mathrm{d}\xi, \tag{20c}$$

and with the substitution $t = (y - h/2)/a$, $\bar{\xi} = (\xi - h/2)/a$, and $b(t) = B(t)/\sqrt{1-t^2}$, it gives

$$(\mu_1 - i)\lim_{t\to 1}\sqrt{2\pi a(1-t)}\left[A_{11}\int_{-1}^{+1}\frac{B(\bar{\xi})}{t-\bar{\xi}}d\bar{\xi} + A_{12}\int_{-1}^{+1}\frac{\overline{B(\bar{\xi})}}{t-\bar{\xi}}d\bar{\xi}\right] =$$

$$(\mu_1 - i)\lim_{t\to 1}\sqrt{2\pi a(1-t)}\pi\left[A_{11}\frac{B(t)}{\sqrt{1-t^2}} + A_{12}\frac{\overline{B(t)}}{\sqrt{1-t^2}}\right] = \tag{20d}$$

$$(\mu_1 - i)\pi\sqrt{\pi a}\left[A_{11}B(+1) + A_{12}\overline{B(+1)}\right].$$

A similar contribution exists from the μ_2 term (second term of 20b).

Therefore, the stress intensity factors at $y = h/2 + a$, can be related to the dislocation densities from the following expression:

$$(K_I + iK_{II})\mid_{h/2+a} = \pi\sqrt{\pi a}\{[(\mu_1 - i)A_{11} + (\mu_2 - i)A_{21} + (\bar{\mu}_1 - i)\bar{A}_{12} +$$

$$+(\bar{\mu}_2 - i)\bar{A}_{22}]B(+1) + [(\mu_1 - i)A_{12} + (\mu_2 - i)A_{22} + (\bar{\mu}_1 - i)\bar{A}_{11} + (\bar{\mu}_2 - i)\bar{A}_{21}]\overline{B(+1)}\}, \tag{21a}$$

where A_{11}, A_{12}, A_{21}, A_{22} are defined in Appendix A and $B(+1)$ is given in (19a). Notice that $\overline{B(+1)}$ is the complex conjugate of $B(+1)$.

Similarly, the stress intensity factors at the other crack tip, $y = h/2 - a$, are:

$$(K_I + iK_{II})\mid_{h/2-a} = -\pi\sqrt{\pi a}\{[(\mu_1 - i)A_{11} + (\mu_2 - i)A_{21} + (\bar{\mu}_1 - i)\bar{A}_{12} +$$

$$+(\bar{\mu}_2 - i)\bar{A}_{22}]B(-1) + [(\mu_1 - i)A_{12} + (\mu_2 - i)A_{22} + (\bar{\mu}_1 - i)\bar{A}_{11} + (\bar{\mu}_2 - i)\bar{A}_{21}]\overline{B(-1)}\}. \tag{21b}$$

3. Discussion of Results

The formulation described above can be easily implemented numerically. First, let us compare the results of this approach with some examples in the literature for isotropic material. Readily available formulas exist in Tada et al. (1985).

The mode I stress intensity factor for a central crack of length $2a$ in a strip of width h, under uniform tension in the direction normal to the crack, is expressed as:

$$K_I = F\sigma\sqrt{\pi a};\quad F = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}}, \tag{22}$$

where $\alpha = 2a/h$ is the crack length ratio. Fig. 3 compares the values of F for different crack length ratios, obtained by the present approach (discrete data points), with those obtained by applying the Tada et al. (1985) formula, Equation (22). The comparison indicates an excellent agreement. It should be mentioned that the isotropic solutions were calculated from the present fully anisotropic formulation by setting the complex parameters $\mu_1 = 1.0001i$ and $\mu_2 = 0.9999i$.

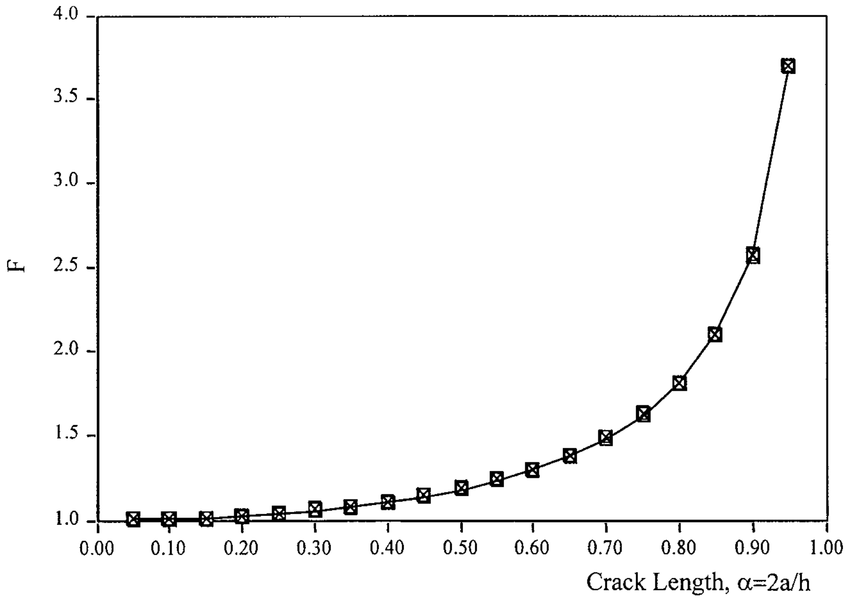

Figure 3. Mode I stress intensity factor parameter, F, in $K_I = F\sigma\sqrt{\pi a}$, for an isotropic, centrally cracked strip under uniform tension, σ. The line is the Tada et al. (1985) relationship for an isotropic crack and the discrete data points are from the present anisotropic formulation when taken at the limit of isotropy.

Table 1. Comparison between the present method and the method presented by Delale et al. (1979) for an orthotropic Thin Strip

$\alpha = \frac{2a}{h}$	Present		Delale et al. (1979)	
	$\bar{K}_I$	$\bar{K}_{II}$	$\bar{K}_I$	$\bar{K}_{II}$
0.2	1.0177	0	1.018	0
0.4	1.0807	0	1.081	0
0.6	1.2260	0	1.226	0
0.8	1.6231	0	1.624	0
0.9	2.2454	0	2.249	0

For orthotropic thin strip, an example given by Delale et al. (1979) is revisited. In their study, the transversal direction of the laminate and the crack are parallel and are at an angle $\pi - \theta$ with the thin strip boundary. When $\theta = 0$, the thin strip is orthotropic with fiber orientation align with x axis. Thus, the crack is perpendicular to the thin strip boundary. A boron-epoxy composite sheet with the following material constants is considered: $E_{11} = 170.65Gpa$, $E_{22} = 55.6Gpa$, $G_{12} = 4.83Gpa$, and $\mu_{12} = 0.1114$. The results calculated from present approach are compared with those of Delale et al. (1979). Again we achieved excellent agreement between these two methods. The data for Delale et al. (1979) are taken from Table 1 in their paper with $\theta = 0$. $\bar{K}_I$ and $\bar{K}_{II}$ are normalized stress intensity factors. The definition of K_I and K_{II} differs by a factor of $\sqrt{\pi}$ between this two papers, $\bar{K}_I = (K_I)/(\sigma_m\sqrt{c})$ and $\bar{K}_{II} = (K_{II})/(\sigma_m\sqrt{c})$ in Delale et al. (1979), while $\bar{K}_I = (K_I)/(\sigma\sqrt{\pi a})$ and $\bar{K}_{II} = (K_{II})/(\sigma\sqrt{\pi a})$ in this paper. c and a are the crack half length.

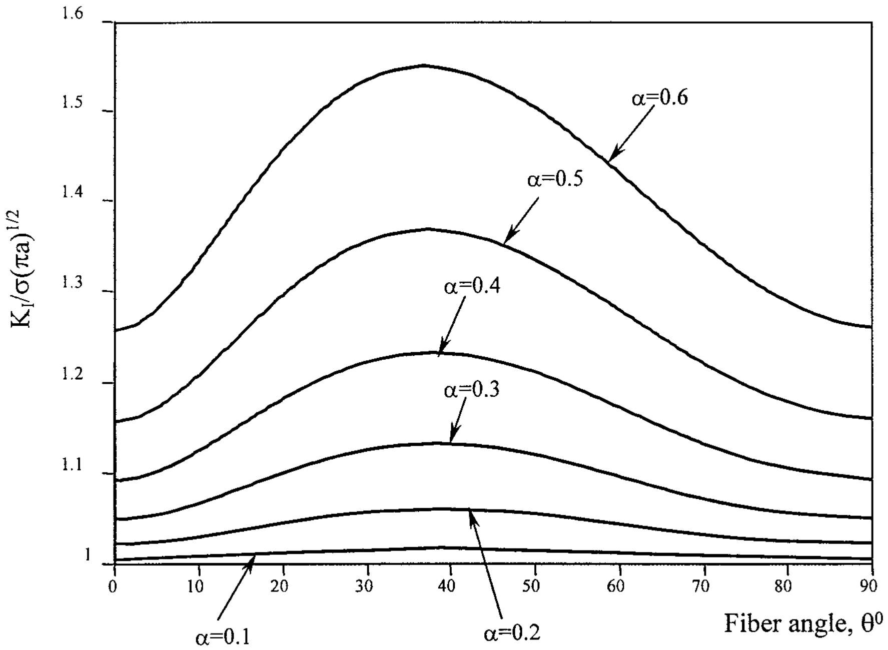

Figure 4. The effect of anisotropy on the mode I stress intensity factor of a center crack in a strip under uniform tension for a unidirectional graphite/epoxy with fiber orientation, θ, measured from the direction of the applied load. The stress intensity factor, K_I, is normalized with the corresponding stress intensity factor of an isotropic, infinite plate, $\sigma\sqrt{\pi a}$.

The effect of anisotropy is shown in Figs. 4 and 5. Again, a central crack of length $2a$ in an infinite strip of width h under uniformly distributed stress, σ, perpendicular to the crack, is studied. Typical data for graphite/epoxy were used, i.e., moduli in GPa: $E_L = 130$, $E_T = 10.5$, $G_{LT} = 6$, and Poisson's ratio $\mu_{LT} = 0.28$, where L and T are the directions along and perpendicular to the fibers, respectively. A unidirectional construction was considered with the fiber orientation angle, θ, varying from 0 to 90 deg. The orientation angle θ is measured from the x direction, i.e., $\theta = 0^0$ is when the crack is perpendicular to the fibers and $\theta = 90^0$ is when the crack is parallel with the fibers. Obviously, the limits of $\theta = 0^0$ and 90^0 are the orthotropic cases. As shown in Fig. 4, the effect of anisotropy on the mode I stress intensity factors is seen to be significant between 30 and 40 deg and depends also strongly on the relative crack length $\tilde{a} = 2a/h$, being larger for cracks of relative larger length. The mode mixity ψ in Fig. 5 is defined as

$$\psi = \tan^{-1}\left(\frac{K_{II}}{K_I}\right), \tag{23}$$

and expresses the relative amounts of mode I and mode II components. A pure mode I state corresponds to ψ = 0 degrees and a pure mode II state corresponds to ψ = ±90 degrees. As expected, the mode II stress intensity factors are zero for orthotropic materials. The effect of anisotropy on the mode mixity is dependent on both the fiber orientation and the relative crack length. The fiber angle at which the mode mixity is maximum shifts to larger values as the relative crack length increases.

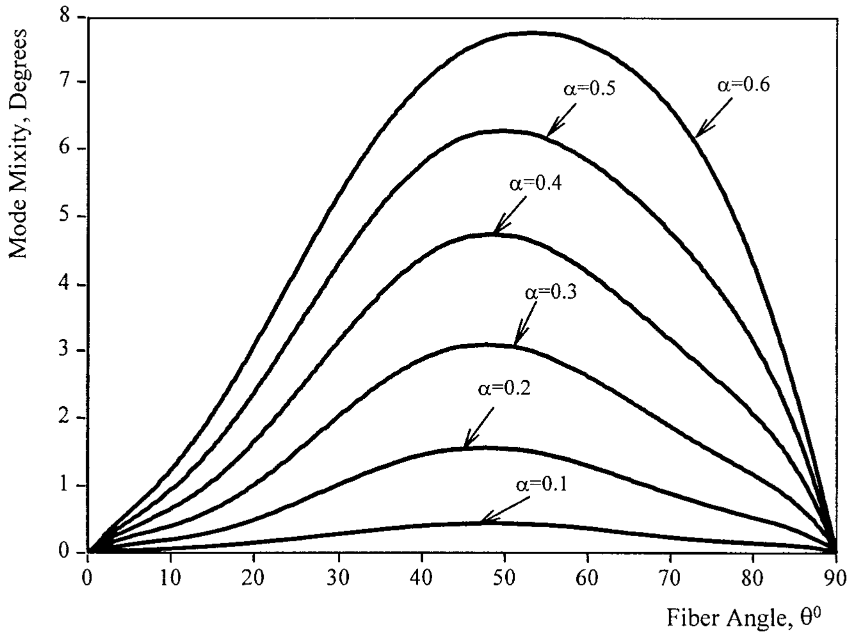

Figure 5. The effect of anisotropy on the mode mixity, ψ, of a center crack in a strip under uniform tension for a unidirectional graphite/epoxy with fiber orientation, θ, measured from the direction of the applied load. The zero degrees correspond to pure mode I. A pure mode II case would give 90 deg.

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Appendix. A dislocation in an anisotropic half plane

Let us consider a state of plane strain, i.e., $\epsilon_{zz} = \gamma_{yz} = \gamma_{xz} = 0$. In this case, the stress-strain relations for the anisotropic body are (Lekhnitskii, 1963):

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{26} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \alpha_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{36} & \alpha_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{bmatrix}, \tag{A1a}$$

where α_{ij} are the compliance constants (we have used the notation $1 \equiv x$, $2 \equiv y$, $3 \equiv z$, $6 \equiv xy$).

Using the condition of plane strain, which requires that $\epsilon_{zz} = 0$, allows elimination of σ_{zz}, i.e.

$$\sigma_{zz} = -\frac{1}{\alpha_{33}}(\alpha_{13}\sigma_{xx} + \alpha_{23}\sigma_{yy}). \tag{A1b}$$

The Equations (1a) can then be written in the form

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}, \tag{A1c}$$

where

$$\beta_{ij} = \alpha_{ij} - \frac{\alpha_{i3}\alpha_{j3}}{\alpha_{33}} \quad (i, j = 1, 2, 6). \tag{A1d}$$

Problems of this type can be formulated in terms of two complex analytic functions $\phi_k(z_k)$ ($k = 1, 2$) of the complex variables $z_k = x + \mu_k y$, where $\mu_k, \bar{\mu}_k, k = 1, 2$ are the roots of the algebraic equation:

$$\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22} = 0. \tag{A2}$$

It was proven by Lekhnitskii (1963) that these roots $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$ are either complex or purely imaginary, i.e., Equation (A2) cannot have real roots. Here, μ_1 and μ_2 are chosen to be the ones with positive imaginary parts.

The stress and displacement components can be expressed in terms of $\Phi_k(z_k)$ as (Lekhnitskii, 1963):

$$\sigma_{xx} = 2Re[\mu_1^2\phi_1'(z_1) + \mu_2^2\phi_2'(z_2)], \tag{A3a}$$

$$\sigma_{yy} = 2Re[\phi_1'(z_1) + \phi_2'(z_2)], \tag{A3b}$$

$$\tau_{xy} = -2Re[\mu_1\phi_1'(z_1) + \mu_2\phi_2'(z_2)]. \tag{A3c}$$

Now, the complex stress potentials at a point $z = x + iy$ due to a single dislocation at $z_0 = x_0 + iy_0$ in an anisotropic half plane are given by Lee (1990). Only the results are presented here.

The derivatives of the complex stress potentials at z due to a dislocation at z_0, are given as:

$$\phi_1'(z_1, z_0) = \frac{A_1}{z_1 - z_{10}} + \frac{1}{\Delta}\left[(\bar{\gamma}_1\gamma_2 - \bar{\delta}_1\delta_2)\frac{\bar{A}_1}{z_1 - \bar{z}_{10}} + (\gamma_2\bar{\gamma}_2 - \delta_2\bar{\delta}_2)\frac{\bar{A}_2}{z_1 - \bar{z}_{20}}\right], \tag{A4a}$$

$$\phi_2'(z_2, z_0) = \frac{A_2}{z_2 - z_{20}} - \frac{1}{\Delta}\left[(\gamma_1\bar{\gamma}_1 - \delta_1\bar{\delta}_1)\frac{\bar{A}_1}{z_2 - \bar{z}_{10}} + (\gamma_1\bar{\gamma}_2 - \delta_1\bar{\delta}_2)\frac{\bar{A}_2}{z_2 - \bar{z}_{20}}\right]. \tag{A4b}$$

The first terms in $\phi_1'(z_1, z_0), \phi_2'(z_2, z_0)$ are the singular solutions for an infinite domain and the second terms are the regular solutions pertinent to a half plane. The material coefficients are:

$$\gamma_k = 1 - i\mu_k; \quad \delta_k = 1 + i\mu_k, \quad k = 1, 2 \quad \text{and} \quad \Delta = \gamma_1\delta_2 - \gamma_2\delta_1. \tag{A4c}$$

Also,

$$z_i = [(1 - i\mu_i)z + (1 + i\mu_i)\bar{z}]/2, \quad i = 1, 2. \tag{A4d}$$

The complex coefficients A_1 and A_2 are related to dislocation densities $b = b_x + ib_y$ as:

$$A_1 = A_{11}b + A_{12}\bar{b}, \tag{A5a}$$

$$A_2 = A_{21}b + A_{22}\bar{b}. \tag{A5b}$$

Then, $\tilde{\phi}_1^{(x)'}(z_1), \tilde{\phi}_2^{(x)'}(z_2)$ and $\tilde{\phi}_1^{(y)'}(z_1), \tilde{\phi}_2^{(y)'}(z_2)$ can be calculated from Equations (A4), (A5) by setting $b = 1$ and $b = i$, respectively. Subsequently, the stresses G_{xij} and G_{yij} can be determined form the complex stress potentials. For example,

$$G_{xxx} = 2Re\left[\mu_1^2\tilde{\phi}_1^{(x)'} + \mu_2^2\tilde{\phi}_2^{(x)'}\right].$$

The complex parameters A_{ij} in Equations (A5) are material properties, which can be found as follows. A_j constitute the solution of the following equations:

$$\begin{bmatrix} \delta_1 & -\bar{\gamma}_1 & \delta_2 & -\bar{\gamma}_2 \\ -\gamma_1 & \bar{\delta}_1 & -\gamma_2 & \bar{\delta}_2 \\ p(\mu_1) & -p(\bar{\mu}_1) & p(\mu_2) & -p(\bar{\mu}_2) \\ -\bar{p}(\mu_1) & \bar{p}(\bar{\mu}_1) & -\bar{p}(\mu_2) & \bar{p}(\bar{\mu}_2) \end{bmatrix} \begin{Bmatrix} A_1 \\ \bar{A}_1 \\ A_2 \\ \bar{A}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ b/2\pi i \\ -\bar{b}/2\pi i \end{Bmatrix}, \tag{A6a}$$

where

$$p(\mu_k) = (\beta_{12} - \beta_{16}\mu_k + \beta_{11}\mu_k^2) + i(\beta_{22} - \beta_{26}\mu_k + \beta_{12}\mu_k^2)/\mu_k \quad \text{and} \quad \bar{p}(\mu_k) = \overline{p(\bar{\mu}_k)}. \tag{A6b}$$

Therefore, if we denote by A_r the solution to (A6) for $b = 1$ and by A_i the solution to (A6) for $b = i$, then from (A5), for $b = 1$,

$$A_1 = A_{11} + A_{12} = A_r(1); \quad A_2 = A_{21} + A_{22} = A_r(3)$$

and for $b = i$,

$$A_1 = A_{11}i - A_{12}i = A_i(1); \quad A_2 = A_{21}i - A_{22}i = A_i(3)$$

and these four equations can be solved for A_{ij}, $i, j = 1, 2$. For example,

$$A_{11} = [A_r(1) - iA_i(1)]/2. \tag{A7}$$