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Elasticity Solutions for Sandwich Orthotropic Cylindrical Shells Under External/Internal Pressure or Axial Force

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The elasticity solution is constructed for a cylindrical sandwich shell under external and/or internal pressure and for the same shell under axial load. The solution is an extension of the one for a homogeneous, monolithic shell and is provided in closed form. All three phases, that is, the two face sheets and the core, are assumed to be orthotropic. Moreover, there are no restrictions as far as the individual thicknesses of the face sheets and the sandwich construction may be asymmetric. These solutions can be used as benchmarks for assessing the performance of various sandwich shell theories. Illustrative results are provided in comparison to the sandwich shell theory.

Introduction

THE need for lightweight yet stiff and durable aerospace structures has made the sandwich composite configuration a leading-edge technology with promise for innovative high-performance structural designs. Sandwich construction is indeed increasingly employed in sections of rotorcraft and fixed-wing aircraft fuselages. A typical sandwich structure is composed of two thin composite laminated faces and a thick soft core made of foam or low-strength honeycomb.

The majority of the literature in sandwich composites is on the plate configuration.[1,2] Among the smaller number of studies on the sandwich shell geometry, we mention the papers by Reissner,[3] Bienek and Freudenthal,[4] Baker and Herrmann,[5] Kollar,[6] Greenberg et al.,[7] Birman and Simonyan,[8] and Frostig.[9] Sandwich shell theories are extensions of the well-known shell theories for monolithic structures such as Reissner's[3] or Love's[10] or Sanders's[11,12] shell theory, with a set of additional assumptions imposed, usually that the core carries only shear stresses and that the face sheets carry the normal stresses. Therefore, the extensional and bending stiffnesses of the shell are calculated exclusively from the face-sheet stiffnesses, whereas the transverse shear stress resultants are based exclusively on the shear stiffnesses of the core.

Elasticity solutions are significant because they provide a benchmark for assessing the performance of the various shell theories. To this extent, the geometry of a circular cylindrical shell is particularly attractive for constructing elasticity solutions due to the axisymmetry that simplifies the analysis. Elasticity solutions for monolithic orthotropic cylindrical shells have been provided by Lekhnitskii.[13] However, elasticity solutions for sandwich cylindrical shell configurations are essentially nonexistant. Like the sandwich shell theories, elasticity solutions for sandwich shells can be obtained by properly extending the solutions for monolithic structures. (This implies, among others, enforcing the proper conditions at the interface of the constituent phases, that is, face sheets and core.)

In this paper we formulate the elasticity solution for a circular cylindrical sandwich shell, first under external and/or internal pressure and then under axial loading, where the face sheets and the core are all orthotropic. No restrictions are imposed as far as the individual thicknesses of the face sheets, and the sandwich construction may be asymmetric. A generalized plane deformation assumption is made for the case of external and/or internal pressure, that is, the shell is assumed essentially to have infinite length. Then, not only the stresses, but also the displacements do not depend on the axial coordinate. For the case of axial loading, we assume that the axial force at the ends is applied so that a uniformly distributed constant axial strain exists throughout the section. The solutions are derived in the form of directly applicable expressions. Subsequently, the resulting stress distribution is compared with the one from sandwich shell theory.

Formulation

We consider the elastic equilibrium of a body in the form of a hollow round cylinder (a tube) of sandwich construction that consists of two face sheets and a core (Fig. 1). All three phases are made from a material with cylindrical orthotropy. The body is under the influence of stresses distributed along the lateral surface and on the ends. Let us assume that 1) the axis of orthotropy coincides with the geometric axis of the body; 2) there are planes of elastic symmetry normal to the axis of the cylinder; 3) the stresses acting on the outer and inner surfaces are normal and distributed uniformly; and 4) the stresses that act on the end surfaces reduce to forces that are directed along the axis and to twisting moments. We denote the thickness of the outer face sheet by f_1, that of the inner face sheet by f_2, and that of the core by c. The inner radius is a and the outer b where $b = a + f_2 + c + f_1$.

Let us denote each phase by i where $i = f_1$ for the outer face sheet, $i = c$ for the core, and $i = f_2$ for the inner face sheet. Then, for each phase, the orthotropic strain–stress relations are

$$\begin{bmatrix} \epsilon_{rr}^{(i)} \\ \epsilon_{\theta\theta}^{(i)} \\ \epsilon_{zz}^{(i)} \\ \gamma_{\theta z}^{(i)} \\ \gamma_{rz}^{(i)} \\ \gamma_{r\theta}^{(i)} \end{bmatrix} = \begin{bmatrix} a_{11}^i & a_{12}^i & a_{13}^i & 0 & 0 & 0 \\ a_{12}^i & a_{22}^i & a_{23}^i & 0 & 0 & 0 \\ a_{13}^i & a_{23}^i & a_{33}^i & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44}^i & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55}^i & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66}^i \end{bmatrix} \begin{bmatrix} \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \sigma_{zz}^{(i)} \\ \tau_{\theta z}^{(i)} \\ \tau_{rz}^{(i)} \\ \tau_{r\theta}^{(i)} \end{bmatrix}$$

$$i = f_1, c, f_2 \quad (1)$$

where a_{ij}^i are the compliance constants. (We have used the notation $1 \equiv r$, $2 \equiv \theta$, and $3 \equiv z$.)

We have taken the axis of the body as the z axis of the cylindrical coordinate system, and let us direct the polar x axis arbitrarily in the plane of one of the end sections. The following notations are introduced: a and b are the inner and outer radii; p and q are the inner and outer pressures per unit area, respectively; P is the axial force; M is the twisting moment.

An elasticity solution for a monolithic, homogeneous, orthotropic body has been provided by Lekhnitskii.[13] This solution will be used in the formulation of the corresponding one for the sandwich shell. Let us introduce the following notation for constants that enter into the stress formulas and depend on the elastic properties:

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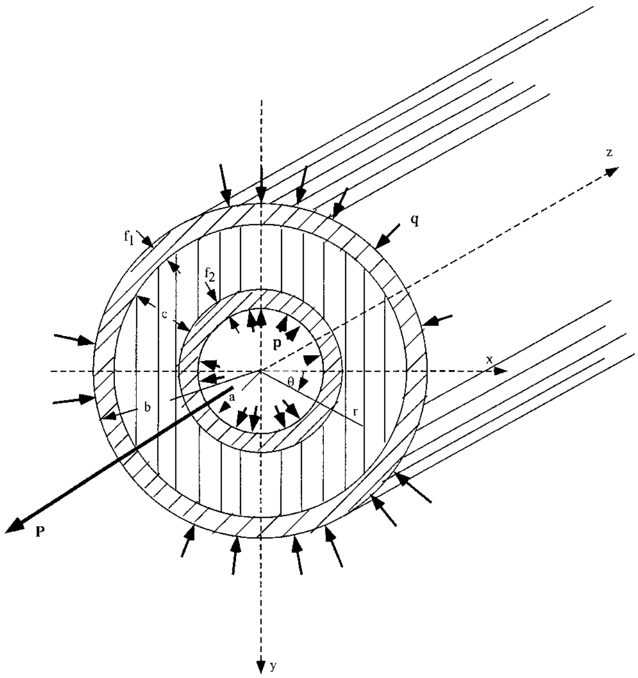

Fig. 1 Cylindrical sandwich shell.

$$\beta_{11}^i = a_{11}^i - a_{13}^{i2}/a_{33}^i, \qquad \beta_{22}^i = a_{22}^i - a_{23}^{i2}/a_{33}^i$$
$$i = f_1,\ c,\ f_2 \qquad (2a)$$

$$\beta_{12}^i = a_{12}^i - a_{13}^i a_{23}^i / a_{33}^i, \qquad i = f_1,\ c,\ f_2 \qquad (2b)$$

$$k_i = \sqrt{\beta_{11}^i/\beta_{22}^i}, \qquad \mu_i = 1/a_{44}^i$$

$$\xi_i = \left(a_{13}^i - a_{23}^i\right)/\left(\beta_{22}^i - \beta_{11}^i\right), \qquad i = f_1,\ c,\ f_2 \qquad (3)$$

One remark: in the case of isotropy ($a_{13}^i = a_{23}^i$ and $\beta_{22}^i = \beta_{11}^i$), we should take ξ to be zero, and all of the formulas in this paper will still be valid.

If we assume that the distribution of the external stresses is identical in all of the cross sections and, in addition, that the stresses depend only on the distance r from the axis, then Lekhnitskii[13] provides a solution for the stresses in each of the orthotropic phases in terms of two stress functions, $F^{(i)}(r)$ and $\Psi^{(i)}(r)$, $i = f_1, c, f_2$, so that

$$\sigma_{rr}^{(i)}(r) = \ F^{(i)\prime}(r)]/r, \qquad \sigma_{\theta\theta}^{(i)}(r) = F^{(i)\prime\prime}(r), \qquad i = f_1, c, f_2 \qquad (4a)$$

$$\tau_{r\theta}^{(i)} = 0, \qquad \tau_{rz}^{(i)} = 0, \qquad \tau_{\theta z}^{(i)} = -\Psi^{(i)\prime}(r), \qquad i = f_1, c, f_2 \qquad (4b)$$

$$\sigma_{zz}^{(i)} = C^{(i)} - (1/a_{33})\left[a_{13}^i\sigma_{rr}^{(i)} + a_{23}^i\sigma_{\theta\theta}^{(i)}\right], \qquad i = f_1, c, f_2 \qquad (4c)$$

Under the aforementioned assumptions, the equations of equilibrium and the condition that the displacements are single-valued functions of the coordinates will be satisfied if

$$F^{(i)}(r) = (C^{(i)}/2)\xi_i r^2 + \left[C_2^{(i)}/(1+k_i)\right]r^{1+k_i}$$
$$+ \left[C_3^{(i)}/(1-k_i)\right]r^{1-k_i}, \qquad i = f_1, c, f_2 \qquad (5a)$$

$$\Psi^{(i)}(r) = -\left(\bar{\theta}^{(i)}\mu_i/2\right)r^2, \qquad i = f_1, c, f_2 \qquad (5b)$$

The constants $C^{(i)}$, $C_2^{(i)}$, $C_3^{(i)}$, and $\bar{\theta}^{(i)}$ are found from the conditions on the cylindrical lateral surfaces, for example, applied uniform internal and/or external pressure, and the conditions on the ends, for example, applied axial load or axial strain or twisting moment. From Eqs. (4), the stresses in each of the phases are

$$\sigma_{rr}^{(i)}(r) = C^{(i)}\xi_i + C_2^{(i)}r^{k_i-1} + C_3^{(i)}r^{-k_i-1}, \qquad i = f_1, c, f_2 \qquad (6a)$$

$$\sigma_{\theta\theta}^{(i)}(r) = C^{(i)}\xi_i + C_2^{(i)}k_i r^{k_i-1} - C_3^{(i)}k_i r^{-k_i-1}, \qquad i = f_1, c, f_2 \qquad (6b)$$

$$\tau_{\theta z}^{(i)}(r) = \bar{\theta}^{(i)}\mu_i r, \qquad i = f_1, c, f_2 \qquad (6c)$$

$$\sigma_{zz}^{(i)}(r) = C^{(i)}\left[1 - \frac{\left(a_{13}^i + a_{23}^i\right)}{a_{33}^i}\xi_i\right] - C_2^{(i)}\frac{\left(a_{13}^i + a_{23}^i k_i\right)}{a_{33}^i}r^{k_i-1}$$
$$- C_3^{(i)}\frac{\left(a_{13}^i - a_{23}^i k_i\right)}{a_{33}^i}r^{-k_i-1}, \qquad i = f_1, c, f_2 \qquad (6d)$$

Denoting by $u_r^{(i)}$, $u_\theta^{(i)}$, and $w^{(i)}$ the displacements in the radial, circumferential, and axial directions, respectively, the displacement field excluding rigid body translation and rotation for this case is given as

$$u_r^{(i)}(r,z) = U^{(i)}(r), \qquad u_\theta^{(i)}(r,z) = \bar{\theta}^{(i)}zr + V^{(i)}(r)$$
$$w^{(i)}(r,z) = C^{(i)}a_{33}^i z + W^{(i)}(r) \qquad (7a)$$

where $U^{(i)}$, $V^{(i)}$, and $W^{(i)}$ are found from the strain–displacement relations and the stress field, Eqs. (4), from the following:

$$\frac{\partial U^{(i)}}{\partial r} = \beta_{11}^i\sigma_{rr}^{(i)} + \beta_{12}^i\sigma_{\theta\theta}^{(i)} + a_{13}^i C^{(i)} \qquad (7b)$$

$$\frac{1}{r}\frac{\partial V^{(i)}}{\partial \theta} + \frac{U^{(i)}}{r} = \beta_{12}^i\sigma_{rr}^{(i)} + \beta_{22}^i\sigma_{\theta\theta}^{(i)} + a_{23}^i C^{(i)} \qquad (7c)$$

$$\frac{1}{r}\frac{\partial U^{(i)}}{\partial \theta} + \frac{\partial V^{(i)}}{\partial r} - \frac{V^{(i)}}{r} = 0, \qquad \frac{\partial W^{(i)}}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial W^{(i)}}{\partial \theta} = 0 \qquad (7d)$$

Therefore, with the definitions (3) for k_i and ξ_i, the displacement field that satisfies these equations and would result in strains is found by integrating Eqs. (7):

$$U^{(i)}(r) = C^{(i)}\left[a_{13}^i + \xi_i\left(\beta_{11}^i + \beta_{12}^i\right)\right]r + C_2^{(i)}\left[\left(\beta_{11}^i + k_i\beta_{12}^i\right)/k_i\right]r^{k_i}$$
$$- C_3^{(i)}\left[\left(\beta_{11}^i - k_i\beta_{12}^i\right)/k_i\right]r^{-k_i} \qquad (8a)$$

$$V^{(i)}(r) = 0, \qquad W^{(i)}(r) = 0 \qquad (8b)$$

Next, we consider the case of a sandwich orthotropic cylinder under external and/or internal pressures and, to enable solving the problem directly, we make the additional assumption that a state of generalized plane deformation exists.

Generalized Plane Deformation of a Sandwich Orthotropic Tube Subjected to Internal and/or External Pressures

Let us assume that the sandwich cylinder considered in the preceding section is deformed by the pressures p and q distributed uniformly on the inner and outer surfaces, respectively, and has infinite length (generalized plane deformation assumption). Then, not only the stresses, but also the displacements, do not depend on z. Alternatively, this is the assumption we would make if the cylinder were securely fixed at the ends. Consequently, we can assume

$$C^{(i)} = \bar{\theta}^{(i)} = 0 \qquad (9)$$

The traction continuity conditions at the core/face sheet interfaces give

$$\sigma_{rr}^{(f2)}\Big|_{r=a+f_2} = \sigma_{rr}^{(c)}\Big|_{r=a+f_2}, \qquad \sigma_{rr}^{(c)}\Big|_{r=b-f_1} = \sigma_{rr}^{(f1)}\Big|_{r=b-f_1} \qquad (10)$$

Applying Eqs. (6) and (9), this gives

$$C_2^{(f2)}(a+f_2)^{k_{f2}-1}+C_3^{(f2)}(a+f_2)^{-k_{f2}-1} = C_2^{(c)}(a+f_2)^{k_c-1}+C_3^{(c)}(a+f_2)^{-k_c-1} \tag{11a}$$

$$C_2^{(c)}(b-f_1)^{k_c-1}+C_3^{(c)}(b-f_1)^{-k_c-1} = C_2^{(f1)}(b-f_1)^{k_{f1}-1}+C_3^{(f1)}(b-f_1)^{-k_{f1}-1} \tag{11b}$$

The displacement continuity at the core/face sheet interfaces is, in turn,

$$U^{(f2)}\Big|_{r=a+f_2}=U^{(c)}\Big|_{r=a+f_2}, \qquad U^{(c)}\Big|_{r=b-f_1}=U^{(f1)}\Big|_{r=b-f_1} \tag{12}$$

which, by use of Eqs. (8a) and (9), gives

$$C_2^{(f_2)}\frac{\left(\beta_{11}^{f_2}+k_{f2}\beta_{12}^{f_2}\right)}{k_{f2}}(a+f_2)^{k_{f2}}-C_3^{(f_2)}\frac{\left(\beta_{11}^{f_2}-k_{f2}\beta_{12}^{f_2}\right)}{k_{f2}}(a+f_2)^{-k_{f2}}$$

$$=C_2^{(c)}\frac{\left(\beta_{11}^{c}+k_c\beta_{12}^{c}\right)}{k_c}(a+f_2)^{k_c} - C_3^{(c)}\frac{\left(\beta_{11}^{c}-k_c\beta_{12}^{c}\right)}{k_c}(a+f_2)^{-k_c} \tag{13a}$$

$$C_2^{(c)}\frac{\left(\beta_{11}^{c}+k_c\beta_{12}^{c}\right)}{k_c}(b-f_1)^{k_c}-C_3^{(c)}\frac{\left(\beta_{11}^{c}-k_c\beta_{12}^{c}\right)}{k_c}(b-f_1)^{-k_c}$$

$$=C_2^{(f_1)}\frac{\left(\beta_{11}^{f_1}+k_{f1}\beta_{12}^{f_1}\right)}{k_{f1}}(b-f_1)^{k_{f1}} - C_3^{(f_1)}\frac{\left(\beta_{11}^{f_1}-k_{f1}\beta_{12}^{f_1}\right)}{k_{f1}}(b-f_1)^{-k_{f1}} \tag{13b}$$

The conditions of applied internal and external pressure at the bounding surfaces are

$$\sigma_{rr}^{(f2)}\Big|_{r=a}=-p, \qquad \sigma_{rr}^{(c)}\Big|_{r=b}=-q \tag{14}$$

which gives

$$C_2^{(f2)}a^{k_{f2}-1}+C_3^{(f2)}a^{-k_{f2}-1}=-p$$

$$C_2^{(f1)}b^{k_{f1}-1}+C_3^{(f1)}b^{-k_{f1}-1}=-q \tag{15}$$

The six unknowns $C_2^{(i)}$, $C_3^{(i)}$, $i=f_1, c, f_2$, are found from solving this system of six linear equations, namely, Eqs. (11a), (11b), (13a), (13b), and (15), in terms of p and q. Then the stresses are found by Eqs. (6).

Because there is no shear stress $\tau_{\theta z}$, there is no resultant twisting moment. The stresses σ_{zz} on the ends and at any cross section reduce to an axial force P, which can be found from

$$\frac{P}{2\pi}=\int_a^b \sigma_{zz} r\,\mathrm{d}r=\int_a^{a+f_2}\sigma_{zz}^{(f2)} r\,\mathrm{d}r+\int_{a+f_2}^{b-f_1}\sigma_{zz}^{(c)} r\,\mathrm{d}r + \int_{b-f_1}^{b}\sigma_{zz}^{(f1)} r\,\mathrm{d}r \tag{16}$$

Using Eq. (6d), this becomes

$$P/2\pi=-(g_2+g_3) \tag{17a}$$

where

$$g_2=C_2^{(f_1)}\frac{\left(a_{13}^{f_1}+a_{23}^{f_1}k_{f1}\right)}{a_{33}^{f_1}(k_{f1}+1)}\left[b^{(k_{f1}+1)}-(b-f_1)^{(k_{f1}+1)}\right] + C_2^{(c)}\frac{\left(a_{13}^{c}+a_{23}^{c}k_c\right)}{a_{33}^{c}(k_c+1)}\left[(b-f_1)^{(k_c+1)}-(a+f_2)^{(k_c+1)}\right] + C_2^{(f_2)}\frac{\left(a_{13}^{f_2}+a_{23}^{f_2}k_{f2}\right)}{a_{33}^{f_2}(k_{f2}+1)}\left[(a+f_2)^{(k_{f2}+1)}-a^{(k_{f2}+1)}\right] \tag{17b}$$

$$g_3=C_3^{(f_1)}\frac{\left(a_{13}^{f_1}-a_{23}^{f_1}k_{f1}\right)}{a_{33}^{f_1}(-k_{f1}+1)}\left[b^{(-k_{f1}+1)}-(b-f_1)^{(-k_{f1}+1)}\right] + C_3^{(c)}\frac{\left(a_{13}^{c}-a_{23}^{c}k_c\right)}{a_{33}^{c}(-k_c+1)}\left[(b-f_1)^{(-k_c+1)}-(a+f_2)^{(-k_c+1)}\right] + C_3^{(f_2)}\frac{\left(a_{13}^{f_2}-a_{23}^{f_2}k_{f2}\right)}{a_{33}^{f_2}(-k_{f2}+1)}\left[(a+f_2)^{(-k_{f2}+1)}-a^{(-k_{f2}+1)}\right] \tag{17c}$$

Of course, the stress σ_{zz} is nonuniformly distributed over the cross section, but a uniform (zero) axial strain exists.

Sandwich Shell Theory: External or Internal Pressure

We are referring to a coordinate system z, θ, and r, in which z and θ are in the axial and circumferential directions, and r is in the (radial) direction of the outward normal to the middle surface. The corresponding displacements at any point are denoted by w, v, and u.

In addition to Eq. (1), which is in terms of the compliance constants, we shall use the stress–strain relations in terms of the stiffness constants, as follows:

$$\begin{bmatrix}\sigma_{rr}^{(i)}\\ \sigma_{\theta\theta}^{(i)}\\ \sigma_{zz}^{(i)}\\ \tau_{\theta z}^{(i)}\\ \tau_{rz}^{(i)}\\ \tau_{r\theta}\end{bmatrix}=\begin{bmatrix}c_{11}^i & c_{12}^i & c_{13}^i & 0 & 0 & 0\\ c_{12}^i & c_{22}^i & c_{23}^i & 0 & 0 & 0\\ c_{13}^i & c_{23}^i & c_{33}^i & 0 & 0 & 0\\ 0&0&0&c_{44}^i&0&0\\ 0&0&0&0&c_{55}^i&0\\ 0&0&0&0&0&c_{66}^i\end{bmatrix}\begin{bmatrix}\epsilon_{rr}^{(i)}\\ \epsilon_{\theta\theta}^{(i)}\\ \epsilon_{zz}^{(i)}\\ \gamma_{\theta z}^{(i)}\\ \gamma_{rz}^{(i)}\\ \gamma_{r\theta}^{(i)}\end{bmatrix}$$

$$i=f_1, c, f_2 \tag{18}$$

where again we have used the notation $1\equiv r$, $2\equiv\theta$, and $3\equiv z$.

The sandwich shell theory employed is described by Birman and Simonyan[8] and is a version of Love's[10] shell theory extended to shear deformable structures (but note the absence of shear in this case of orthotropy). The core carries only shear stresses and the face sheets carry the normal stresses; therefore, the extensional and bending stiffnesses of the shell are based exclusively on the face-sheet stiffnesses. On the contrary, transverse shear stress resultants (should they exist) are based exclusively on the shear stiffnesses of the core.

Taking into account the displacement distribution through the thickness assumed in the shell theory,[8] we can easily see that in the generalized plane deformation problems under consideration, the displacement field throughout the shell is

$$u(r,\theta,z)=u_0, \qquad v(r,\theta,z)=0, \qquad w(r,\theta,z)=\epsilon_0 z \tag{19a}$$

where u_0 is a constant, and ϵ_0 is the uniform axial strain.

The relationships for the strains throughout the shell, corresponding to Love's[10] shell theory are

$$\epsilon_{rr}=0, \qquad \epsilon_{\theta\theta}=u_0/R, \qquad \epsilon_{zz}=\epsilon_0 \tag{19b}$$

where R is the midsurface radius. The shear strains are all zero. Notice that in these simplified, axisymmetric, generalized plane defor-mantion problems, there is no difference between first-order shear deformation and classical solutions.

The stress resultants of interest are

$$N_\theta = C_{22}\epsilon^0_{\theta\theta} + C_{23}\epsilon^0_{zz}, \qquad N_z = C_{23}\epsilon^0_{\theta\theta} + C_{33}\epsilon^0_{zz}, \qquad N_{z\theta} = 0 \tag{19c}$$

where ϵ^0_{ij} are the midsurface strains, identical to the ones in Eq. (19b). Moreover, the C_{ij} are the shell stiffness constants, governed by the face sheets (in the context of sandwich shell formulation) by

$$C_{ij} = f_1 c^{f1}_{ij} + f_2 c^{f2}_{ij}, \qquad i, j = 2, 3 \tag{19d}$$

For external pressure, the equilibrium equations in terms of the stress resultants (see, for example, Brush and Almroth[14]) are satisfied if

$$N_\theta = -pR \tag{20a}$$

Furthermore, based on the assumptions of the problem for the external pressure case, $\epsilon_0 = 0$. Then Eqs. (19c) and (19b) give

$$u_0 = -pR^2 \big/ C_{22}, \qquad \epsilon_{\theta\theta} = -pR/C_{22} \tag{20b}$$

Subsequently, by using Eq. (18), the stresses are

$$\sigma_{rr} = -p\left(c^i_{12}R\big/C_{22}\right), \qquad \sigma_{\theta\theta} = -p\left(c^i_{22}R\big/C_{22}\right)$$

$$\sigma_{zz} = -p\left(c^i_{23}R\big/C_{22}\right), \qquad i = f_2, c, f_1 \tag{20c}$$

which would, therefore, undergo a finite change at the face sheet/core interfaces.

Results: External or Internal Pressure

As an illustrative example, the stress and displacement distribution was determined for a sandwich composite circular cylindrical shell of outer radius $b = 1$ m, a ratio of outside over inside radii $b/a = 1.20$, and ratios of face-sheet thicknesses over shell thickness $f_2/h = f_1/h = 0.10$.

The face sheets are made from unidirectional E-glass/polyester with the fiber direction along the circumference, with properties

$$E^{(f1,f2)}_2 = 40 \text{ GPa}, \qquad E^{(f1,f2)}_1 = E^{(f1,f2)}_3 = 10 \text{ GPa}$$

$$G^{(f1,f2)}_{31} = 3.5 \text{ GPa}, \qquad G^{(f1,f2)}_{12} = G^{(f1,f2)}_{23} = 4.5 \text{ GPa}$$

$$\nu^{(f1,f2)}_{31} = 0.40, \qquad \nu^{(f1,f2)}_{21} = \nu^{(f1,f2)}_{23} = 0.26$$

Note that 1 is the radial r, 2 is the circumferential θ, and 3 is the axial z direction.

The core is assumed to be made from cross-linked polyvinyl chloride (PVC) foam, which is isotropic with

$$E^c = 75 \text{ MPa}, \qquad \nu^c = 0.30$$

Notice that by referring to Eq. (1), the compliance constants for each orthotropic phase are

$$a_{11} = 1/E_1, \qquad a_{22} = 1/E_2, \qquad a_{33} = 1/E_3$$

$$a_{44} = 1/G_{23}, \qquad a_{55} = 1/G_{31}, \qquad a_{66} = 1/G_{12}$$

$$a_{12} = -\nu_{21}/E_2, \qquad a_{13} = -\nu_{31}/E_3, \qquad a_{23} = -\nu_{32}/E_3$$

For the case of pure external pressure q, Fig. 2 shows the radial displacement $U(r)$, normalized with qR^2/C_{22}, where C_{22} is defined in Eq. (19d). The elasticity solution predicts a nonumiform displacement as opposed to the shell theory. Figures 3a–3c show the stresses σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz}, normalized with the external pressure q. There are noticable differences between the elasticity and the shell theory predictions. A higher magnitude σ_{rr} is predicted in the shell theory at the face-sheet regions. The hoop stress $\sigma_{\theta\theta}$ at the outer bounding surface is predicted higher in the elasticity solution, whereas $\sigma_{\theta\theta}$ at the inner bounding surface is predicted higher in the shell theory. The shell theory is also predicting a higher axial stress σ_{zz} at the face-sheet regions.

Table 1 Elasticity vs shell for pure external pressure

b/a	$\sigma_{\theta\theta}(a)$ elast / $\sigma_{\theta\theta}(a)$ shell	$\sigma_{\theta\theta}(b)$ elast / $\sigma_{\theta\theta}(b)$ shell	$U(a)$ elast / $U(a)$ shell	$U(b)$ elast / $U(b)$ shell
1.05	1.002	1.014	1.018	1.078
1.10	0.947	1.100	0.940	1.193
1.15	0.859	1.220	0.833	1.348
1.20	0.764	1.348	0.723	1.517

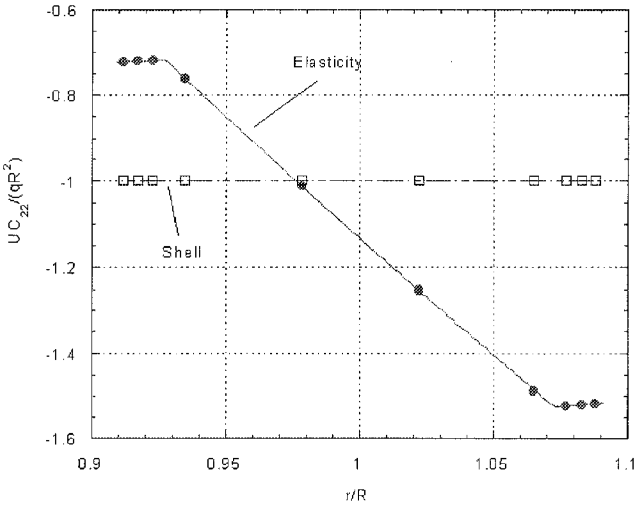

Fig. 2 Radial displacement $U(r)$ for a cylindrical sandwich shell of mean radius R under uniform external pressure q.

These differences are, of course, expected to become smaller as the shell becomes thinner, that is, as the ratio b/a tends closer to unity. This is clearly shown in Table 1, which gives the ratio of the elasticity vs shell theory predictions for $\sigma_{\theta\theta}$ and U at the inner and outer bounding surfaces, that is, at $r = a$ and b, for a range of thicknesses, as expressed by the ratio b/a (case of pure external pressure). In Table 1, the outside radius b and the relative thicknesses of the face sheets, as expressed by the ratios f_1/h and f_2/h, as well as the material properties for the face sheets and the core, have remained the same.

Next, we consider the case of an applied axial force.

Orthotropic Sandwich Hollow Cylinder Loaded by an Axial Force

We now assume that the shell is deformed only by stresses distributed on the ends and these reduce to a tensile force P (Fig. 1). The force at the ends is applied so that a uniformly distributed constant axial strain ϵ_0 exists throughout the section. Note also that no resultant twisting moment is assumed to exist and that $\bar{\theta}^i = 0$.

Because from Eq. (7a) the axial strain is $C^{(i)}a^i_{33}$, the first condition is

$$C^{(f2)}a^{f2}_{33} = C^{(c)}a^c_{33} = C^{(f1)}a^{f1}_{33} = \epsilon_0 \tag{21}$$

that is, the constants $C^{(i)}$ are now nonzero.

Next, the traction conditions (10) at the face sheet/core interfaces give, by use of Eqs. (6a) and (21)

$$\epsilon_0\left(\xi_{f2}\big/a^{f2}_{33}\right) + C^{(f2)}_2(a+f_2)^{k_{f2}-1} + C^{(f2)}_3(a+f_2)^{-k_{f2}-1}$$

$$= \epsilon_0\left(\xi_c\big/a^c_{33}\right) + C^{(c)}_2(a+f_2)^{k_c-1} + C^{(c)}_3(a+f_2)^{-k_c-1} \tag{22a}$$

$$\epsilon_0\left(\xi_c\big/a^c_{33}\right) + C^{(c)}_2(b-f_1)^{k_c-1} + C^{(c)}_3(b-f_1)^{-k_c-1}$$

$$= \epsilon_0\left(\xi_{f1}\big/a^{f1}_{33}\right) + C^{(f1)}_2(b-f_1)^{k_{f1}-1} + C^{(f1)}_3(b-f_1)^{-k_{f1}-1} \tag{22b}$$

The displacement continuity at the face sheet/core interfaces, Eq. (12), becomes, by use of Eqs. (8a) and (21),

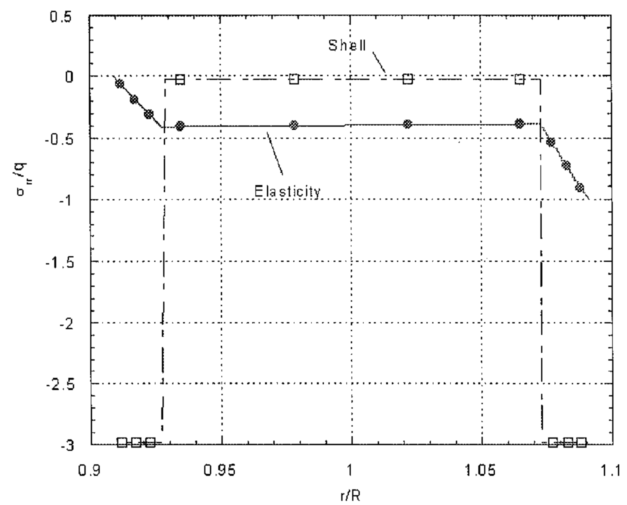

Fig. 3a Radial stress σ_{rr} for a cylindrical sandwich shell of mean radius R under uniform external pressure q.

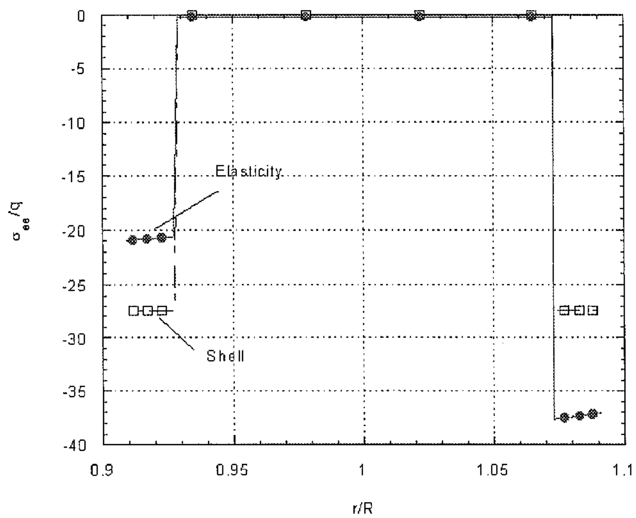

Fig. 3b Hoop stress $\sigma_{\theta\theta}$ for a cylindrical sandwich shell of mean radius R under uniform external pressure q.

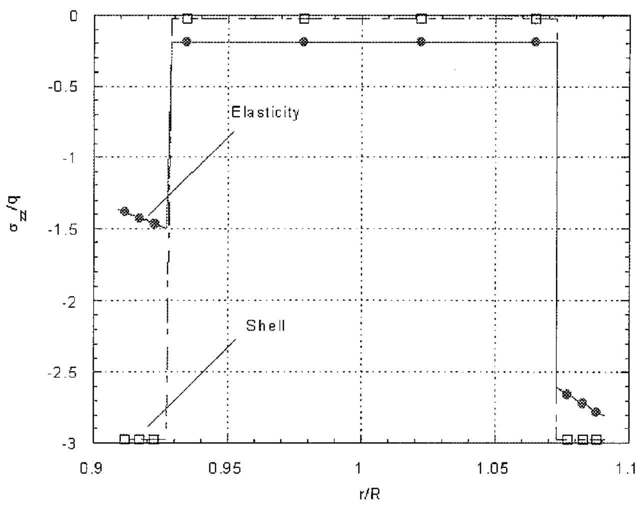

Fig. 3c Axial stress σ_{zz} for a cylindrical sandwich shell of mean radius R under uniform external pressure q.

$$\begin{aligned}
&\epsilon_0 \frac{\left[a_{13}^{f_2} + \xi_{f2}\left(\beta_{11}^{f_2} + \beta_{12}^{f_2}\right)\right]}{a_{33}^{f_2}}(a + f_2) \\
&\quad + C_2^{(f_2)} \frac{\left(\beta_{11}^{f_2} + k_{f2}\beta_{12}^{f_2}\right)}{k_{f2}}(a + f_2)^{k_{f2}} \\
&\quad - C_3^{(f_2)} \frac{\left(\beta_{11}^{f_2} - k_{f2}\beta_{12}^{f_2}\right)}{k_{f2}}(a + f_2)^{-k_{f2}} \\
&\quad = \epsilon_0 \frac{\left[a_{13}^{c} + \xi_{c}\left(\beta_{11}^{c} + \beta_{12}^{c}\right)\right]}{a_{33}^{c}}(a + f_2) \\
&\quad + C_2^{(c)} \frac{\left(\beta_{11}^{c} + k_{c}\beta_{12}^{c}\right)}{k_{c}}(a + f_2)^{k_{c}} \\
&\quad - C_3^{(c)} \frac{\left(\beta_{11}^{c} - k_{c}\beta_{12}^{c}\right)}{k_{c}}(a + f_2)^{-k_{c}} \qquad (23a)
\end{aligned}$$

$$\begin{aligned}
&\epsilon_0 \frac{\left[a_{13}^{c} + \xi_{c}\left(\beta_{11}^{c} + \beta_{12}^{c}\right)\right]}{a_{33}^{c}}(b - f_1) + C_2^{(c)} \frac{\left(\beta_{11}^{c} + k_{c}\beta_{12}^{c}\right)}{k_{c}}(b - f_1)^{k_{c}} \\
&\quad - C_3^{(c)} \frac{\left(\beta_{11}^{c} - k_{c}\beta_{12}^{c}\right)}{k_{c}}(b - f_1)^{-k_{c}} \\
&\quad = \epsilon_0 \frac{\left[a_{13}^{f_1} + \xi_{f1}\left(\beta_{11}^{f_1} + \beta_{12}^{f_1}\right)\right]}{a_{33}^{f_1}}(b - f_1) \\
&\quad + C_2^{(f_1)} \frac{\left(\beta_{11}^{f_1} + k_{f1}\beta_{12}^{f_1}\right)}{k_{f1}}(b - f_1)^{k_{f1}} \\
&\quad - C_3^{(f_1)} \frac{\left(\beta_{11}^{f_1} - k_{f1}\beta_{12}^{f_1}\right)}{k_{f1}}(b - f_1)^{-k_{f1}} \qquad (23b)
\end{aligned}$$

Next, the conditions of traction-free lateral surfaces

$$\sigma_{rr}^{(f2)}\big|_{r=a} = 0, \qquad \sigma_{rr}^{(f1)}\big|_{r=b} = 0 \qquad (24a)$$

give

$$\epsilon_0\left(\xi_{f2}/a_{33}^{f_2}\right) + C_2^{(f_2)}a^{k_{f2}-1} + C_3^{(f_2)}a^{-k_{f2}-1} = 0 \qquad (24b)$$

$$\epsilon_0\left(\xi_{f1}/a_{33}^{f_1}\right) + C_2^{(f_1)}b^{k_{f1}-1} + C_3^{(f_1)}b^{-k_{f1}-1} = 0 \qquad (24c)$$

Again, the solution is found by solving for the six constants $C_2^{(i)}, C_3^{(i)}$, $i = f_1, c, f_2$, in terms of ϵ_0, from the six linear equations (22a–23b), (24b), and (24c).

An expression for the resultant applied force P in terms of ϵ_0 can be found by intergrating σ_{zz} as in Eq. (16), and this now gives, by the use of Eq. (6d),

$$P/2\pi = -(g_1 + g_2 + g_3) \qquad (25a)$$

where g_2 and g_3 are given by Eqs. (17b) and (17c) and

$$\begin{aligned}
\frac{g_1}{\epsilon_0} &= \left[1 - \frac{\left(a_{13}^{f_1} + a_{23}^{f_1}\right)}{a_{33}^{f_1}}\xi_{f1}\right]\frac{\left[b^2 - (b - f_1)^2\right]}{2a_{33}^{f1}} \\
&\quad + \left[1 - \frac{\left(a_{13}^{c} + a_{23}^{c}\right)}{a_{33}^{c}}\xi_{c}\right]\frac{\left[(b - f_1)^2 - (a + f_2)^2\right]}{2a_{33}^{c}} \\
&\quad + \left[1 - \frac{\left(a_{13}^{f_2} + a_{23}^{f_2}\right)}{a_{33}^{f_2}}\xi_{f2}\right]\frac{\left[(a + f_2)^2 - a^2\right]}{2a_{33}^{f2}} \qquad (25b)
\end{aligned}$$

Of course, the axial stress σ_{zz} is distributed along the cross section nonuniformly as opposed to the axial strain ϵ_0, which was assumed to be uniform.

Sandwich Shell Theory: Axial Loading

For axial loading with a uniform axial strain ϵ_0, the equilibrium equations are satisfied if $N_\theta = 0$, which, by using Eqs. (22c) and (22b), gives

$$u_0 = -\epsilon_0 R C_{23}/C_{22}, \qquad \epsilon_{\theta\theta} = -\epsilon_0 C_{23}/C_{22} \tag{26a}$$

Subsequently, N_z can be obtained from Eq. (22c) as

$$N_z = \epsilon_0\left(C_{33} - C_{23}^2/C_{22}\right) \tag{26b}$$

Then the stress field is found by using Eqs. (22b) and (24):

$$\sigma_{rr} = \epsilon_0\left[c_{13}^i - c_{12}^i(C_{23}/C_{22})\right], \qquad \sigma_{\theta\theta} = \epsilon_0\left[c_{23}^i - c_{22}^i(C_{23}/C_{22})\right]$$

$$\sigma_{zz} = \epsilon_0\left[c_{33}^i - c_{23}^i\epsilon_0(C_{23}/C_{22})\right] \tag{26c}$$

that is, the stresses again undego a finite jump at the face sheet/core interfaces.

Results: Axial Loading

For the case of pure axial loading by a uniform applied axial strain ϵ_0, Fig. 4 gives the displacement $U(r)$, normalized with $\epsilon_0 R C_{23}/C_{22}$. Again, the elasticity solution predicts a nonuniform displacement distribution as opposed to the shell theory. Next, Figs. 5a–5c show the stresses σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz}, normalized with $\epsilon_0 C_{33}/(f_1 + f_2)$. There are again noticable differences between the elasticity and the shell theory predictions. Again, a higher σ_{rr} and σ_{zz} is predicted in the shell theory at the face-sheet regions; however, the magnitude of $\sigma_{\theta\theta}$ is predicted higher by the elasticity solution at both the outer and the inner bounding surface.

Also, note that because of the orthotropy and the axisymmetric geometry, there are no shear stresses generated from both loadings of internal/external pressure and axial loading; therefore, even a first-order shear deformation theory would not result in improved shell theory predictions. This would not be the case if the face sheets were fully anisotropic rather than orthotropic, even if the core were to remain isotropic.

Finally, note that the concept of sandwich construction may not be ideal for the loading and structure analyzed. This is because in the case considered there is no shear in the core, and, to really take advantage of the sandwich concept, the core should carry the shear and the face sheet should support the normal stresses. If buckling occurs, on the other hand, then the core would support the shear, and the solution presented can be used as the exact prebuckling state of stress and displacement in the formulation of the buckling problem.

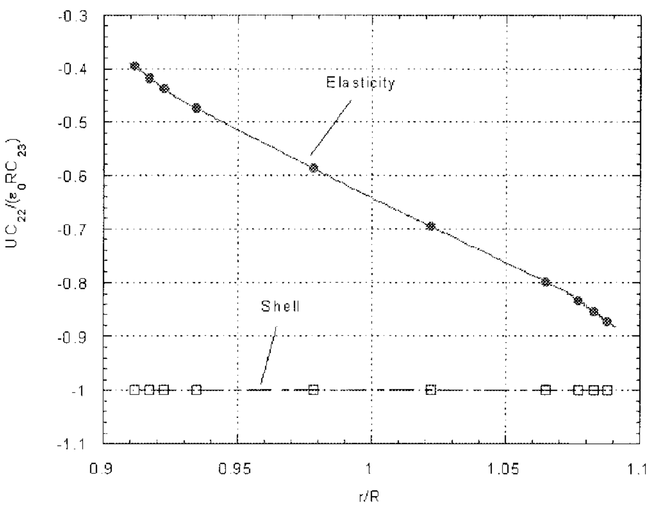

Fig. 4 Radial displacement $U(r)$ for a cylindrical sandwich shell of mean radius R under uniformly applied axial strain ϵ_0.

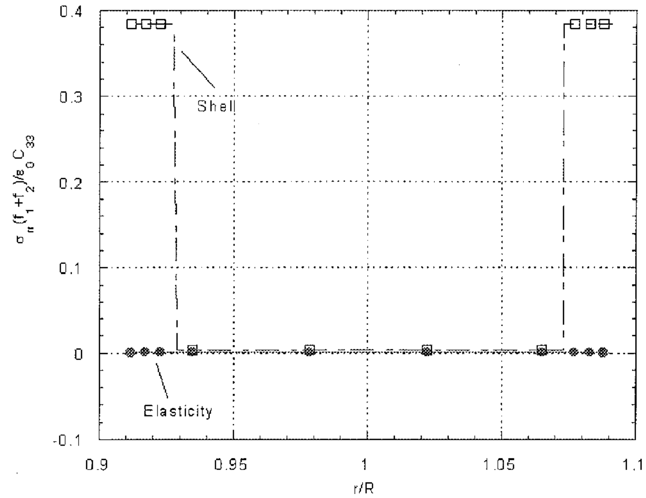

Fig. 5a Radial stress σ_{rr} for a cylindrical sandwich shell of mean radius R under uniformly applied axial strain ϵ_0.

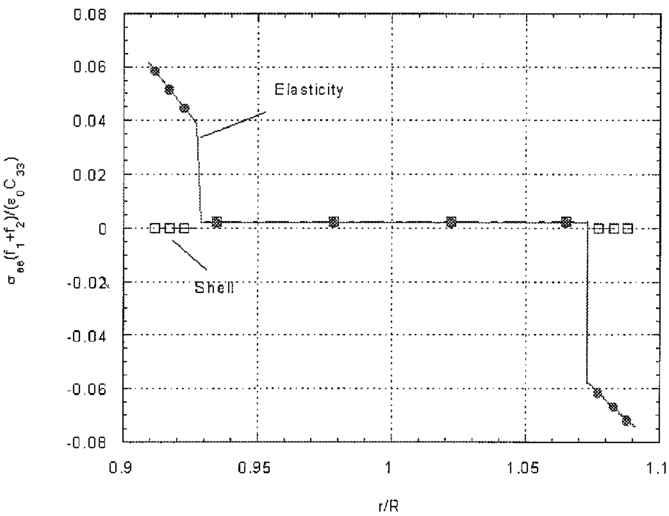

Fig. 5b Hoop stress $\sigma_{\theta\theta}$ for a cylindrical sandwich shell of mean radius R under uniformly applied axial strain ϵ_0.

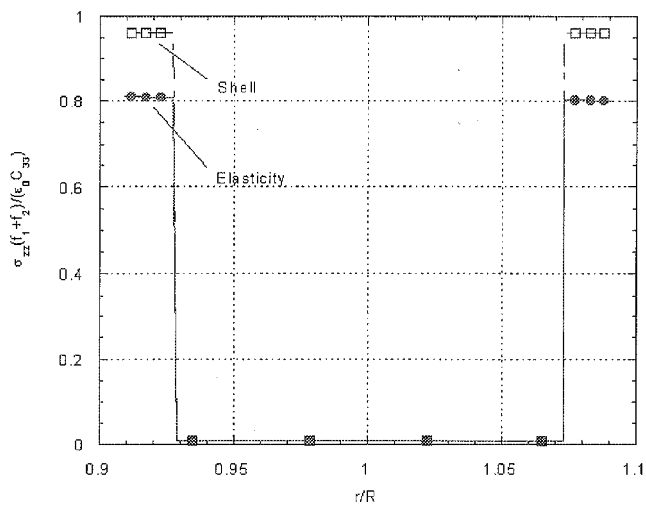

Fig. 5c Axial stress σ_{zz} for a cylindrical sandwich shell of mean radius R under uniformly applied axial strain ϵ_0.

Conclusions

A closed-form elasticity solution is constructed for a cylindrical sandwich shell under external or internal pressure or axial load, in which all three phases, i.e., the two face sheets and the core, are assumed to be orthotropic. A comparison of results for the stresses and displacements with sandwich shell theory predictions shows that differences can be quite noticeable. The present solutions can serve as benchmarks for assessing the performance of various sandwich shell theories.

Acknowledgments

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