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DISPLACEMENT FIELDS FOR MIXED MODE ELASTIC–PLASTIC CRACKS

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SHIH[1] EXTENDED the HRR[2, 3] singularity by giving the dominant singularity solutions governing the asymptotic stress and strain field of a stationary crack for the mixed Mode I and II case. In terms of a stress–strain law of the form $\sigma = \sigma_1\varepsilon^n$, with Mode I mixity parameter M^p, a far-field path independent integral J and the scalar function $I_{1/n}(M^p)$, the displacement and strain components at r, θ can be written[4]:

$$\frac{u_i(r,\theta)}{r} = \left[\frac{J}{\sigma_1 I_{1/n}(M^p)r}\right]^{1/(n+1)} \tilde{u}_i(\theta, 1/n, M^p), \tag{1}$$

$$\varepsilon_{ij}(r,\theta) = \left[\frac{J}{\sigma_1 I_{1/n}(M^p)r}\right]^{1/(n+1)} \tilde{\varepsilon}_{ij}(\theta, 1/n, M^p). \tag{2}$$

The displacement functions u_i are determined from the strain functions for the plane strain case considered here. The radial displacement u_r may be found from the radial strain,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \tag{3}$$

so:

$$u_r = \int_0^r \varepsilon_{rr}\,\mathrm{d}r + u_r(0,\theta). \tag{4}$$

For zero rigid-body translation at $r = 0$, $u_r(0,\theta) = 0$. Eliminating ε_{rr} with (2) and integrating (4) gives

$$\frac{u_r}{r} = \left[\frac{J}{\sigma_1 r I(n, M^p)}\right]^{1/(n+1)} \left(\frac{n+1}{n}\right)\tilde{\varepsilon}_{rr}. \tag{5}$$

Introducing the radial displacement from (1) gives the angular function for the radial displacement $\tilde{u}_r$:

$$\tilde{u}_r(\theta, M^p, n) = \left(\frac{n+1}{n}\right)\tilde{\varepsilon}_{rr}(\theta, M^p, n). \tag{6}$$

The tangential displacement u_θ is found by integrating the definition of $\varepsilon_{\theta\theta}$,

$$\varepsilon_{\theta\theta} = \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{7}$$

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to be

$$u_\theta = u_\theta(r, -\pi) + \int_{-\pi}^{\theta} (r\varepsilon_{\theta\theta} - u_r)\, \mathrm{d}\theta. \tag{8}$$

From (1), the θ-independent term $u_\theta(r, -\pi)$ must be $0(r^{n/(n+1)})$ and thus can be written in terms of a constant C:

$$u_\theta(r, -\pi) = \left[\frac{J}{\sigma_1 I(n, M^p)}\right]^{1/(n+1)} Cr^{n/(n+1)}. \tag{9}$$

The integral of eqn (8) can be converted to a single integral over $\tilde{\varepsilon}_{rr}(\theta)$ by noting $\varepsilon_{\theta\theta} = -\varepsilon_{rr}$ for incompressible plane strain, by using (2) for ε_{rr}, defining

$$F(\theta) = \int_{-\pi}^{\theta} \tilde{\varepsilon}_{rr}\, \mathrm{d}\theta, \tag{10}$$

and introducing (5) for u_r:

$$u_\theta = \left[\frac{J}{\sigma_1 rI(n, M^p)}\right]^{1/(n+1)} r^{n/(n+1)} \left[C - \left(\frac{2n+1}{n}\right) F(\theta)\right]. \tag{11}$$

To determine the constant C, substitute (11) for u_θ and (5) for u_r in the definition of $\varepsilon_{r\theta}$,

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right), \tag{12}$$

to get

$$C = \left(\frac{2n+1}{n}\right) F(\theta) + \frac{(n+1)^2}{n}\frac{\partial \tilde{\varepsilon}_{rr}}{\partial \theta} - 2(n+1)\tilde{\varepsilon}_{r\theta}. \tag{13}$$

Using (11) we can thus find the angular function for the tangential displacement, $\tilde{u}_\theta$, in terms of the angular strain functions $\tilde{\varepsilon}_{rr}$ and $\tilde{\varepsilon}_{r\theta}$:

$$\tilde{u}_\theta(\theta, M^p, n) = \frac{(n+1)^2}{n}\frac{\partial \tilde{\varepsilon}_{rr}}{\partial \theta} - 2(n+1)\tilde{\varepsilon}_{r\theta}. \tag{14}$$

Notice that the integral $F(\theta)$ does not enter into the expression for $\tilde{u}_\theta$. Thus finally

$$\frac{u_\theta}{r} = \left[\frac{J}{\sigma_1 I(n, M^p) r}\right]^{1/(n+1)} \left[\frac{(n+1)^2}{n}\frac{\partial \tilde{\varepsilon}_{rr}}{\partial \theta} - 2(n+1)\tilde{\varepsilon}_{r\theta}\right]. \tag{15}$$

Plots of the displacement functions, $\tilde{u}$, determined numerically by (6), (14) from the strain functions given in Ref. [1] for two strain hardening exponents, $n = 1/13$ and $n = 1/3$, are shown in the figures. It was also verified that these equations give the strain functions (2), when substituted into the definitions (3), (7) and (12), and by using the compatibility equation which, in this case, reduces to

$$\frac{\partial^2 \tilde{\varepsilon}_{rr}}{\partial \theta^2} - \frac{n}{(n+1)^2}\tilde{\varepsilon}_{\theta\theta} + \frac{1}{n+1}\tilde{\varepsilon}_{rr} - \frac{2n}{n+1}\frac{\partial \tilde{\varepsilon}_{r\theta}}{\partial \theta} = 0. \tag{16}$$

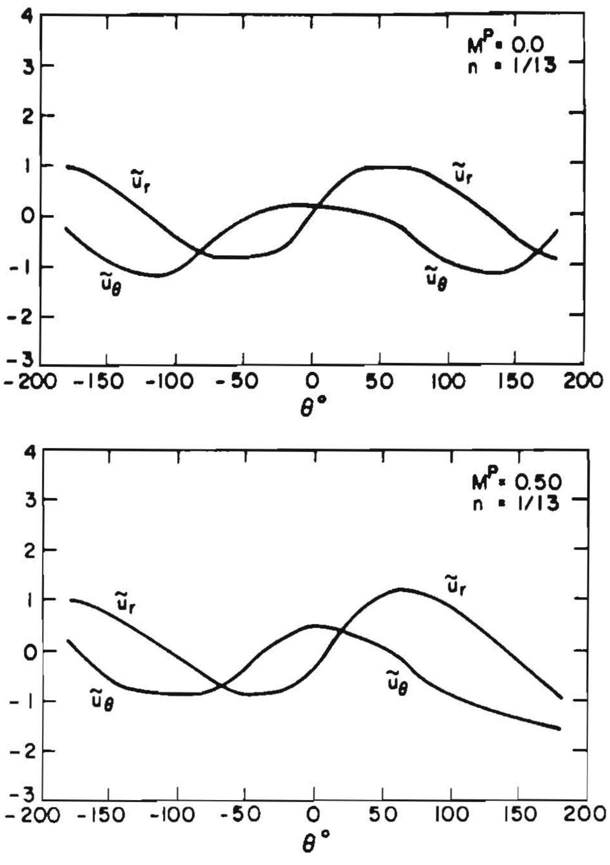

Fig. 1. θ-variations for the displacement functions at the tip of a crack for plane strain, $n = 1/13$, $M^p = 0.0$, 0.50.

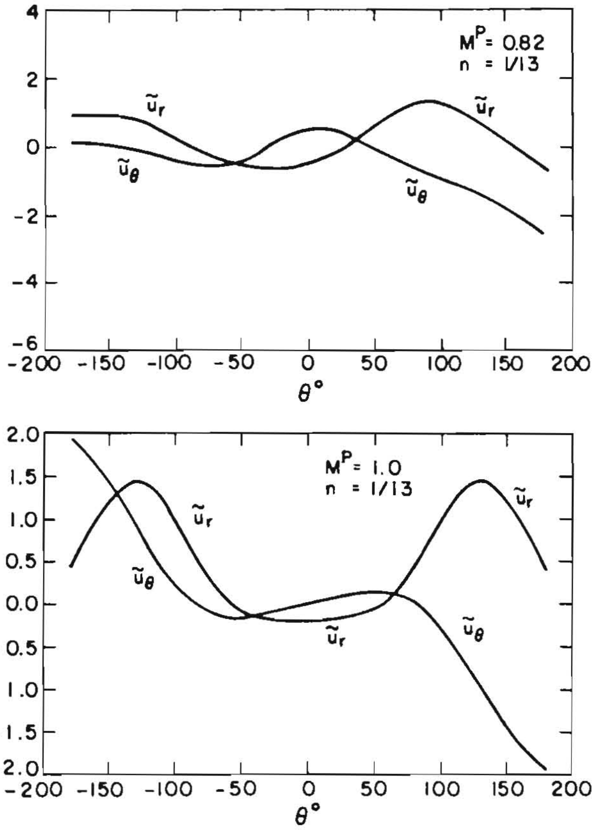

Fig. 2. θ-variations for the displacement functions at the tip of a crack for plane strain, $n = 1/13$, $M^p = 0.82$, 1.0.

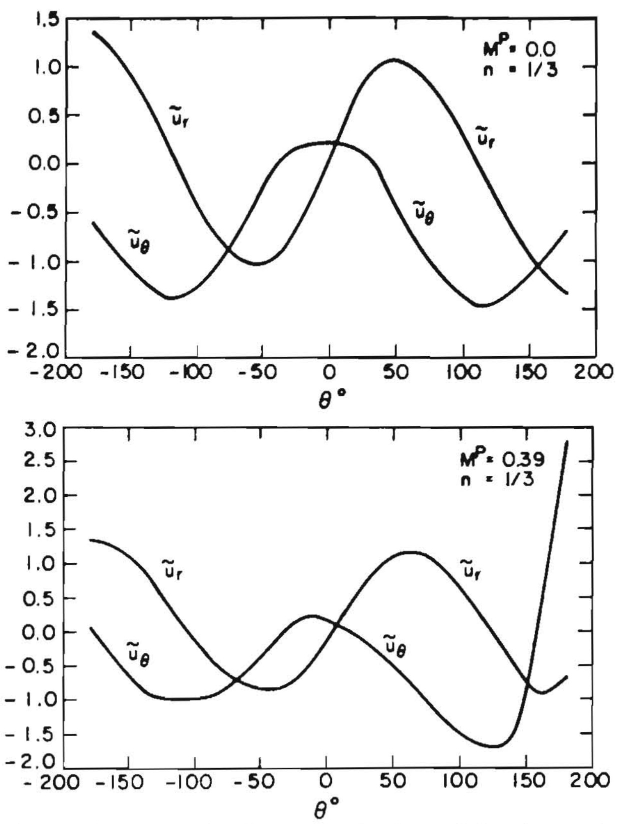

Fig. 3. θ-variations for the displacement functions at the tip of a crack for plane strain, $n = 1/3$, $M^p = 0.0$, 0.39.

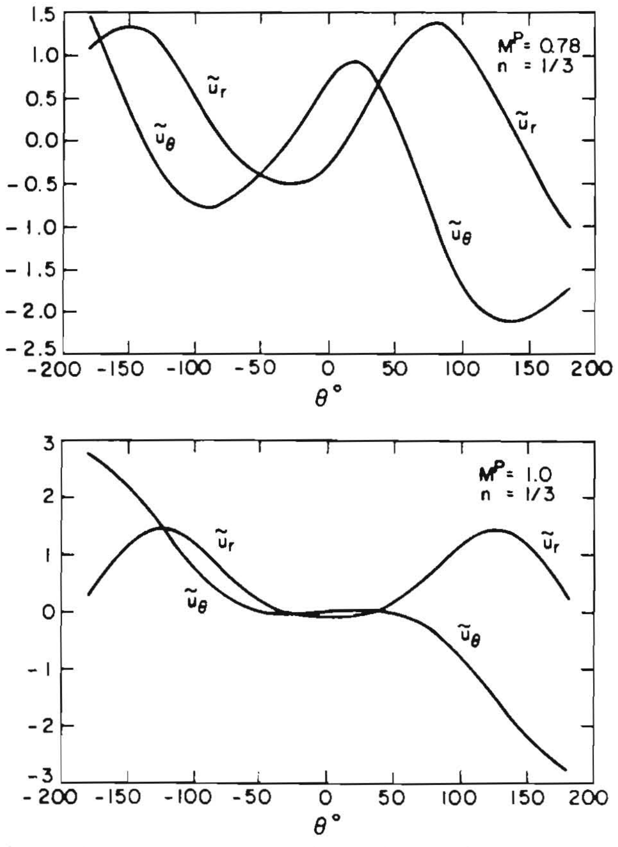

Fig. 4. θ-variations for the displacement functions at the tip of a crack for plane strain, $n = 1/3$, $M^p = 0.78$, 1.0.

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