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Buckling and Initial Postbuckling Behavior of Sandwich Beams Including Transverse Shear

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An asymptotic solution is presented for the buckling and initial postbuckling behavior of sandwich beams. The effect of transverse shear is included, and the shear correction is calculated from energy equivalency. The asymptotic procedure is based on the nonlinear beam equation (with transverse shear included), and closed-form solutions are derived for the critical load and for the load and midpoint deflection and axial shortening vs applied compressive load during the initial postbuckling phase. Illustrative results are presented for a few typical sandwich construction configurations, in particular, with regard to the effect of face sheet and core material system.

Nomenclature

A	=	total cross-sectional area
c	=	subscript for the core
E	=	Young's modulus
e	=	distance of the neutral axis of the section from the core midline
$(EI)_{\mathrm{eq}}$	=	equivalent rigidity
f_1	=	subscript for the top face sheet
f_2	=	subscript for the bottom face sheet
G	=	shear modulus
$\bar{G}$	=	"effective" shear modulus of the section
L	=	beam length
P	=	axial force
P_{cr}	=	critical load
s	=	distance along the deflected beam
V	=	shear force (normal to the deflected beam axis)
v	=	vertical displacement
α	=	shear correction coefficient
β	=	slope of the deflected beam axis
γ_{eq}	=	equivalent for the section shear angle
θ	=	rotation of the cross section due to bending

Introduction

SANDWICH structures have received considerable attention recently, primarily because of their high specific stiffness and strength properties. These structures are typically composed of two thin composite laminated faces and a thick soft core made of foam or low-strength honeycomb. Sandwich construction has been used in aircraft, marine, and other types of structures.

Research into sandwich structural behavior and failure modes can be traced following World War II in a rather sporadic fashion but intensified in the 1990s, especially with regard to proper modeling of the core through high-order theories (for example, Kant and Patil,[1] Hunt and Da Silva,[2,3] Frostig,[4] and Frostig and Baruch[5]).

There are still several questions that need to be addressed regarding the behavior of these structures, among them the issue of buckling and postbuckling. Even in homogeneous beams, critical loads can be overestimated if transverse shear is not included (for example, Timoshenko and Gere[6]). In sandwich beams, this overestimation can be significant due to the contribution of the core, which is expected to carry the shear and which has a very low modulus. The effect of the transverse shear due to the relatively compliant core on the postbuckling behavior can likewise be significant.

In the present work, a nonlinear beam equation including transverse shear is used. The shear effect is accounted for a general unsymmetric sandwich section (meaning face sheets not of the same geometry and/or material). A perturbation procedure is subsequently applied to obtain a closed-form solution for the initial postbuckling behavior. Illustrative results are presented for two common sandwich constructions, namely, glass-polyester/polymeric foam and graphite–epoxy/honeycomb.

Nonlinear Governing Equation for a Sandwich Beam with Shear

Let us consider an elastic sandwich, initially straight beam in a symmetric buckled configuration between sections i and j, as shown schematically in Fig. 1. Initially, the beam is of length L and has a uniform equivalent flexural stiffness, $(EI)_{\mathrm{eq}}$. [The experssion for $(EI)_{\mathrm{eq}}$ is given in the Appendix for an arbitrary asymmetric sandwich construction.] Elastic buckling of the beam is conditioned by the end restraints and the magnitude of the axial load P. In the buckled form, the end moments M_i and $M_j = -M_i$ (assumed positive clockwise) are set up. Because of the symmetry assumed, the equal and opposite shearing forces Q, are zero (to satisfy overall equilibrium).

The moment m at a distance s is given by

$$\frac{1}{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}s} = -\frac{m}{(EI)_{\mathrm{eq}}} \tag{1}$$

where ρ is the radius of curvature and θ is the rotation of the normal to the cross section, measured positive clockwise.

From equilibrium,

$$m = Pv + M_i \tag{2}$$

Now, the sandwich beam consist of a soft core of thickness c, for example, a hardened polymeric foam or honeycomb and two stiff faces (skins). The contribution of the longitudinal normal stresses in the core is small compared with those in the skins. Consequently, the shear stress is nearly uniform through the thickness of the core. Because the skins are thin, the shear stresses in the core carry most of the shear force, and the shear deformation of the core is very important. The shear deformations can be taken into account by relaxing the assumption that the plane cross sections remain normal to the deflected beam axis, that is, the slope β of the deflected beam axis is no longer required to be equal to the rotation θ of the cross section due to bending, the difference being the equivalent for the section shear angle γ_{eq} [Timoshenko beam theory assumption (see Ref. 6)]. If we assume that the stresses are distributed uniformly over the entire section A, then an equivalent shear angle can be defined based on the "effective" shear modulus of the section $\bar{G}$, which is defined from the compliances of the constituent phases:

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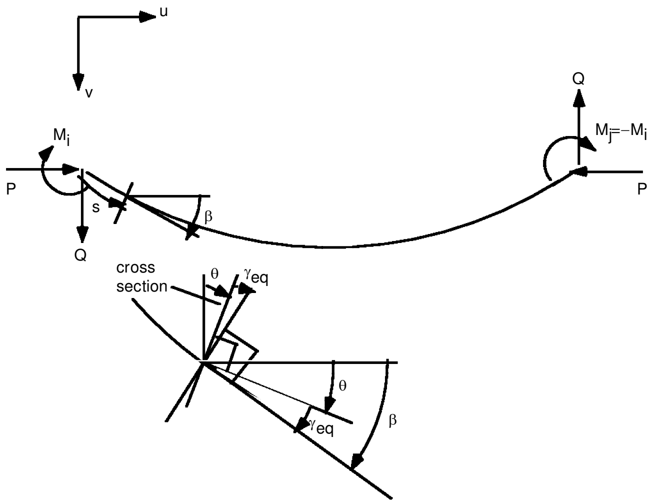

Fig. 1 Buckled sandwich beam.

$$(f_1 + c + f_2)/\bar{G} = f_1/G_{f1} + c/G_c + f_2/G_{f2} \tag{3a}$$

as

$$\gamma_{eq} = \beta - \theta = \alpha V/\bar{G}A \tag{3b}$$

in which V rotates as the beam deflects. Furthermore, α is calculated in the Appendix for an arbitrary unsymmetric sandwich structure from strain energy considerations and takes into account the nonuniform distribution of the shear stresses throughout the cross section and the contribution of the skins. The equivalent rigidity of the entire section, $(EI)_{eq}$ is also given in the Appendix for an arbitrary asymmetric sandwich construction.

Using v to denote the vertical displacement, we can write

$$\frac{dv}{ds} = \sin\beta = \sin(\gamma_{eq} + \theta) \tag{4}$$

Now, V is the component of the axial force P in the direction normal to the cross section, that is,

$$V = P\sin\beta \simeq P\sin\theta \tag{5}$$

Assume that the shear strain is small, which is a reasonable assumption for the buckling and the initial postbuckling states that we are studying, substituting $\sin\gamma_{eq} = \gamma_{eq}$, $\cos\gamma_{eq} = 1$ and Eqs. (3) and (5) into Eq. (4), we have

$$\frac{dv}{ds} = \gamma_{eq}\cos\theta + \sin\theta = \frac{\alpha P}{2A\bar{G}}\sin 2\theta + \sin\theta \tag{6}$$

When Eq. (2) is differentiated with respect to s, Eq. (1) is used, and Eq. (6) is substituted, the governing nonlinear equation that includes transverse shear becomes

$$(EI)_{eq}\frac{d^2\theta}{ds^2} + P\left(\frac{\alpha P}{2A\bar{G}}\sin 2\theta + \sin\theta\right) = 0 \tag{7}$$

For buckling, we can use the usual assumption that θ is small; therefore, $\sin\theta \simeq \theta$, and we can also replace ds with dx (inextensional assumption). Then Eq. (7) becomes a linear differential equation:

$$(EI)_{eq}\frac{d^2\theta}{dx^2} + \left(\frac{\alpha P}{A\bar{G}} + 1\right)P\theta = 0 \tag{8}$$

Then, following the usual procedure for solving for the critical load by using the general trigonometric solution of Eq. (8) and imposing the relevant boundary conditions, for example, see Simitses,[7] we can write the critical load as

$$P_{cr} = \frac{-1 + \sqrt{1 + 4\alpha(EI)_{eq}\lambda_{cr}^2/A\bar{G}}}{(2\alpha/A\bar{G})} \tag{9}$$

where $\lambda_{cr}L = 2\pi$ for a clamped–clamped and $\lambda_{cr}L = \pi$ for a simply supported beam.

For example, the buckling load for a clamped–clamped sandwich beam can be written as

$$P_{cr} = \frac{-1 + \sqrt{1 + 16\alpha(EI)_{eq}\pi^2/A\bar{G}L^2}}{(2\alpha/A\bar{G})}$$

Asymptotic Solution for the Initial Postbuckling Behavior

Now, for studying the initial postbuckling behavior, let us expand $\sin\theta$ according to the Taylor's series $\sin\theta = \theta - \theta^3/6$; then Eq. (7) becomes (again replacing ds with dx for initial postbuckling states)

$$(EI)_{eq}\frac{d^2\theta}{dx^2} + P\left(\frac{\alpha P}{A\bar{G}} + 1\right)\theta - \left(\frac{2\alpha P}{3A\bar{G}} + \frac{1}{6}\right)P\theta^3 = 0 \tag{10}$$

We use the perturbation method to solve this nonlinear differential equation, that is, we expand the load P and the slope θ in the neighborhood of the critical load, in terms of a small perturbation parameter ϵ. Thus, we set

$$\theta = \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \cdots \tag{11a}$$

$$P = P_{cr} + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \cdots \tag{11b}$$

Let us also set the perturbation parameter ϵ to be the slope, $\theta(L/4)$; then we have

$$\theta(L/4) = \epsilon = \epsilon\theta_1 + \epsilon^2\theta_2 + \epsilon^3\theta_3 + \cdots \tag{12}$$

which means that, for $x = L/4$, one has the additional conditions

$$\theta_2(L/4) = \theta_3(L/4) = \cdots = 0, \qquad \theta_1(L/4) = 1 \tag{13}$$

Because the beam is symmetrical, only the left half of the beam needs to be considered. From the symmetry condition, we have $\theta(L/2) = 0$; thus,

$$\theta_1(L/2) = \theta_2(L/2) = \theta_3(L/2) = \cdots = 0 \tag{14}$$

Furthermore, from the clamped boundary conditions, $\theta = 0$ at $x = 0$; therefore,

$$\theta_1(0) = \theta_2(0) = \theta_3(0) = \cdots = 0 \tag{15}$$

Inserting the expansion equations (11a) and (11b) into the nonlinear differential equation (10) and rearranging the terms according to the order of ϵ, one finds the following sequentially solvable set of linear differential equations:

For first order $\mathcal{O}(\epsilon^1)$,

$$(EI)_{eq}\frac{d^2\theta_1}{dx^2} + \left(\frac{\alpha P_{cr}^2}{A\bar{G}} + P_{cr}\right)\theta_1 = 0 \tag{16a}$$

for second order $\mathcal{O}(\epsilon^2)$,

$$(EI)_{eq}\frac{d^2\theta_2}{dx^2} + \left(\frac{\alpha P_{cr}^2}{A\bar{G}} + P_{cr}\right)\theta_2 = -\left(\frac{2\alpha P_{cr}}{A\bar{G}} + 1\right)P_1\theta_1 \tag{16b}$$

and for third order $\mathcal{O}(\epsilon^3)$,

$$(EI)_{eq}\frac{d^2\theta_3}{dx^2} + \left(\frac{\alpha P_{cr}^2}{A\bar{G}} + P_{cr}\right)\theta_3 = -\left(\frac{2\alpha P_{cr}P_1}{A\bar{G}} + P_1\right)\theta_2$$
$$-\left[\frac{\alpha\left(2P_{cr}P_2 + P_1^2\right)}{A\bar{G}} + P_2\right]\theta_1 + \left(\frac{2\alpha P_{cr}^2}{3A\bar{G}} + \frac{P_{cr}}{6}\right)\theta_1^3 \tag{16c}$$

First Order

The first-order equation is the familiar linearized buckling equation. Denoting

$$\lambda_{cr} = \sqrt{\frac{\alpha P_{cr}^2}{(EI)_{eq} A\bar{G}} + \frac{P_{cr}}{(EI)_{eq}}} \tag{17}$$

and applying the boundary condition (15), $\theta_1(0) = 0$, we obtain the solution for Eq. (16a) as

$$\theta_1(x) = A_1 \sin \lambda_{cr} x \tag{18}$$

Applying Eq. (14), $\theta_1(L/2) = 0$, gives the critical load from

$$\lambda_{cr} L/2 = \pi \tag{19}$$

By applying Eq. (13), $\theta_1(L/4) = 1$, we find that $A_1 = 1$. Therefore, the solution for the first-order linear differential equation (16a) is

$$\theta_1(x) = \sin \lambda_{cr} x \tag{20}$$

Second Order

The solution for Eq. (16b) after inserting Eq. (20) is

$$\theta_2(x) = A_2 \sin \lambda_{cr} x + B_2 \cos \lambda_{cr} x + \frac{P_1}{2\lambda_{cr}(EI)_{eq}}\left(\frac{2\alpha P_{cr}}{A\bar{G}} + 1\right) x \cos \lambda_{cr} x \tag{21}$$

Applying the boundary conditions (13), (14), and (15), that is, $\theta_2 = 0$ at $x = 0$, $L/4$, and $L/2$, and using Eq. (19), one finds that

$$A_2 = 0, \qquad B_2 = 0, \qquad P_1 = 0 \tag{22}$$

This leads to

$$\theta_2(x) = 0 \tag{23}$$

Third Order

Because $P_1 = 0$ and $\theta_2 = 0$, the second-order linear differential equation (16c) is simplified as follows:

$$(EI)_{eq}\frac{d^2\theta_3}{dx^2} + \left(\frac{\alpha P_{cr}^2}{A\bar{G}} + P_{cr}\right)\theta_3 = -\left(\frac{2\alpha P_{cr}}{A\bar{G}} + 1\right) P_2 \theta_1 + \left(\frac{2\alpha P_{cr}^2}{3A\bar{G}} + \frac{P_{cr}}{6}\right)\theta_1^3 \tag{24}$$

Let us denote

$$C_1 = \frac{1}{(EI)_{eq}}\left(\frac{2\alpha P_{cr}}{A\bar{G}} + 1\right), \qquad C_2 = \frac{1}{(EI)_{eq}}\left(\frac{2\alpha P_{cr}^2}{3A\bar{G}} + \frac{P_{cr}}{6}\right) \tag{25}$$

Then, the solution for Eq. (24) is

$$\theta_3(x) = A_3 \sin \lambda_{cr} x + B_3 \cos \lambda_{cr} x + (C_1 P_2 / 2\lambda_{cr} - 3C_2/8\lambda_{cr}) x \cos \lambda_{cr} x + \left(C_2 \big/ 32\lambda_{cr}^2\right) \sin 3\lambda_{cr} x \tag{26}$$

The constants A_3, B_3, and the load P_2 are found by applying the boundary conditions (13), (14), and (15), that is, $\theta_3 = 0$ at $x = 0$, $L/4$, and $L/2$, and using Eq. (19):

$$A_3 = C_2/32\lambda_{cr}, \qquad B_3 = 0, \qquad P_2 = 3C_2/4C_1 \tag{27}$$

Thus, the solution to Eq. (24) is

$$\theta_3(x) = \left(C_2 \big/ 32\lambda_{cr}^2\right)(\sin \lambda_{cr} x + \sin 3\lambda_{cr} x) \tag{28}$$

Complete Synthesized Solution

Now, the initial postbuckling solution can be written as follows:

$$\theta(x) = \epsilon \sin \lambda_{cr} x + \epsilon^3 \left(C_2 \big/ 32\lambda_{cr}^2\right)(\sin \lambda_{cr} x + \sin 3\lambda_{cr} x) + \cdots \tag{29a}$$

$$P = P_{cr} + \epsilon^2 (3C_2/4C_1) + \cdots \tag{29b}$$

where C_1 and C_2 are defined in Eq. (25). If the external load $\bar{P}$ is known, for example, measured in an experiment, then the perturbation parameter ϵ can be calculated from Eq. (29b) as

$$\epsilon = \sqrt{\frac{4C_1(P - P_{cr})}{3C_2}} \tag{30}$$

The vertical deformation of the beam can be obtained from Eq. (6) by expanding the trigonometric terms and integrating the slope, that is,

$$v(x) = \left(\frac{\alpha P}{A\bar{G}} + 1\right)\int_0^x \theta(\xi)\, d\xi - \left(\frac{4\alpha P}{A\bar{G}} + 1\right)\int_0^x \frac{\theta^3(\xi)}{6}\, d\xi \tag{31a}$$

Substituting the asymptotic expression for $\theta(x)$ from Eq. (29a) gives the deflection:

$$v(x) = \epsilon\left(\frac{\alpha P}{A\bar{G}} + 1\right)\frac{(1 - \cos \lambda_{cr} x)}{\lambda_{cr}} + \epsilon^3\left\{\left(\frac{\alpha P}{A\bar{G}} + 1\right)\frac{C_2}{32\lambda_{cr}^3}\left[(1 - \cos \lambda_{cr} x) + \frac{(1 - \cos 3\lambda_{cr} x)}{3}\right] - \left(\frac{4\alpha P}{A\bar{G}} + 1\right)\frac{1}{6\lambda_{cr}}\left[(1 - \cos \lambda_{cr} x) + \frac{\left(\cos^3 \lambda_{cr} x - 1\right)}{3}\right]\right\} \tag{31b}$$

The midpoint deflection v_m, which may be of interest in testing, is

$$v_m = \epsilon\left(\frac{\alpha P}{A\bar{G}} + 1\right)\frac{2}{\lambda_{cr}} + \epsilon^3\left[\left(\frac{\alpha P}{A\bar{G}} + 1\right)\frac{C_2}{12\lambda_{cr}^3} - \left(\frac{4\alpha P}{A\bar{G}} + 1\right)\frac{2}{9\lambda_{cr}}\right] \tag{31c}$$

Finally, the shortening of the beam, by use of Eq. (6) after expanding the trigonometric terms, is

$$\delta = \frac{1}{2}\int_0^L v'^2\, dx = \left(1 + \frac{\alpha P}{A\bar{G}}\right)^2 \int_0^L \frac{\theta^2(x)}{2}\, dx - \left(1 + \frac{\alpha P}{A\bar{G}}\right)\left(1 + \frac{4\alpha P}{A\bar{G}}\right)\int_0^L \frac{\theta^4(x)}{6}\, dx + \left(1 + \frac{4\alpha P}{A\bar{G}}\right)^2 \int_0^L \frac{\theta^6(x)}{72}\, dx \tag{32a}$$

which becomes, after substituting the asymptotic expression for $\theta(x)$ [Eq. (29a)],

$$\delta = \frac{\epsilon^2}{4}\left(1 + \frac{\alpha P}{A\bar{G}}\right)^2\left(L - \frac{\sin 2\lambda_{cr} L}{2\lambda_{cr}}\right) + \epsilon^4\left[\left(1 + \frac{\alpha P}{A\bar{G}}\right)^2 \frac{C_2}{64\lambda_{cr}^2}\left(L - \frac{\sin 4\lambda_{cr} L}{4\lambda_{cr}}\right) - \frac{1}{6}\left(1 + \frac{\alpha P}{A\bar{G}}\right) \times \left(1 + \frac{4\alpha P}{A\bar{G}}\right)\left(\frac{3L}{8} - \frac{\sin 2\lambda_{cr} L}{4\lambda_{cr}} + \frac{\sin 4\lambda_{cr} L}{32\lambda_{cr}}\right)\right] + \cdots \tag{32b}$$

For higher accuracy, the sixth-order term can be added, which is

$$+\epsilon^6\left\{\left(1+\frac{\alpha P}{A\bar{G}}\right)^2\frac{C_2^2}{4.8^3.\lambda_{cr}^4}\left(L+\frac{\sin 2\lambda_{cr}L}{4\lambda_{cr}}-\frac{\sin 4\lambda_{cr}L}{4\lambda_{cr}}-\frac{\sin 6\lambda_{cr}L}{12\lambda_{cr}}\right)-\left(1+\frac{\alpha P}{A\bar{G}}\right)\left(1+\frac{4\alpha P}{A\bar{G}}\right)\frac{C_2}{8.48\lambda_{cr}^2}\times\left(2L-\frac{\sin 2\lambda_{cr}L}{2\lambda_{cr}}-\frac{\sin 4\lambda_{cr}L}{2\lambda_{cr}}+\frac{\sin 6\lambda_{cr}L}{6\lambda_{cr}}\right)+\frac{1}{72}\left(1+\frac{4\alpha P}{A\bar{G}}\right)^2\left[\frac{5}{6}\left(\frac{3L}{8}-\frac{\sin 2\lambda_{cr}L}{4\lambda_{cr}}+\frac{\sin 4\lambda_{cr}L}{32\lambda_{cr}}\right)-\frac{\sin^5\lambda_{cr}L\cos\lambda_{cr}L}{6\lambda_{cr}}\right]\right\} \tag{32c}$$

Discussion of Results

For an illustration of the results from the preceding analysis, consider a sandwich beam with (in millimeters) $f_1 = f_2 = 4$ and $c = 20$, that is, of total thickness $h = 28$ mm and of width $w = h$. Two types of core were used: 1) a PVC core with (in mega pascal) $E_c = 93$ and $G_c = 35$ and 2) a honeycomb core with $E_c = 2000$ and $G_c = 460$. The corresponding face sheets were 1) E-glass/polyester unidirectional with (in gigapascal) $E_{f1} = E_{f2} = 26$ and $G_{f1} = G_{f2} = 3$ and 2) graphite–epoxy unidirectional with $E_{f1} = E_{f2} = 140$ and $G_{f1} = G_{f2} = 5$. In the results presented, the case of no transverse shear effect can be treated by simply setting $\alpha = 0$.

Figure 2 shows the critical load P_{cr} normalized with the Euler load $P_{Eul} = 4\pi^2(EI)_{eq}/L^2$ for a beam length over thickness ratio L/h, from 10 to 50. It is seen that the normalized critical load is lower for the glass-polyester/PVC material system, the difference between the two material systems being significant, especially at smaller L/h ratios. Notice that if transverse shear is not included then $P_{cr}/P_{Eul} = 1$.

The initial postbuckling results that follow are produced for a beam length over thickness ratio $L/h = 20$. Figure 3 shows the midpoint deflection v_m, and Fig. 4 shows the axial shortening δ, both normalized with the beam length, vs the applied load P/P_{cr} for the two material systems. Both displacements are higher for the glass-polyester/PVC case. It can also be seen that if transverse shear is not included, these displacements can be substantially underestimated. Finally, because the solution presented is an asymptotic solution, the results would tend to be less accurate as we move away from the critical point, that is, at the higher P/P_{cr} values, but, of course, additional terms of the expansion can be derived in a similar manner and, therefore, increase the accuracy.

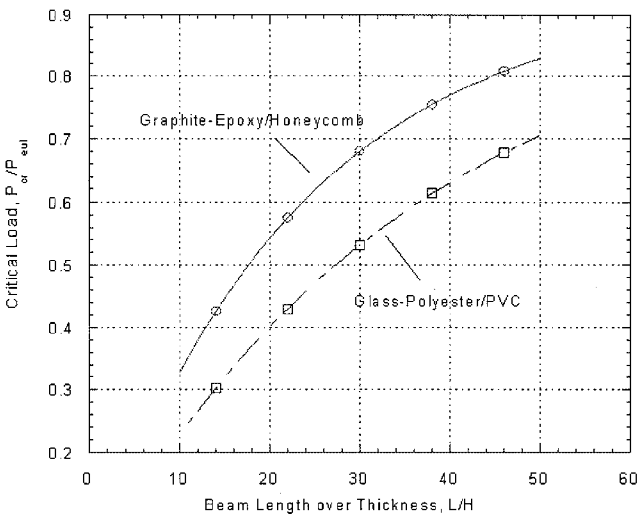

Fig. 2 Critical load, P_{cr}/P_{Euler}, vs beam length over thickness L/h, for the two material sandwich systems; if transverse shear is not included, the critical load is the Euler load for both materials.

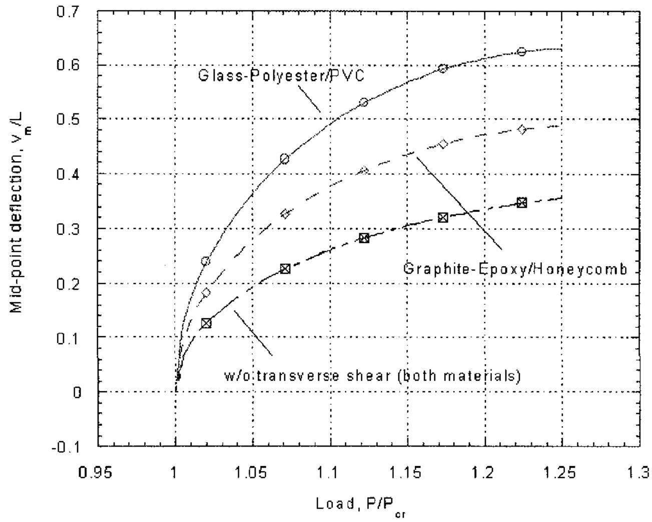

Fig. 3 Midpoint deflection, v_m/L, vs load P/P_{cr} during the initial postbuckling phase, for the two material sandwich systems (graphite–epoxy/honeycomb and glass-polyester/PVC).

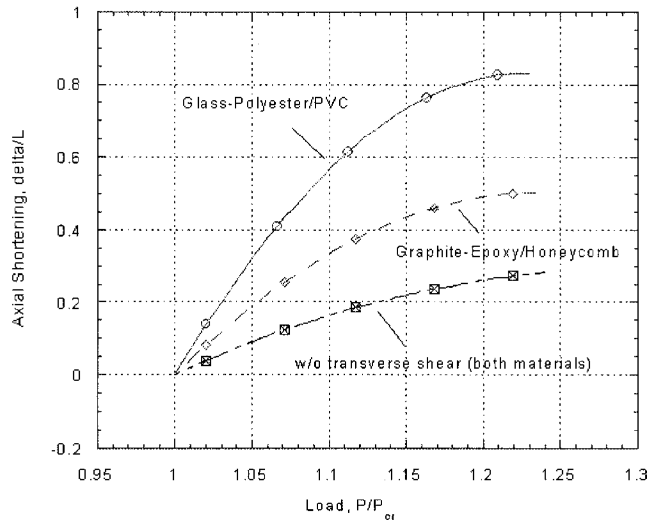

Fig. 4 Axial shortening, δ/L, vs load P/P_{cr} during the initial postbuckling phase for the two material sandwich systems.

Conclusions

A closed-form solution is presented for the buckling and initial postbuckling behavior of sandwich beams, including the transverse shear effect. The solution is derived by applying the perturbation procedure on the nonlinear beam equation. Results for the critical load and the midpoint transverse deflection and axial shortening vs applied load indicate the significance of the face sheet/core material system, which is reflected in the transverse shear effect.

Appendix: Transverse Shear Correction Factor

An unsymmetric sandwich section is shown in Fig. A1. The section consists of two face sheets of thickness f_1 and f_2, extensional moduli E_{f1} and E_{f2}, and shear modul G_{f1} and G_{f2}, respectively. The core, of thickness c, has an extensional modulus E_c, and shear modulus G_c. The width is uniform, w.

With respect to the reference axis y through the middle of the core, the neutral axis of the section is defined at a distance e, as

$$e = \frac{E_{f2}f_2}{E_{f1}f_1 + E_{f2}f_2 + E_c c}\left(\frac{f_2}{2}+\frac{c}{2}\right) - \frac{E_{f1}f_1}{E_{f1}f_1 + E_{f2}f_2 + E_c c}\left(\frac{f_1}{2}+\frac{c}{2}\right) \tag{A1}$$

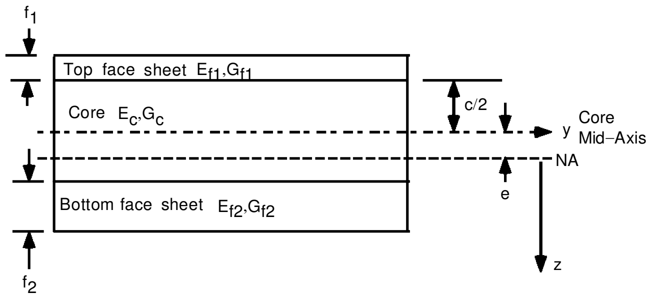

Fig. A1 Section of an unsymmetric sandwich beam.

and the equivalent flexural rigidity of the sandwich section $(EI)_{\mathrm{eq}}$ is

$$(EI)_{\mathrm{eq}} = w\left[E_{f1}\frac{f_1^3}{12} + E_{f1}f_1\left(\frac{f_1}{2}+\frac{c}{2}+e\right)^2 + E_{f2}\frac{f_2^3}{12} + E_{f2}f_2\left(\frac{f_2}{2}+\frac{c}{2}-e\right)^2 + E_c\frac{c^3}{12} + E_c c e^2\right] \tag{A2}$$

The shear stress can be written in terms of the shear force $V(x)$ in the form

$$\tau(x, z) = V(x)q(z)$$

Also, due to the sandwich construction, $G = G(z)$; therefore, the shear energy is

$$w\iint \frac{\tau^2(x, z)}{2G}\,\mathrm{d}x\,\mathrm{d}z = \frac{w}{2}\int V^2(x)\left[\int \frac{q^2(z)}{G(z)}\,\mathrm{d}z\right]\mathrm{d}x$$

Now, assume that the shear stresses are distributed in a uniform fashion over the entire section A, then the corresponding equivalent shear stress and strain are

$$\tau_{\mathrm{eq}} = V(x)/A, \qquad \gamma_{\mathrm{eq}} = \alpha V(x)/\bar{G}A \tag{A3}$$

where α takes into account the nonuniform distribution of shear stresses due to the sandwich construction throughout the entire cross section, and the shear strain is based on the effective shear modulus of the sandwich section $\bar{G}$ defined in Eq. (3a) from a rule-of-mixtures calculation on the compliances of the constituent phases (because a uniform shear stress is assumed in the definition of the shear correction). This definition of $\bar{G}$ is also compatible with usual practice regarding effective shear moduli, for example, see Tsai.[8] However, note that what actually enters into the theory is the ratio $\alpha/(\bar{G}A)$; thus, it would not matter if the shear strain expression was based on an effective shear modulus defined in a different way because then the shear correction factor would be different but the ratio the same.

Then, the energy due to shear is

$$A\int \frac{1}{2}\tau_{\mathrm{eq}}\gamma_{\mathrm{eq}}\,\mathrm{d}x = \frac{\alpha}{2\bar{G}A}\int V^2(x)\,\mathrm{d}x \tag{A4}$$

therefore,

$$\alpha = \bar{G}Aw\int \frac{q^2(z)}{G(z)}\,\mathrm{d}z \tag{A5}$$

The shear stresses, from simple bending theory, are as follows.

Upper face sheet, $z < 0$:

$$\tau_{f1}(z) = \frac{V}{(EI)_{\mathrm{eq}}}\frac{E_{f1}}{2}\left[\left(f_1+\frac{c}{2}+e\right)^2 - z^2\right]$$

Core, $z < 0$:

$$\tau_c(z) = \frac{V}{(EI)_{\mathrm{eq}}}\left\{E_{f1}f_1\left(\frac{f_1}{2}+e+\frac{c}{2}\right) + \frac{E_c}{2}\left[\left(\frac{c}{2}+e\right)^2 - z^2\right]\right\}$$

Core, $z > 0$:

$$\tau_c(z) = \frac{V}{(EI)_{\mathrm{eq}}}\left\{E_{f2}f_2\left(\frac{f_2}{2}-e+\frac{c}{2}\right) + \frac{E_c}{2}\left[\left(\frac{c}{2}-e\right)^2 - z^2\right]\right\}$$

Lower face sheet, $z > 0$:

$$\tau_{f2}(z) = \frac{V}{(EI)_{\mathrm{eq}}}\frac{E_{f2}}{2}\left[\left(f_2+\frac{c}{2}-e\right)^2 - z^2\right]$$

Therefore, if we define for $i = 1, 2$,

$$a_i = f_i + c/2 + (-1)^{i+1}e, \qquad b_i = c/2 + (-1)^{i+1}e$$

$$c_i = f_i/2 + c/2 + (-1)^{i+1}e$$

then the shear correction coefficient is found from Eq. (A5) as

$$\alpha = \bar{G}Aw\sum_{i=1,2}\frac{E_{fi}^2}{4(EI)_{\mathrm{eq}}^2 G_{fi}}\left[a_i^4 f_i - \frac{2}{3}a_i^2\left(a_i^3 - b_i^3\right) + \frac{1}{5}\left(a_i^5 - b_i^5\right)\right] + \frac{E_{fi}^2}{(EI)_{\mathrm{eq}}^2 G_c}\left[f_i^2 c_i^2 b_i + \frac{2}{15}\frac{E_c^2}{E_{fi}^2}b_i^5 + \frac{2}{3}\frac{E_c}{E_{fi}}f_i c_i b_i^3\right] \tag{A6}$$

For a homogeneous part, that is, same material for face sheets and core, the calculations reduce to the simple value $\alpha = \frac{6}{5}$. This coincides with the classical results first derived by Goens[9] and later reintroduced by Reissner.[10]

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