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Comparative Studies on the Buckling of Isotropic, Orthotropic, and Sandwich Columns

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ABSTRACT

Advanced composite and sandwich construction has raised the issue of accuracy of the column buckling formulas currently used in structural design. In these advanced material systems, transverse shear effects are significant and cannot be ignored. The objective of this paper is to answer the question of how accurate the simple column buckling formulas by Euler or the transverse shear correction formulas by Engesser and by Haringx or other direct column buckling formulas in the literature are when composite or sandwich construction and moderate thickness are involved. For this purpose, a three-dimensional elasticity solution is presented along with finite element results. For the elasticity solution, which is performed for the monolithic orthotropic material, the column is considered to be in the form of a hollow, circular cylinder and the direct column buckling formulas are based on the axial modulus. As an example, the cases of an orthotropic material with stiffness constants typical of glass/epoxy or graphite/epoxy and the reinforcing direction along the periphery or along the cylinder axis are considered. Finite element results are presented for the sandwich columns, which are of metallic (aluminum) and laminated (boron/epoxy, graphite/epoxy, and Kevlar/epoxy) facings and alloy-foam or glass/phenolic honeycomb core. Sandwich columns are especially critical with the Euler load being, in some cases of typical design, as much as almost five times the critical load from the finite elements and, therefore, in these cases of sandwich construction, the classical Euler load calculations cannot be relied upon.

§1. INTRODUCTION

In composite structural members, the buckling strength is an important design parameter because of the large strength-to-weight ratio and the lack of extensive plastic yielding in these materials. Columns made out of composite materials for structural applications are envisioned in the form of a hollow cylinder of moderate thickness, produced mainly by filament winding or pultrusion. Such designs can be used, for example, as support members in civil and offshore structures or in space vehicles as a primary load-carrying structure. Recently, considerable attention has been paid to another advanced structural concept, namely sandwich construction, which consists typically of two thin composite laminated

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faces and a thick soft core made of foam or low-strength honeycomb. Sandwich construction has already been used in aircraft, marine, and other types of structures.

The case of a slender, ideal column, which is built in vertically at the base, free at the upper end, and subjected to an axial force P, constitutes the first problem of bifurcation buckling, the one that was originally solved by Euler [1]. The Euler solution is based on the well-known Euler–Bernoulli assumptions (i.e., plane sections remain plane after bending, no effect of transverse shear deformation) and for an isotropic elastic material. Nontrivial solutions (nonzero transverse deflections) are then sought for the equations governing bending of the column under an axial compressive load and subject to the particular set of boundary conditions; thus, the problem is reduced to an eigen-boundary-value problem [2].

Regarding formulas for the stability loss of elastic bars, the only alternative direct expressions to the Euler load that exist in the literature are the Engesser [3] and the Haringx [4, 5] formulas. (Haringx actually obtained the formula in connection with helical springs.) These formulas are also in the book by Timoshenko and Gere [6]. (Timoshenko also referred to the Haringx analysis as the "modified" approach.) These formulas were intended to account for the influence of transverse shearing deformations. The specific load expressions, denoted by P_{Engssr} and P_{Hrngx}, are given in the Results section. Despite the simplicity of the derivation of these formulas, it will be seen that they perform remarkably well in accounting for the thickness effects as well as for the effects of a low ratio of shear versus extensional modulus.

Composite materials have one important distinguishing feature: namely, an extensional-to-transverse shear modulus ratio much larger than that of their metal counterparts. In sandwich beams, this ratio is even larger due to the contribution of the core which is expected to carry the transverse shear and which has a very low modulus. The resulting effects of transverse shear may render the calculations of the critical load from simple classical column formulas highly nonconservative. Moreover, an additional deviation is expected because composites are anisotropic and these classical column formulas are based on isotropic material assumption. The objective of the present paper is to investigate the accuracy of the classical Euler load, and the simple transverse shear correction formulas by Engesser and Haringx, with regard to predicting the critical load. To this extent, in the first part of the paper, a three-dimensional elasticity analysis for a generally orthotropic rod with no restrictive assumptions regarding the cross-sectional dimensions is performed for the homogeneous composite cases; in the second part of the paper, finite element results are presented for the sandwich composite cases. For the sandwich construction, transverse shear is accounted for in direct formulas given by Bazant and Cedolin [7], Huang and Kardomateas [8], and Allen [9]. It should be mentioned that transverse shear effects are expected to be even more significant in sandwich columns due to the low transverse shear modulus of the core.

Three-dimensional elasticity solutions for buckling of composites have been derived by Kardomateas [10] and Kardomateas and Chung [11] for a cylindrical orthotropic shell subjected to external pressure. In these studies, it was shown that the critical load predicted by shell theory can be quite nonconservative for thick construction. For axial compression, a related study was conducted by Kardomateas [12] for the case of a transversely isotropic column. The reason for restricting the material to a transversely isotropic one was the desire to produce closed-form analytical solutions. By performing a series expansion of the terms of the resulting characteristic equation from the elasticity formulation for the isotropic case, the Euler load was proven to be the solution in the first approximation; consideration of the second approximation gave a direct expression for the correction to the Euler load, therefore defining a new, yet simple formula for column buckling, which herein will be referred to as the Euler load with a second term. In a subsequent paper [13], the study was extended to the case of a generally orthotropic moderately thick shell under axial compression. An orthotropic column under axial loading was studied by Kardomateas and Dancila [14] and some results reported herein are from that study.

Therefore, the first part of the study conducted in this paper includes specific results for the critical load of a column in the form of a hollow cylinder under axial compression for various ratios of length over external radius, L/R_2, and ratios of external over internal radii, R_2/R_1. The nonlinear three-dimensional theory of elasticity is appropriately formulated and reduced to a standard eigenvalue problem for ordinary linear differential equations in terms of a single variable (the radial distance r) with the applied axial load P as the eigenvalue. The formulation employs the exact elasticity solution by Lekhnitskii [15] for the prebuckling state. The results from the elasticity formulation will be compared with the classical Euler load predictions and with the Engesser or Haringx column buckling with transverse shear correction formulas, as well as with the Euler load with a second term, as derived by Kardomateas [12]. The effect of the material orthotropy is examined by considering two material cases, glass/epoxy and graphite/epoxy, and with reinforcing direction either along the circumferential (θ) or along the axial (z) direction.

The second part of this paper presents results from these direct formulas compared with ABAQUS [16] finite element results. Some finite element results for both static and dynamic buckling of sandwich columns had also been presented by Kardomateas et al. [17], but that study did not include a comparison with Allen's [9] formulas; the present study includes a more thorough comparison with the direct column buckling formulas and a more comprehensive discussion regarding their performance.

It should be noted that the only type of buckling considered here is what is typically referred to [18] as "long-wavelength buckling" (also called general instability) in contrast with "short-wavelength buckling" (also called face wrinkling).

§2. Homogeneous Orthotropic Column

2.1. *Buckling from three-dimensional elasticity*

Following Kardomateas [10], we obtain the following buckling equations from the equilibrium of the column, considered a three-dimensional elastic body:

$$\frac{\partial}{\partial r}(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta) + \frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta) + \frac{\partial}{\partial z}(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)$$
$$+ \frac{1}{r}(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z) = 0 \tag{1a}$$

$$\frac{\partial}{\partial r}(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r) + \frac{\partial}{\partial z}(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)$$
$$+ \frac{1}{r}(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r) = 0 \tag{1b}$$

$$\frac{\partial}{\partial r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r) + \frac{\partial}{\partial z}(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r)$$
$$+ \frac{1}{r}(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r) = 0 \tag{1c}$$

In the preceding equations, σ^0_{ij} and ω^0_j are the values of the stresses σ_{ij} and linear rotations ω_j at the initial equilibrium position, and σ'_{ij} and ω'_j are the values at the perturbed (buckled) position.

The associated boundary conditions for the lateral and end surfaces can be expressed as follows, again following Kardomateas [10]:

$$(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta)\hat{l} + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)\hat{m} + (\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)\hat{n} = 0 \tag{2a}$$

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r)\hat{l} + (\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)\hat{m} + (\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)\hat{n} = 0 \tag{2b}$$

$$(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta)\hat{l} + (\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta)\hat{m} + (\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta)\hat{n} = 0 \tag{2c}$$

where $(\hat{l}, \hat{m}, \hat{n})$ is the outward unit normal on the surface (before any deformation).

2.1.1. *Prebuckling state*

The column is considered to be in a cylindrical form; therefore, problem under consideration is that of an orthotropic hollow cylinder compressed by an axial force applied at one end. The stress–strain relations for the orthotropic body are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \tag{3}$$

where c_{ij} are the stiffness constants (using the notation $1 \equiv r, 2 \equiv \theta, 3 \equiv z$).

Let R_1 be the internal and R_2 the external radius. Lekhnitskii [15] gave the stress field for an applied compressive load of absolute value P in terms of the following quantities:

$$k = \sqrt{\frac{a_{11}a_{33} - a_{13}^2}{a_{22}a_{33} - a_{23}^2}} \tag{4a}$$

$$\tilde{h} = \frac{(a_{23} - a_{13})a_{33}}{(a_{11} - a_{22})a_{33} + (a_{23}^2 - a_{13}^2)} \tag{4b}$$

$$\tilde{T} = \pi(R_2^2 - R_1^2) - \frac{2\pi\tilde{h}}{a_{33}}\left[\frac{R_2^2 - R_1^2}{2}(a_{13} + a_{23}) - \frac{\left(R_2^{k+1} - R_1^{k+1}\right)^2}{R_2^{2k} - R_1^{2k}}\frac{a_{13} + ka_{23}}{k+1} - \frac{\left(R_2^{k-1} - R_1^{k-1}\right)^2 (R_1R_2)^2}{R_2^{2k} - R_1^{2k}}\frac{a_{13} - ka_{23}}{k-1}\right] \tag{4c}$$

The stress field for orthotropy is as follows:

$$\sigma^0_{rr} = P(C_0 + C_1 r^{k-1} + C_2 r^{-k-1}) \tag{5a}$$

$$\sigma^0_{\theta\theta} = P(C_0 + C_1 k r^{k-1} - C_2 k r^{-k-1}) \tag{5b}$$

$$\sigma^0_{zz} = -\frac{P}{\tilde{T}} - P\left(C_0\frac{a_{13} + a_{23}}{a_{33}} + C_1\frac{a_{13} + ka_{23}}{a_{33}}r^{k-1} + C_2\frac{a_{13} - ka_{23}}{a_{33}}r^{-k-1}\right) \tag{5c}$$

$$\tau^0_{r\theta} = \tau^0_{rz} = \tau^0_{\theta z} = 0 \tag{5d}$$

where

$$C_0 = -\frac{\tilde{h}}{\tilde{\tilde{T}}} \qquad C_1 = \frac{R_2^{k+1} - R_1^{k+1}}{R_1^{2k} - R_1^{2k}}\frac{\tilde{h}}{\tilde{\tilde{T}}} \tag{5e}$$

$$C_2 = \frac{R_2^{k-1} - R_1^{k-1}}{R_2^{2k} - R_1^{2k}}(R_1R_2)^{k+1}\frac{\tilde{h}}{\tilde{\tilde{T}}} \tag{5f}$$

Notice that, for orthotropy, both σ^0_{rr} and $\sigma^0_{\theta\theta}$ are nonzero. For an isotropic or a transversely isotropic body with the plane of isotropy normal to the $3 \equiv z$ axis, these two stress components are zero.

In the previous equations, a_{ij} are the compliance constants, found by taking the inverse of the stiffness matrix in Eq. (3).

2.1.2. *Perturbed state*

Then, using constitutive relations (3) for the stresses σ'_{ij} in terms of the linear strains ϵ'_{ij}, and taking into account the strain–displacement relations for the strains ϵ'_{ij} and the rotations ω'_j in terms of the displacements u_1, v_1, w_1 at the buckled state are

$$\epsilon_{rr} = u_{1,r} \qquad \epsilon_{\theta\theta} = \frac{v_{1,\theta} + u_1}{r} \qquad \epsilon_{zz} = w_{1,z} \tag{6a}$$

$$\gamma_{r\theta} = \frac{u_{1,\theta}}{r} + v_{1,r} - \frac{v_1}{r} \qquad \gamma_{rz} = u_{1,z} + w_{1,r} \qquad \gamma_{\theta z} = v_{1,z} + \frac{w_{1,\theta}}{r} \tag{6b}$$

and the linear rotations are

$$2\omega_r = \frac{w_{1,\theta}}{r} - v_{1,z} \quad 2\omega_\theta = u_{1,z} - w_{1,r} \quad 2\omega_z = v_{1,r} + \frac{v_1}{r} - \frac{u_{1,\theta}}{r} \tag{6c}$$

and taking into account Eqs. (5), buckling Eq. (1*a*) for the problem at hand is written in terms of the displacements at the perturbed state as follows:

$$\begin{aligned}
&c_{11}\left(u_{1,rr} + \frac{u_{1,r}}{r}\right) - c_{22}\frac{u_1}{r^2} + \left(c_{66} + \frac{\sigma^0_{\theta\theta}}{2}\right)\frac{u_{1,\theta\theta}}{r^2} + \left(c_{55} + \frac{\sigma^0_{zz}}{2}\right)u_{1,zz}\\
&+ \left(c_{12} + c_{66} - \frac{\sigma^0_{\theta\theta}}{2}\right)\frac{v_{1,r\theta}}{r} - \left(c_{22} + c_{66} + \frac{\sigma^0_{\theta\theta}}{2}\right)\frac{v_{1,\theta}}{r^2}\\
&+ \left(c_{13} + c_{55} - \frac{\sigma^0_{zz}}{2}\right)w_{1,rz} + (c_{13} - c_{23})\frac{w_{1,z}}{r} = 0
\end{aligned} \tag{7a}$$

The second buckling equation, Eq. (2*b*), gives

$$\begin{aligned}
&\left(c_{66} + \frac{\sigma^0_{rr}}{2}\right)\left(v_{1,rr} + \frac{v_{1,r}}{r} - \frac{v_1}{r^2}\right) + \left(\frac{\sigma^0_{rr} - \sigma^0_{\theta\theta}}{2}\right)\left(\frac{v_{1,r}}{r} + \frac{v_1}{r^2}\right) + c_{22}\frac{v_{1,\theta\theta}}{r^2}\\
&+ \left(c_{44} + \frac{\sigma^0_{zz}}{2}\right)v_{1,zz} + \left(c_{66} + c_{12} - \frac{\sigma^0_{rr}}{2}\right)\frac{u_{1,r\theta}}{r} + \left(c_{66} + c_{22} + \frac{\sigma^0_{\theta\theta}}{2}\right)\frac{u_{1,\theta}}{r^2}\\
&+ \left(c_{23} + c_{44} - \frac{\sigma^0_{zz}}{2}\right)\frac{w_{1,\theta z}}{r} + \frac{1}{2}\frac{d\sigma^0_{rr}}{dr}\left(v_{1,r} + \frac{v_1}{r} - \frac{u_{1,\theta}}{r}\right) = 0
\end{aligned} \tag{7b}$$

In a similar fashion, the third buckling equation, Eq. (1c), gives

$$\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right)\left(w_{1,rr}+\frac{w_{1,r}}{r}\right)+\left(c_{44}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{w_{1,\theta\theta}}{r^2}+c_{33}w_{1,zz}$$
$$+\left(c_{13}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)u_{1,rz}+\left(c_{23}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,z}}{r}$$
$$+\left(c_{23}+c_{44}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta z}}{r}+\frac{1}{2}\frac{d\sigma_{rr}^0}{dr}(w_{1,r}-u_{1,z})=0 \qquad (7c)$$

In the perturbed position, we seek equilibrium modes in the form

$$u_1(r,\theta,z)=U(r)\cos\theta\sin\frac{\pi z}{L} \qquad v_1(r,\theta,z)=V(r)\sin\theta\sin\frac{\pi z}{L}$$
$$w_1(r,\theta,z)=W(r)\cos\theta\cos\frac{\pi z}{L} \qquad (8)$$

where the functions $U(r)$, $V(r)$, $W(r)$ are uniquely determined. These equilibrium modes are the "column-type" buckling modes of a single axial half-wave and circumferential wave. Notice that the equilibrium modes in Eq. (8) are a special case of the general shell buckling modes that had been considered in the three-dimensional elasticity shell buckling formulation of Kardomateas [13]. It should also be mentioned that these modes correspond to the condition of "simply supported" ends since u_1 varies as $\sin\lambda z$ and $u_1=u_{1,zz}=0$ at $z=0, L$.

Now let $U^{(i)}(r)$, $V^{(i)}(r)$, and $W^{(i)}(r)$ denote the ith derivative of $U(r)$, $V(r)$, and $W(r)$, respectively, with the additional notation $U^{(0)}(r)=U(r)$, $V^{(0)}(r)=V(r)$, and $W^{(0)}(r)=W(r)$.

Substituting in Eq. (7a), we obtain the following linear homogeneous ordinary differential equation:

$$U(r)''c_{11}+U(r)'\frac{c_{11}}{r}+U(r)[(b_{00}+b_{01}P)r^{-2}+b_{02}Pr^{k-3}$$
$$+b_{03}Pr^{-k-3}+(b_{04}+b_{05}P)+b_{06}Pr^{k-1}+b_{07}Pr^{-k-1}]$$
$$+\sum_{i=0}^{1}V^{(i)}(r)[(d_{i0}+d_{i1}P)r^{i-2}+d_{i2}Pr^{k-3+i}+d_{i3}Pr^{-k-3+i}]$$
$$+\sum_{i=0}^{1}W^{(i)}(r)[(f_{i0}+f_{i1}P)r^{i-1}+f_{i2}Pr^{k-2+i}+f_{i3}Pr^{-k-2+i}]=0$$
$$R_1\le r\le R_2 \qquad (9a)$$

The second differential equation, Eq. (7b), gives

$$V(r)[(g_{04}+g_{05}P)+g_{06}Pr^{k-1}+g_{07}Pr^{-k-1}]$$
$$+\sum_{i=0}^{2}V^{(i)}(r)[(g_{i0}+g_{i1}P)r^{i-2}+g_{i2}Pr^{k-3+i}+g_{i3}Pr^{-k-3+i}]$$

$$+\sum_{i=0}^{1} U^{(i)}(r)[(h_{i0}+h_{i1}P)r^{i-2}+h_{i2}Pr^{k-3+i}+h_{i3}Pr^{-k-3+i}]$$
$$+W(r)[(t_{00}+t_{01}P)r^{-1}+t_{02}Pr^{k-2}+t_{03}Pr^{-k-2}]=0$$
$$R_1 \le r \le R_2 \tag{9b}$$

In a similar fashion, Eq. (7*c*) gives

$$W(r)q_{04}+\sum_{i=0}^{2} W^{(i)}(r)[(q_{i0}+q_{i1}P)r^{i-2}+q_{i2}Pr^{k-3+i}+q_{i3}Pr^{-k-3+i}]$$
$$+\sum_{i=0}^{1} U^{(i)}(r)[(s_{i0}+s_{i1}P)r^{i-1}+s_{i2}Pr^{k-2+i}+s_{i3}Pr^{-k-2+i}]$$
$$+V(r)[(\beta_{00}+\beta_{01}P)r^{-1}+\beta_{02}Pr^{k-2}+\beta_{03}Pr^{-k-2}]=0$$
$$R_1 \le r \le R_2 \tag{9c}$$

All of Eqs. (9) are linear, homogeneous, ordinary differential equations of the second order for $U(r)$, $V(r)$, and $W(r)$. In these equations, the constants $b_{ij}, d_{ij}, f_{ij}, g_{ij}, h_{ij}, t_{ij}, q_{ij}, s_{ij}$, and β_{ij} are given in the Appendix and depend on the material stiffness coefficients c_{ij} and k.

Now we proceed to the boundary conditions on the lateral surfaces $r = R_j (j = 1, 2)$. These will complete the formulation of the eigenvalue problem for the critical load.

From Eqs. (2), we obtain for $\hat{l} = \pm 1, \hat{m} = \hat{n} = 0$,

$$\sigma'_{rr}=0 \qquad \tau'_{r\theta}+\sigma^0_{rr}\omega'_z=0 \qquad \tau'_{rz}-\sigma^0_{rr}\omega'_\theta=0 \qquad \text{at } r=R_1, R_2 \tag{10}$$

Substituting in Eqs. (3), (6), (8), and (5), the boundary condition $\sigma'_{rr} = 0$ at $r = R_j (j = 1, 2)$ gives

$$U'(R_j)c_{11}+[U(R_j)+V(R_j)]\frac{c_{12}}{R_j}-c_{13}\frac{\pi}{L}W(R_j)=0 \qquad j=1,2 \tag{11a}$$

The boundary condition $\tau'_{r\theta} + \sigma^0_{rr}\omega'_z = 0$ at $r = R_j (j = 1, 2)$ gives

$$V'(R_j)\left[\left(c_{66}+\frac{C_0}{2}P\right)+\frac{C_1}{2}PR_j^{k-1}+\frac{C_2}{2}PR_j^{-k-1}\right]$$
$$+[V(R_j)+U(R_j)]\left[\left(-c_{66}+\frac{C_0}{2}P\right)R_j^{-1}\right.$$
$$\left.+\frac{C_1}{2}PR_j^{k-2}+\frac{C_2}{2}PR_j^{-k-2}\right] \qquad j=1,2 \tag{11b}$$

In a similar fashion, the condition $\tau'_{rz} - \sigma^0_{rr}\omega'_\theta = 0$ at $R_j (j = 1, 2)$ gives

$$U(R_j)\frac{\pi}{L}\left[\left(c_{55}-\frac{C_0}{2}P\right)-\frac{C_1}{2}PR_j^{k-1}-\frac{C_2}{2}PR_j^{-k-1}\right]$$
$$+W'(R_j)\left[\left(c_{55}+\frac{C_0}{2}P\right)+\frac{C_1}{2}PR_j^{k-1}+\frac{C_2}{2}PR_j^{-k-1}\right] \qquad j=1,2 \tag{11c}$$

Equations (9) and (11) constitute an eigenvalue problem for differential equations, with the parameter of applied compressive load P, which can be solved by standard numerical methods (two-point boundary-value problem).

Before discussing the numerical procedure used for solving this eigenvalue problem one final point will be addressed. To completely satisfy all the elasticity requirements, we should discuss the boundary conditions at the ends. From Eqs. (2), the boundary condition on the ends are

$$\tau'_{rz} + \sigma^0_{zz}\omega'_\theta = 0 \qquad \tau'_{\theta z} - \sigma^0_{zz}\omega'_r = 0 \qquad \sigma'_{zz} = 0 \qquad \text{at } z = 0, L \tag{12}$$

Since σ'_{zz} varies as $\sin\frac{\pi}{L}z$, the condition $\sigma'_{zz} = 0$ on both the lower end, $z = 0$, and the upper end, $z = L$, is satisfied. It can be proved that the remaining two conditions are satisfied on the average [13, 14]. At this point, it should be noted that for some of the boundary conditions a form of resultant instead of pointwise conditions has been frequently used in elasticity treatments and can be considered to be based on some form of Saint-Venant's principle. For this reason, they are sometimes referred to as relaxed end conditions of the Saint-Venant type [19].

As has already been stated, Eqs. (9) and (11) constitute an eigenvalue problem for ordinary second-order linear differential equations in the r variable, with the applied compressive load P, the eigenvalue. This is essentially a standard two-point boundary-value problem. The relaxation method was used [20], which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the section. For this purpose, an equally spaced mesh of 241 points was employed and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the classical column theory solution was used. In the solution scheme, seven functions of r are defined: $y_1 = U$, $y_2 = U'$, $y_3 = V$, $y_4 = V'$, $y_5 = W$, $y_6 = W'$, and $y_7 = P$. The seven differential equations are $y'_1 = y_2$, Eq. (9*a*), $y'_3 = y_4$, Eq. (9*b*), $y'_5 = y_6$, Eq. (9*c*), and $y'_7 = 0$. The corresponding seven boundary conditions are, at $r = R_2$, Eqs. (11*a*)–(11*c*); at $r = R_2$, $U = 1.0$; and at $r = R_1$, Eqs. (11*a*)–(11*c*). The solution gives the eigenfunctions y_1, y_3, and y_5, as well as the eigenvalue y_7.

An investigation of the convergence showed that essentially the same results were produced with even three times as many mesh points. It is also first verified that the structure behaves as a column rather than a shell (which would buckle at multiple axial half-waves or circumferential waves). This is accomplished by considering the structure as a shell and using the Kardomateas [13] solution to find if it would buckle at multiple axial half-waves or multiple circumferential waves.

2.2. *Buckling from simple, direct formulas*

The Euler critical load for a compressed simply supported column is

$$P_{\text{Euler}} = \frac{\pi^2 E_3 I}{L^2} = E_3 I \lambda^2 \qquad \lambda = \frac{\pi}{L} \tag{13}$$

where I is the moment of inertia of the cross section.

Two formulas provide a correction to the Euler load due to the influence of transverse shearing deformations. The first formula is the Engesser [3] formula,

$$P_{\text{Engssr}} = \frac{P_{\text{Euler}}}{1 + \beta P_{\text{Euler}}/AG} \tag{14}$$

and the second is the formula obtained by Haringx [3, 4] in connection with helical springs,

$$P_{\mathrm{Hrngre}} = \frac{\sqrt{1 + 4\beta P_{\mathrm{Euler}}/AG} - 1}{2\beta/AG} \tag{15}$$

where β is a numerical factor depending on the shape of the transverse section, A is the cross-sectional area [$= \pi(R_2^2 - R_1^2)$], and G is the shear modulus. For a tubular cross section, $\beta = 2.0$ [21].

By performing a series expansion of the terms of the resulting characteristic equation from the elasticity formulation for an isotropic column of solid circular cross section, Kardomateas [12] proved that the Euler load is the solution in the first approximation; consideration of the second approximation gave a direct expression for the correction to the Euler load, therefore defining a revised, yet simple, formula for column buckling. Although this formula was derived by considering a solid cylinder, it can be heuristically extended for the case of a hollow cylinder. In terms of

$$\tilde{\lambda} = \lambda 2\sqrt{I/A} = \frac{\pi}{L}\sqrt{\frac{R_2^4 - R_1^4}{R_2^2 - R_1^2}} \tag{16a}$$

and Poisson's ratio v_{32}, the Euler load with a second term is

$$P_{E2} = \lambda^2 E_3 I - \frac{\epsilon_2}{16(1 - v_{32}^2)} E_3 A \tag{16b}$$

where

$$\epsilon_2 = \sqrt{\Delta} - 4 - \frac{\tilde{\lambda}^2}{6}(5 + 2v_{32} + 12v_{32}^2) \tag{16c}$$

and

$$\Delta = 16 + \frac{\tilde{\lambda}^2}{3}(20 + 8v_{32} + 48v_{32}^2) + \frac{\tilde{\lambda}^4}{36}(409 + 212v_{32} - 356v_{32}^2 - 48v_{32}^3 + 144v_{32}^4) \tag{16d}$$

2.3. *Results for homogeneous orthotropic columns*

Results are produced for two common polymeric composites: namely, the mildly orthotropic glass/epoxy and the strongly orthotropic graphite/epoxy. The elastic constants of the materials are given in the tables of results, with the following notation: 1 is the radial r, 2 is the circumferential θ, and 3 is the axial (z) direction. Two reinforcing configurations are considered with each material, namely along the circumferential (θ) or along the axial (z) direction.

Regarding the glass/epoxy material, Tables 1*a* and 1*b* give the predictions of the Euler, P_{Euler}, and Engesser, P_{Engssr}, the Haringx, P_{Hrngx}, and the Euler with a second term, P_{E2}, formulas as a ratio over the elasticity solution, P_{elast}, for radii ratio $R_2/R_1 = 1.20$ and column length ratios L/R_2 ranging from 10 to 20. Tables 2*a* and 2*b* give the same data for graphite/epoxy material and Table 3 for isotropic material with Poisson's ratio $v = 0.3$. The calculations for the critical loads from these formulas are based on the axial modulus E_3.

Table 1*a*

Comparison with column buckling formulas: Glass/epoxy with axial reinforcement

L/R_2	P_{Euler}/P_{elast}	P_{Engssr}/P_{elast}	P_{Hrngx}/P_{elast}	P_{E2}/P_{elast}
10	1.598	0.870	1.036	1.502
12	1.414	0.894	1.002	1.354
14	1.304	0.914	0.986	1.263
16	1.232	0.929	0.978	1.203
18	1.183	0.941	0.976	1.161
20	1.149	0.950	0.975	1.131

$R_2/R_1 = 1.20$ ($R_2 = 1.0$ m); Moduli in GN/m^2: $E_2 = E_1 = 14$, $E_3 = 57$, $G_{31} = 5.7$, $G_{12} = 5.0$, $G_{23} = 5.7$; Poisson's ratios: $\upsilon_{12} = 0.400$, $\upsilon_{23} = 0.068$, $\upsilon_{31} = 0.277$.

One important issue is that of the relation of compression strength to buckling strength. Indeed, in practical applications, the strength in compression has to be considered in conjunction with the results on the critical load, because compressive failure may precede buckling. For example, for the graphite/epoxy with circumferential reinforcement (Table 2*b*), assuming a typical compressive strength of $\sigma_{cf} = 0.246$ GPa, the critical load P_{elast} is below the load corresponding to the compressive strength σ_{cf}, for length ratios L/R_2 beyond 12, which means that buckling would precede compressive failure. In some of the other configurations, compressive failure would precede buckling. Although this simple calculation does not take into account the complex phenomena of composite failure that could involve, among other things, the influence of layer/fiber waviness, it illustrates the importance of considering buckling in compressively loaded composite structures.

Next follows a list of conclusions, drawn from the results of Tables 1–3.

1. In all cases the elasticity solution predicts a lower value than the Euler critical load; that is, P_{Euler} is a nonconservative estimate. Moreover, the degree of nonconservatism of the Euler formula is strongly dependent on the reinforcing direction; the axially

Table 1*b*

Comparison with column buckling formulas: Glass/epoxy with circumferential reinforcement

L/R_2	P_{Euler}/P_{elast}	P_{Engssr}/P_{elast}	P_{Hrngx}/P_{elast}	P_{E2}/P_{elast}
10	1.145	0.950	0.974	1.081
12	1.100	0.963	0.976	1.057
14	1.073	0.971	0.979	1.042
16	1.056	0.977	0.982	1.032
18	1.044	0.982	0.985	1.025
20	1.035	0.985	0.987	1.020

$R_2/R_1 = 1.20$ ($R_2 = 1.0$ m); Moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$; Poisson's ratios: $\upsilon_{12} = 0.068$, $\upsilon_{23} = 0.277$, $\upsilon_{31} = 0.400$.

Table 2a

Comparison with column buckling formulas: Graphite/epoxy with axial reinforcement

L/R_2	$P_{\mathrm{Euler}}/P_{\mathrm{elast}}$	$P_{\mathrm{Engssr}}/P_{\mathrm{elast}}$	$P_{\mathrm{Hrngx}}/P_{\mathrm{elast}}$	$P_{\mathrm{E2}}/P_{\mathrm{elast}}$	
10	(3.948)	(1.061)	(1.775)	(3.711)	Buckles as a shell (2,3)†
12	(2.751)	(0.952)	(1.401)	(2.634)	Buckles as a shell (2,3)†
14	(2.023)	(0.847)	(1.136)	(1.959)	Buckles as a shell (2,4)†
16	1.774	0.860	1.078	1.731	
18	1.612	0.876	1.044	1.581	
20	1.495	0.890	1.021	1.472	

$R_2/R_1 = 1.20$ ($R_2 = 1.0$ m); Moduli in GN/m^2: $E_2 = 9.1$, $E_1 = 9.9$, $E_3 = 140.0$, $G_{31} = 4.7$, $G_{12} = 5.9$, $G_{23} = 4.3$; Poisson's ratios: $\upsilon_{12} = 0.533$, $\upsilon_{23} = 0.020$, $\upsilon_{31} = 0.283$.

†(n, m) in the general shell buckling modes:

$$u_1(r, \theta, z) = U(r)\cos n\theta \sin\frac{m\pi z}{L} \qquad v_1(r, \theta, z) = V(r)\sin n\theta \sin\frac{m\pi z}{L}$$

$$w_1(r, \theta, z) = W(r)\cos n\theta \cos\frac{m\pi z}{L}$$

Table 2b

Comparison with column buckling formulas: Graphite/epoxy with circumferential reinforcement

L/R_2	$P_{\mathrm{Euler}}/P_{\mathrm{elast}}$	$P_{\mathrm{Engssr}}/P_{\mathrm{elast}}$	$P_{\mathrm{Hrngx}}/P_{\mathrm{elast}}$	$P_{\mathrm{E2}}/P_{\mathrm{elast}}$
10	1.121	0.952	0.972	1.060
12	1.081	0.963	0.974	1.040
14	1.058	0.970	0.976	1.028
16	1.042	0.975	0.979	1.020
18	1.032	0.978	0.981	1.014
20	1.024	0.981	0.982	1.010

$R_2/R_1 = 1.20$ ($R_2 = 1.0$ m); Moduli in GN/m^2: $E_2 = 140$, $E_1 = 9.9$, $E_3 = 9.1$, $G_{31} = 5.9$, $G_{12} = 4.7$, $G_{23} = 4.3$; Poisson's ratios: $\upsilon_{12} = 0.020$, $\upsilon_{23} = 0.300$, $\upsilon_{31} = 0.490$.

Table 3

Camparison with column buckling formulas: Isotropic

L/R_2	$P_{\mathrm{Euler}}/P_{\mathrm{elast}}$	$P_{\mathrm{Engssr}}/P_{\mathrm{elast}}$	$P_{\mathrm{Hrngx}}/P_{\mathrm{elast}}$	$P_{\mathrm{E2}}/P_{\mathrm{elast}}$
10	1.137	0.934	0.960	1.068
12	1.095	0.951	0.966	1.048
14	1.069	0.963	0.972	1.036
16	1.053	0.971	0.976	1.028
18	1.042	0.976	0.980	1.022
20	1.034	0.981	0.983	1.018

$\upsilon = 0.3$, $R_2/R_1 = 1.20$ ($R_2 = 1.0$ m).

reinforced columns show the highest deviation from the elasticity value. The degree of nonconservatism of the Euler load for the circumferentially reinforced columns is much smaller and is comparable to that of isotropic columns.

2. The strongly orthotropic graphite/epoxy material show much higher deviations from the elasticity solution than the glass/epoxy in the axially reinforced configuration; however, the deviations from the elasticity solution for both the graphite/epoxy and glass/epoxy are comparable in the circumferentially reinforced case.
3. For the small length ratios (L/R_2 between 10 and 14), the graphite/epoxy with axial reinforcement buckles as a shell; this is not the case with the glass/epoxy material.
4. The Engesser shear correction formula is, in all cases examined, conservative; that is, it predicts a lower critical load than the elasticity solution.
5. The Haringx shear correction formula is, in most cases (but not always), conservative. For the isotropic case (Table 3) it is conservative. However, for a strongly orthotropic material (graphite/epoxy with axial reinforcement, Table 2*a*) or for relatively short columns (Table 1*a*) it may be nonconservative. Also, in all cases considered, the Haringx (second Timoshenko) shear correction estimate is always closer to the elasticity solution than the first one.
6. The Euler load with a second term formula, Eq. (25*b*), which is supposed to account for thickness effects, is a nonconservative estimate; it performs better than the Euler load, but in general no better than the Engesser/Haringx formulas for moderate thickness.

§3. Sandwich column

3.1. *Buckling of a sandwich column from direct formulas*

The structural geometry is of a column of length L, depth $c + 2h$, and width B. The column is of sandwich construction, symmetric about the midsurface, and the depth of the facings is h, while the depth of the core is c. The boundary conditions are (a) clamped-free-cantilever, (b) simply supported at both ends, and (c) clamped at both ends. For the composite facings, all plies have 0° orientation with respect to the column axis. The material properties are given in Table 4.

One of the closed-form solutions has been developed and was reported by Bazant and Cedolin [7]. Another sandwich column buckling formula is the expression by Huang and Kardomateas [8]. Finally, two other sandwich column buckling formulas are the ones in Allen's book [9]; one is for thin and the other for thick face sheets. Moreover, static critical conditions were also obtained by ABAQUS [16] for several configurations.

Table 4
Material properties for the sandwich columns

Material	E_{11} (kPa)	E_{22} (kPa)	ν_{12}	ν_{21}	G_{13} (kPa)
Aluminum	6.90E+07	6.90E+07	0.35	0.35	2.59E+07
Boron/epoxy	2.21E+08	2.07E+07	0.23	0.0216	5.79E+06
Kevlar/epoxy	7.59E+07	5.52E+06	0.34	0.0247	2.28E+06
Graphite/epoxy	1.81E+08	1.03E+07	0.28	0.0159	7.17E+06
Alloy/foam	4.59E+04	4.59E+04	0.33	0.33	1.72E+04
Honeycomb	3.90E+05	3.20E+04	0.25	0.0205	4.80E+04

For Bazant and Cedolin's [7] formula,

$$P_{\text{cr1}} = (EI)_a k_{\text{cr}}^2 \bigg/ \left[1 + \frac{k_{\text{cr}}^2 (EI)_b}{G_c(h+c)B}\right] \tag{17a}$$

where

$$(EI)_a = \frac{E_f}{(1-\upsilon_{12}\upsilon_{21})} B\left[\frac{h}{2}(c+h)^2 + \frac{h^3}{12}\right] \tag{17b}$$

$$(EI)_b = \frac{E_f}{(1-\upsilon_{12}\upsilon_{21})}\frac{h}{2}(c+h)^2 B \tag{17c}$$

and

$$k_{\text{cr}} = \begin{cases} \pi/2L & \text{for cantilever} \\ \pi/L & \text{for simply supported} \\ 2\pi/L & \text{for clamped} \end{cases} \tag{18}$$

Note that the symbols used herein are not exactly those used by Bazant and Cedolin [7].

Regarding the other closed-form formula for sandwich buckling, by Huang and Kardomateas [8], in addition to the previous definitions, that is, face sheets of thickness h and extensional modulus E_f, and core of thickness c, extensional modulus E_c, and shear modulus G_c, we denote by G_f the shear moduli of the face sheets. The width is uniform, B, and the total cross-sectional area is denoted by $A = B(2h + c)$.

Because the section under consideration is symmetric, the neutral surface is at the middle surface, and the equivalent flexural rigidity of the sandwich section, $(EI)_{\text{eq}}$, is

$$(EI)_{\text{eq}} = B\left[E_f\frac{h^3}{6} + 2E_f h\left(\frac{h}{2}+\frac{c}{2}\right)^2 + E_c\frac{c^3}{12}\right] \tag{19a}$$

Haung and Kardomateas [8] presented a solution for the buckling and initial post-buckling behavior of sandwich beams including transverse shear effects (for a general unsymmetric construction). The linearized differential equation for the beam is [8]

$$(EI)_{\text{eq}}\frac{d^2\theta}{dx^2} + \left(\frac{\alpha P}{A\tilde{G}} + 1\right)P\theta = 0 \tag{19b}$$

where $\tilde{G}$ is the "effective" shear modulus of the sandwich section, defined by Huang and Kardomateas [8] from a rule-of-mixtures calculation on the compliances of the constituent phases:

$$\frac{2h+c}{\tilde{G}} = \frac{2h}{G_f} + \frac{c}{G_c} \tag{19c}$$

where α is the shear correction factor, which takes into account the nonuniform distribution of shear stresses due to the sandwich construction throughout the entire cross section; it is found in the Huang and Kardomateas [8] from energy equivalency. If we define

$$a = h + \frac{c}{2} \qquad b = \frac{c}{2} \qquad d = \frac{h}{2} + \frac{c}{2} \tag{19d}$$

then the shear correction factor is found as [8]

$$\alpha = 2\tilde{G}ABe \tag{19e}$$

where

$$\begin{aligned} e = {} & \frac{E_f^2}{4(EI)_{\text{eq}}^2 G_f}\left[a^4 h - \frac{2}{3}a^2(a^3 - b^3) + \frac{1}{5}(a^5 - b^5)\right] \\ & + \frac{E_f^2}{(EI)_{\text{eq}}^2 G_c}\left[h^2 d^2 b + \frac{2}{15}\frac{E_c^2}{E_f^2}b^5 + \frac{2}{3}\frac{E_c}{E_f}hdb^3\right] \end{aligned} \tag{19f}$$

Notice that for a homogeneous part (i.e., same material for face sheets and core) it can be proved that this formula reduces to the simple and familiar value of $\alpha = 6/5$. Also, notice that the shear correction factor is given in [8] for a general unsymmetric construction (different properties of top and bottom face sheets).

Returning to Eq. (19*b*), following the usual procedure for solving for the critical load by using the general trigonometric solution of Eq. (5) and imposing the relevant boundary conditions (e.g., [2]) we can write the critical load as

$$P_{\text{cr2}} = \frac{-1 + \sqrt{1 + 4\alpha(EI)_{\text{eq}}k_{\text{cr}}^2/(A\tilde{G})}}{(2\alpha/A\tilde{G})} \tag{19g}$$

Static critical loads are computed for several facing materials, boundary conditions, and column lengths. To this end, the two closed-form expressions, Eqs. (1) and (6)–(8), are employed. Moreover, the classical (Euler) critical value was computed via the following formula:

$$P_{\text{Eul}} = k_{\text{cr}}^2(EI)_{\text{eq}} \tag{20}$$

This is shown only for comparison purposes. It is expected that Eq. (20) overestimates the critical load because it does not account for transverse shear effects, and this is seen in the results that follow. However, the question we are researching is the degree of conservatism of the Euler formula.

The last two formulas for sandwich column buckling are the ones in Allen's book [9]. The first one is for thin face sheets, as follows:

$$\frac{1}{P_{\text{cr}A1}} = \frac{1}{P_{\text{Eul}}} + \frac{c}{B(c+h)^2 G_c} \tag{21a}$$

For thick faces, Allen [9] suggests

$$P_{\text{cr}A2} = P_{\text{Eul}A}\frac{1 + \frac{P_{\text{Ef}}}{P_c} - \frac{P_{\text{Ef}}}{P_c}\frac{P_{\text{Ef}}}{P_{\text{Eul}A}}}{1 + \frac{P_{\text{Eul}A}}{P_c} - \frac{P_{\text{Ef}}}{P_c}} \tag{21b}$$

where P_{Ef} represents the sum of the Euler loads of the two faces when they buckle as independent struts; that is, when the core is absent and P_c may be described as the shear buckling load

$$P_c = BG_c\frac{(c+h)^2}{c} \qquad P_{\mathrm{Ef}} = E_f k_{\mathrm{cr}}^2 \frac{Bh^3}{6} \tag{21c}$$

The Euler load used in Allen's formulas is without the core; that is,

$$P_{\mathrm{Eul}A} = E_f k_{\mathrm{cr}}^2 B\left[\frac{h^3}{6} + \frac{h(c+h)^2}{2}\right] \tag{21d}$$

Finally, results are also generated by employing ABAQUS [16] for several geometries. For this study, eight-node brick elements were used.

3.2. *Results for sandwich columns*

Table 5 shows the effect of boundary conditions and it depicts static critical loads for $L = 2032$ mm and using various sources. There are several observations that need to be pointed out.

1. It is seen from the results of this table that the effect of transverse shear is more pronounced as we move from the cantilever column to the simply supported column to the clamped/clamped one. One reason for this is that the effective simply supported length decreases (*EI* remains constant) and transverse shear effects are more pronounced for shorter columns. It is also seen that for the same core material and geometry, and for the same thickness of the facings, the construction in going from the

Table 5
Critical loads of sandwich columns for alloy-foam core in Newtons

		Al	Boron/epoxy	Grpahitc/epoxy	Kevlar/epoxy
Cantilever	Euler	3,090	9,890	8,100	3,398
	ABAQUS	3,278	8,086	6,853	3,197
	[7]	3,205	7,803	6,645	3,127
	[8]	2,883	8,216	6,915	3,152
	Allen1	2,869	7,934	6,739	3,133
	Allen2	2,867	7,942	6,744	3,132
Simply supported	Euler	12,358	39,558	32,400	13,593
	ABAQUS	10,543	20,215	18,189	10,340
	[7]	10,148	19,021	17,195	9,952
	[8]	9,919	24,579	21,228	10,735
	Allen1	9,448	19,919	17,925	10,153
	Allen2	9,456	19,997	17,983	10,164
Clamped/clamped	Euler	49,434	158,232	129,601	54,373
	ABAQUS	22,864	37,311	36,523	25,292
	[7]	22,139	29,694	28,512	21,903
	[8]	28,829	62,227	54,893	30,817
	Allen1	22,147	32,007	30,638	23,087
	Allen2	22,246	32,422	30,971	23,201

$L = 2032$ mm, $c = 25.3$ mm, $h = 2.53$ mm, and $B = 76.2$ mm.

stronger (higher static critical load) configuration to the weaker one is: boron/epoxy, graphite/epoxy, to Kevlar/epoxy and aluminum (virtually tied). If specific buckling strength is considered, boron/epoxy and graphite/epoxy are virtually tied. They are followed by Kevlar/epoxy, which is better than aluminum.

2. The cantilever and metallic face sheet is the only case where the Euler load and the load in [7] are lower than the ABAQUS results. This is a matter of concern that needs to be investigated further. The underestimation is on the order of 6%.
3. In all other cases, although the geometry is not very demanding in terms of length and thickness, the Euler load is higher than the ABAQUS results by amounts ranging from +6% for the Kevlar/epoxy facings and cantilever case to a factor of more than 4 for the boron/epoxy and clamped/clamped case. This shows clearly that the Euler load cannot be relied on for design of sandwich columns.
4. In all cases, the formula in [8] and the two Allen formulas are below the Euler load.
5. The two formulas by Allen are in all cases close to each other but this was expected because our geometry does not feature thick face sheets; it is expected that differences would become pronounced if a construction with thicker face sheets is adopted.
6. With the exception of the metallic face sheets, the two formulas by Allen are within 5% of the results in [7]; the latter are the lowest.
7. For the composite face sheets and the simply supported and the clamped/clamped cases, the two formulas by Allen seem to be the closest to the ABAQUS results; for the cantilever case, the results in [8] are the closest to the ABAQUS results.
8. For the clamped/clamped case, the Euler load is in all cases higher than the ABAQUS results by more than a factor of 2, which again underscores the significance of transverse shear effects.
9. Results from [8] are in most cases higher than the ABAQUS results, which indicates that this formula is in general nonconservative, although it is even in some cases the most accurate.
10. The two formulas by Allen and the formula in [7] are in all cases below the ABAQUS results, which indicates that these formulas are conservative.

Table 6 gives the critical loads for the same geometry as in Table 5 but with a honeycomb core instead of alloy foam. The results in Table 6 lead to observations similar to those in Table 5. Additional observations are as follows:

11. The overestimation of the Euler load is less than in the alloy-foam core case. For example, in the clamped/clamped case and graphite/epoxy cases and in the alloy-foam case, the Euler load is higher than the ABAQUS by a factor of 3.5, and in the Honeycomb core case this factor is 1.9.
12. The formula in [8] seems to perform better in the honeycomb core case. For all cases of Kevlar/epoxy face sheets, these results [8] are the closest to the ABAQUS results (in addition to the cantilever case and all composite face sheets as in the alloy-foam core).

The effect of transverse shear is significant and can be most easily seen by comparing the results in the literature [7, 8] for the clamped/clamped case in the case of alloy-foam core and boron/epoxy face sheets. In Table 5, for a column length of 2032 mm, the result from [7] is only 18.7% of the Euler load and the result from [8] is only 39.3% of the Euler load. For a shorter column, this effect would be even more pronounced. For example, for a column 1270 mm long, the result from [7] would be only 8.2% of the Euler load and the present result from reference [8] would be only 26.8% of the Euler load.

Table 6

Critical loads of sandwich columns for Honeycomb core in Newtons

		Al	Boron/epoxy	Grpahite/epoxy	Kevlar/epoxy
Cantilever	Euler	3,111	9,911	8,121	3,419
	ABAQUS	3,475	9,256	7,696	3,386
	[7]	3,396	9,042	7,523	3,309
	[8]	3,029	9,164	7,607	3,321
	Allen1	3,027	9,105	7,572	3,318
	Allen2	3,004	9,089	7,554	3,296
Simply supported	Euler	12,443	39,643	32,485	13,678
	ABAQUS	12,690	29,581	25,458	12,400
	[7]	12,349	28,567	24,637	12,060
	[8]	11,306	31,063	26,324	12,325
	Allen1	11,199	29,277	25,180	12,189
	Allen2	11,127	29,274	25,158	12,120
Clamped/clamped	Euler	49,772	158,571	129,939	54,711
	ABAQUS	35,154	71,211	66,801	39,191
	[7]	36,229	62,076	57,125	35,603
	[8]	37,356	88,667	77,094	40,267
	Allen1	34,456	65,628	60,144	36,753
	Allen2	34,342	65,923	60,341	36,661

$L = 2032$ mm, $c = 25.3$ mm, $h = 2.53$ mm, and $B = 76.2$ mm.

Since the formula in [7] and the two formulas by Allen are based on an Engesser-type [3] derivation, whereas the critical load formula in [8] is based on a Haringx-type [4, 5] derivation, it should be mentioned at this point that the study of column buckling for monolithic composites from three-dimensional elasticity, which is outlined in the first part of this paper, showed that the Engesser formula would predict in general smaller values for the critical load, therefore is expected to be the most conservative, but not in general the most accurate, and indeed the Haringx formula results were found to be in general closer to the elasticity results. The complexity of sandwich composites notwithstanding, this conclusion is not contrary to the general observations made in this second part of the paper (with the exceptions noted in the detailed discussion of the tables).

At this point, it is important that we also refer to the recent work by Bazant [22], in which the discrepancy between Engesser-type and Haringx-type formulas is explained by the dependence of the shear modulus on the initial stresses, which is in turn influenced by the choice of the finite strain measure. In fact, the shear stiffness of the core is in general a linear function of the axial forces carried by the skins, and this function is different for stability theories associated with different strain measures. (The corresponding definitions of the shear force caused by the applied axial force are also different.) Therefore, the Engesser-type buckling formula and the Haringx-type buckling formula are, in principle, mutually equivalent, but different shear stiffness of the core must be used in each. However, as explained by Bazant [22], the Haringx-type formula represents a special case in which the shear modulus of the core can be taken as independent of the axial force in the skins and equal to the shear modulus measured in a simple test (e.g., the torsional test of a thin-walled tube made out of the core material). Therefore, the Haringx-type formula is more convenient for practice, and it is the formula where the assumption of constant shear modulus would be, strictly speaking, most appropriate. In our work, a constant shear modulus of the core is used in all buckling formulas. Thus, our general finding that the Haringx-type formula is closer to the three-dimensional elasticity results is in agreement with the conclusions reached by Bazant [22].

§4. Concluding comments

In closing, it is recommended that more work is needed in investigating the issues and sources of disagreement among the different direct formulas. There is additional concern that the loads computed by ABAQUS in some cases are above the Euler load. Further work should also focus on extending the three-dimensional elasticity formulation for monolithic composites to the sandwich construction.

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APPENDIX

For convenience, define

$$D_0 = -\frac{1}{\tilde{T}} - C_0\frac{a_{13}+a_{23}}{a_{33}} \qquad \lambda = \pi/L \tag{A1}$$

$$D_1 = -C_1\frac{a_{13}+ka_{23}}{a_{33}} \qquad D_2 = -C_2\frac{a_{13}-ka_{23}}{a_{33}} \tag{A2}$$

The coefficients of the first differential equation, Eq. (9*a*), are

$$\begin{aligned}
&b_{00} = -(c_{22}+c_{66}) \quad b_{01} = -C_0/2.0 \quad b_{02} = -C_1k/2 \quad b_{03} = C_2k/2\\
&b_{04} = -c_{55}\lambda^2 \quad b_{05} = -D_0\lambda^2/2 \quad b_{06} = -D_1\lambda^2/2 \quad b_{07} = -D_2\lambda^2/2
\end{aligned}\tag{A3}$$

$$\begin{aligned}
&d_{10} = (c_{12}+c_{66}) \quad d_{11} = -C_0/2 \quad d_{12} = -kC_1/2 \quad d_{13} = kC_2/2\\
&d_{00} = -(c_{22}+c_{66}) \quad d_{01} = -C_0/2 \quad d_{02} = -kC_1/2 \quad d_{03} = kC_2/2
\end{aligned}\tag{A4}$$

$$\begin{aligned}
&f_{10} = -\lambda(c_{13}+c_{55}) \quad f_{11} = \lambda D_0/2 \quad f_{12} = \lambda D_1/2 \quad f_{13} = \lambda D_2/2\\
&f_{00} = \lambda(c_{23}-c_{13}) \quad f_{01} = f_{02} = f_{03} = 0
\end{aligned}\tag{A5}$$

The coefficients of the second differential equation, Eq. (9*b*), are given as follows:

$$\begin{aligned}
&g_{20} = c_{66} \quad g_{21} = C_0/2 \quad g_{22} = C_1/2 \quad g_{23} = C_2/2\\
&g_{10} = c_{66} \quad g_{11} = C_0/2 \quad g_{12} = C_1/2 \quad g_{13} = C_2/2\\
&g_{00} = -(c_{22}+c_{66}) \quad g_{01} = -C_0/2 \quad g_{02} = -C_1/2 \quad g_{03} = -C_2/2\\
&g_{04} = -c_{44}\lambda^2 \quad g_{05} = -D_0\lambda^2/2 \quad g_{06} = -D_1\lambda^2/2 \quad g_{07} = -D_2\lambda^2/2
\end{aligned}\tag{A6}$$

$$\begin{aligned}
&h_{10} = -(c_{66}+c_{12}) \quad h_{11} = C_0/2 \quad h_{12} = C_1/2 \quad h_{13} = C_2/2\\
&h_{00} = -(c_{22}+c_{66}) \quad h_{01} = -C_0/2 \quad h_{02} = -C_1/2 \quad h_{03} = -C_2/2
\end{aligned}\tag{A7}$$

$$t_{00} = (c_{23}+c_{44})\lambda \quad t_{01} = -\lambda D_0/2 \quad t_{02} = -\lambda D_1/2 \quad t_{03} = -\lambda D_2/2 \tag{A8}$$

Finally, the coefficients of the third differential equation, Eq. (9*c*), are

$$\begin{aligned}
&q_{20} = c_{55} \quad q_{21} = C_0/2 \quad q_{22} = C_1/2 \quad q_{23} = C_2/2\\
&q_{10} = c_{55} \quad q_{11} = C_0/2 \quad q_{12} = kC_1/2 \quad q_{13} = -kC_2/2\\
&q_{00} = -c_{44} \quad q_{01} = -C_0 \quad q_{02} = -kC_1/2 \quad q_{03} = kC_2/2 \quad q_{04} = -c_{33}\lambda^2
\end{aligned}\tag{A9}$$

$$\begin{aligned}
&s_{10} = (c_{55}+c_{13})\lambda \quad s_{11} = -\lambda C_0/2 \quad s_{12} = -\lambda C_1/2 \quad s_{13} = -\lambda C_2/2\\
&s_{00} = (c_{23}+c_{55})\lambda \quad s_{01} = -\lambda C_0/2 \quad s_{02} = -k\lambda C_1/2 \quad s_{03} = k\lambda C_2/2
\end{aligned}\tag{A10}$$

$$\beta_{00} = (c_{23}+c_{44})\lambda \quad \beta_{01} = -\lambda C_0/2 \quad \beta_{02} = -k\lambda C_1/2 \quad \beta_{03} = k\lambda C_2/2 \tag{A11}$$