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Thermo-elastic crack branching in general anisotropic media

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Abstract

Thermal fields may exist in addition to mechanical loading, for example, due to short term exposure to fire. In this paper, the branching of cracks in the presence of combined thermal and mechanical loads is investigated for general anisotropic media by employing the theory of Stroh's dislocation formalism, extended to thermo-elasticity in matrix notation. A general solution to the thermo-elastic crack problem for an anisotropic material under arbitrary loading is obtained in a compact form. Green's functions are also presented for a thermal dislocation (heat vortex) and a conventional dislocation (or, referred as mechanical dislocation), which are formulated considering the cuts located at an arbitrary angle with respect to the x_1 axis of the coordinate system (x_1, x_2, x_3). Using the derived compact expressions, the interaction between the crack and the dislocation is studied and a closed form solution for this interaction is obtained. The branching portion of the thermo-elastic crack is modelled as a continuous distribution of dislocations. This problem is then converted into a set of singular integral equations. Numerical results are presented to illustrate the possible effects of thermal loading on the propagation of the branched crack.

Keywords: Anisotropic; Composite; Crack; Branching; Kinking; Fire; Thermal; Composite

1. Introduction

Cracks and defects in engineering structures may be introduced during manufacturing or during service, e.g. from impact loading. A local temperature gradient, which would induce a local thermal stress concentration around these defects, could be generated when such structures are exposed to the flow of heat such as from short-term fire.

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Within the last decade, the thermal stress effects on structures have drawn great interest because of the widespread application of composites or metallic materials to high speed flight engine components, aerospace structures, nuclear source energy generators, etc. All these structures often operate under extremely high temperature variations. Studies for steady thermo-elastic effects on solids with defects can be generally classified as falling into the categories of either monolithic isotropic/anisotropic media or isotropic/anisotropic bi-material media. Florence and Goodier (1963) used the method of dual integral equations to obtain a solution for isotropic infinite plane with penny-shaped insulated crack. Atkinson and Clements (1972, 1983) employed the Fourier transformation method and complex variable techniques, respectively, to study the thermo-elastic problem of monolithic anisotropic materials (Atkinson and Clements, 1972) and the thermo-elastic effects on the dissimilar anisotropic media with an assumed interface crack (Atkinson and Clements, 1983). Sturia and Barber (1988) extended Dundurs and Comninou's (1979) isotropic thermo-elastic Green's functions to simulate the crack in an anisotropic medium under thermal loading. There are also many other authors who have done extensive research on the thermo-elastic effect on cracked bodies. Among them are Hwu (1990) on an insulated elliptic hole or crack in a monolithic anisotropic medium, Herrmann and Loboda (2001) on an interface crack in anisotropic bi-materials, etc.

However, in contrast to the thermo-elastic straight crack or interface crack problem, very few papers can be found on the thermo-elastic crack branching or kinking problems due to the complicated coupling or interaction between thermal effects and mechanically loading. In our literature search, only two papers were available for this problem, one by Hasebe et al. (1986) using rational mapping to the curved crack in an isotropic infinite plate and the other by Chao and Shen (1993) using the extended Muskhelishvili's (1953) techniques to the curved interface crack of dissimilar isotropic media. It can be seen that both these papers dealt with isotropic media only and actually, it may not be possible to be directly extended to anisotropic media because of the difficulty of finding a similar rational mapping function for anisotropic media. Therefore, the crack branching problem for either monolithic anisotropic or dissimilar anisotropic bi-media under thermo-mechanical loading has not been adequately studied yet.

In this paper, an analysis of thermo-elastic crack branching for a general anisotropic material is presented based on the extended Stroh's (1958) dislocation formalism and the approaches developed by Li and Kardomateas (2001). A solution to the stress and temperature functions for arbitrary thermo-mechanical loading is derived by using the analytic continuation principle of complex functions. It should be mentioned that in the literature, the Green's functions for dislocations were usually given in the (x_1, x_2, x_3) coordinate system, i.e. the dislocation components were evaluated along the x_i $(i = 1, 2, 3)$ direction of a Cartesian coordinate system. This is apparently not convenient for the study of crack branching. In this work, the Green's functions for thermal dislocation (heat vortex) and conventional dislocation are formulated in the (r, θ, x_3) coordinate system. Thus, a closed form solution is obtained for the interaction between the crack and these dislocations. The modelling of the crack branched portion by a continuous distribution of heat vortex and mechanical dislocations leads to two sets of coupled singular integral equations in terms of the heat vortex density and the mechanical dislocation density. Numerical results for some typical materials are given to demonstrate the thermal effects on the propagation of the crack branch. It should also be mentioned that in the derivations in this paper, the dual coordinate systems are used alternatively to deal with the complexity of the crack branching problem.

2. Basic thermo-anisotropic elasticity formulas

Summarized in this section are some basic equations for thermo-anisotropic elasticity. In a fixed Cartesian coordinate system (x_1, x_2, x_3), let us consider an anisotropic elastic medium, in which the displacements u_i, the stresses σ_{ij} and the temperature fields are independent of x_3. The heat flux can be expressed as

$$h_i = -k_{ij}\frac{\partial T}{\partial x_j} \quad (i,j=1,2,3), \tag{1}$$

where $k_{ij} = k_{ji}$ are the coefficients of heat conduction.

The stress–strain law in the presence of thermal fields can be expressed in the following form:

$$\sigma_{ij} = c_{ijkl}\frac{\partial u_k}{\partial x_l} - \beta_{ij}T \quad (i,j,k,l=1,2,3), \tag{2}$$

where c_{ijkl} is the elastic moduli tensor with properties of $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ and β_{ij} are the stress-temperature coefficients; the repeated indices imply summation. Equilibrium and conservation of energy lead to

$$\sigma_{ij,j} = 0, \quad \text{i.e.,} \quad c_{ijkl}\,\frac{\partial^2 u_k}{\partial x_l \partial x_j} - \beta_{ij}\frac{\partial T}{\partial x_j} = 0, \tag{3}$$

and

$$\frac{\partial h_i}{\partial x_i} = 0, \quad \text{i.e.,} \quad k_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} = 0. \tag{4}$$

For the plane system, a non-trivial displacement and temperature distribution $T(x_1, x_2)$ solving Eqs. (3) and (4) with the corresponding stress function may be written as

$$\begin{aligned} \mathbf{u} &= \mathbf{A}\ \phi(z_\alpha) + \overline{\mathbf{A}}\,\overline{\phi(z_\alpha)} + \mathbf{C}\chi(z_\tau) + \overline{\mathbf{C}}\,\overline{\chi(z_\tau)}, \quad T(x_1,x_2) = \chi'(z_\tau) + \overline{\chi'(z_\tau)}, \\ \varphi &= \mathbf{B}\phi(z_\alpha) + \overline{\mathbf{B}}\,\overline{\phi(z_\alpha)} + \mathbf{D}\chi(z_\tau) + \overline{\mathbf{D}}\ \overline{\chi(z_\tau)}, \end{aligned} \tag{5}$$

where $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ are 3×3 matrices; $\mathbf{C}$ and $\mathbf{D}$ are 3×1 vectors; $\chi(z_\tau)$ is a scalar function and $\phi(z_\alpha)$ is a function vector such that

$$\phi(z_\alpha) = \langle\langle \mathbf{f}(z_\alpha)\rangle\rangle\mathbf{q}; \quad \langle\langle \mathbf{f}(z_\alpha)\rangle\rangle = \mathrm{diag}[\mathbf{f}(z_1), \mathbf{f}(z_2), \mathbf{f}(z_3)], \tag{6}$$

in which, $\mathbf{f}(z_\alpha)$ and $\mathbf{q}$, respectively, are unknown functions and constants to be determined for a given problem; $z_\alpha = x_1 + p_\alpha x_2 (\alpha = 1,2,3)$ and $z_\tau = x_1 + \tau x_2$; the overbar $\overline{(\ )}$ denotes the conjugate of a complex variable, the prime ($'$) denotes differentiation with respect to z_α or z_τ. The constant τ is the root with positive imaginary part of the equation

$$k_{22}\tau^2 + 2k_{12}\tau + k_{11} = 0. \tag{7}$$

The p_α, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constants which satisfy the following equations:

$$N\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix} = p\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}, \quad N = \begin{bmatrix}N_1 & N_2\\ N_3 & N_1^{\mathrm{T}}\end{bmatrix}, \quad N\begin{bmatrix}\mathbf{c}\\ \mathbf{d}\end{bmatrix} = \tau\begin{bmatrix}\mathbf{c}\\ \mathbf{d}\end{bmatrix} - \begin{bmatrix}0 & N_2\\ I & N_1^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}, \tag{8}$$

where $N_1 = -T^{-1}R^{\mathrm{T}}$, $N_2 = T^{-1}$, $N_3 = RT^{-1}R^{\mathrm{T}} - Q$; the superscript "T" stands for the transpose of a matrix and

$$(\beta_{\mathbf{1}})_i = \beta_{i1}, \quad (\beta_{\mathbf{2}})_i = \beta_{i2} \quad Q_{ik} = c_{i1k1}, \quad R_{ik} = c_{i1k2}, \quad \mathrm{T}_{ik} = c_{i2k2}. \tag{9}$$

The stresses can be written in term of stress functions (Stroh, 1958) as

$$\sigma_{i1} = -\frac{\partial \varphi_i}{\partial x_2}, \quad \sigma_{i2} = \frac{\partial \varphi_i}{\partial x_1}. \tag{10}$$

The heat flux then becomes

$$h_i = -(k_{i1} + \tau k_{i2})\chi''(z_\tau) - (k_{i1} + \bar{\tau}k_{i2})\overline{\chi''(z_\tau)}. \tag{11}$$

Let $k = k_{22}(\tau - \bar{\tau})/2\mathrm{i}$, then $k = \sqrt{k_{11}k_{22} - k_{12}^2}$ and

$$h_1 = \mathrm{i}k\tau\chi''(z_\tau) - \mathrm{i}\overline{k\tau\chi''(z_\tau)}, \quad h_2 = -\mathrm{i}k\chi''(z_\tau) + \mathrm{i}\overline{k\chi''(z_\tau)}. \tag{12}$$

Next, we define three matrices as

$$H = 2\mathrm{i}AA^{\mathrm{T}}, \quad L = -2\mathrm{i}BB^{\mathrm{T}}, \quad S = \mathrm{i}(2AB^{\mathrm{T}} - I), \tag{13}$$

which are real as shown in Stroh (1958); also $I = \mathrm{diag}[1, 1, 1]$ is a unit matrix.

From the orthogonality of the eigenvectors of Eq. (8), it is easy to verify the following identity:

$$\begin{bmatrix} B^{\mathrm{T}} & A^{\mathrm{T}} \\ \overline{B}^{\mathrm{T}} & \overline{A}^{\mathrm{T}} \end{bmatrix} \times \begin{bmatrix} A & \overline{A} \\ B & \overline{B} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{14}$$

3. The general solution to the thermo-elastic crack in an anisotropic material

In this section, a solution as well as the method leading to the solution, for a crack in an anisotropic medium under a combined thermal and mechanical loading, is presented in detail. And it can be seen that the general solution given here, lays the foundation for the study of the thermo-elastic crack branching phenomena (Fig. 1).

Let us assume a crack to be located in the region $a \leqslant x_1 \leqslant b$, $-\infty < x_3 < \infty$ of the plane $x_2 = 0$. A heat flux h_0 and σ_{ij}^{∞} is applied at infinity. By the superposition principle, the boundary conditions for this problem can be written as

$$\begin{aligned}
&h_2(x_1, x_2 = 0^+) = -h(x_1); \quad h_2(x_1, x_2 = 0^-) = -h(x_1), \quad a \leqslant x_1 \leqslant b, \ x_2 = 0, \\
&\sigma_{i2}(x_1, x_2 = 0^+) = -\sigma_{i2}^{\infty}(x_1) = -p(x_1), \quad a \leqslant x_1 \leqslant b, \ x_2 = 0, \\
&\sigma_{i2}(x_1, x_2 = 0^-) = -\sigma_{i2}^{\infty}(x_1) = -p(x_1), \quad a \leqslant x_1 \leqslant b, \ x_2 = 0, \\
&h_2 = 0, \quad \sigma_{ij} = 0 \quad \text{at infinity.}
\end{aligned} \tag{15}$$

If we denote $\chi''(x_1, x_2) = \chi''_{\pm}(x_1)$ as $x_2 \to \pm 0$, then substitution of Eq. (11) in Eqs. $(15)_1$ and $(15)_2$ yields

$$-\mathrm{i}k\chi''_{+}(x_1) + \mathrm{i}k\overline{\chi}''_{-}(x_1) = -h(x_1), \quad -\mathrm{i}k\chi''_{-}(x_1) + \mathrm{i}k\overline{\chi}''_{+}(x_1) = -h(x_1), \quad a \leqslant x_1 \leqslant b. \tag{16}$$

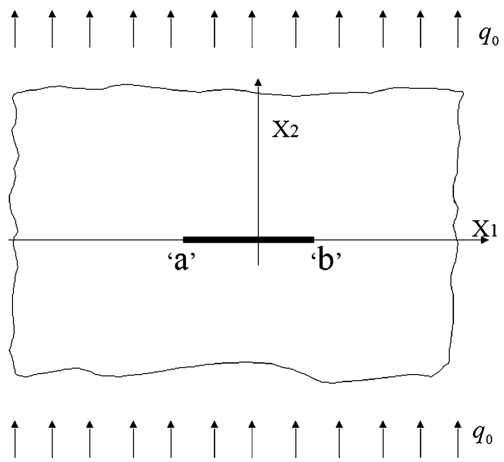

Fig. 1. A thermo-elastic crack in an anisotropic medium.

Subtraction of $(16)_1$ from $(16)_2$ leads to

$$\chi''_+(x_1) + \overline{\chi}''_+(x_1) = \chi''_-(x_1) + \overline{\chi}''_-(x_1), \quad a \leqslant x_1 \leqslant b. \tag{17}$$

This equation implies that if we define a function

$$\Theta(z) = \chi''(z) + \overline{\chi}''(z), \tag{18}$$

which is analytical in $x_2 > 0$ and $x_2 < 0$, respectively, then, it is continuous across the plane $x_2 = 0$. Thus this function is continuous in the whole space. From the principle of analytic continuation (Muskhelishvili, 1992), $\Theta(z)$ is analytical in the entire space. Since $\Theta(z)$ is finite at infinity, by the Liouville theorem (Rudin, 1987) we conclude that

$$\Theta(z) = 0 \quad \text{for all } z. \tag{19}$$

Therefore, the summation of $(16)_1$ and $(16)_2$ gives

$$k[\chi''_+(x_1) + \chi''_-(x_1)] = -\mathrm{i}h(x_1), \quad a \leqslant x_1 \leqslant b. \tag{20}$$

A general solution to this equation may take the following form (Muskhelishvili, 1953):

$$\chi''(z) = \frac{\mathbf{x}(z)}{2\pi\mathrm{i}} \int_a^b \frac{\mathbf{x}_+^{-1}(x_1)}{x_1 - z} \frac{(-\mathrm{i})h(x_1)}{k} \,\mathrm{d}x_1 = \frac{1}{2\pi\mathrm{i}k\sqrt{(z-a)(z-b)}} \int_a^b \frac{\sqrt{(x_1-a)(b-x_1)}}{x_1 - z} h(x_1)\,\mathrm{d}x_1, \tag{21}$$

where Eqs. (18) and (19) were used for $x_2 \to \pm 0$, and

$$\mathbf{x}(z) = \frac{1}{\sqrt{(z-a)(z-b)}}. \tag{22}$$

For constant heat flux, $h(x_1) = h_0$, the above solution becomes

$$\chi''(z) = \frac{h_0}{\mathrm{i}2k}\left[1 - \frac{z-(a+b)/2}{\sqrt{(z-a)(z-b)}}\right]. \tag{23}$$

Integrating this equation gives

$$\chi'(z) = \frac{h_0}{\mathrm{i}2k}[z - \sqrt{(z-a)(z-b)}],$$

$$\chi(z) = \frac{h_0}{\mathrm{i}4k}\left[z^2 - \left(z - \frac{a+b}{2}\right)\sqrt{(z-a)(z-b)} - \left(\frac{a-b}{2}\right)^2 \log\left(z - \frac{(a+b)}{2} + \sqrt{(z-a)(z-b)}\right)\right], \tag{24}$$

where a constant which plays no role in the sequel section, was omitted. The corresponding temperature field is

$$T(x_1, x_2) = 2\mathrm{Re}[\chi'(z)] = \frac{h_0}{k}\mathrm{Im}\left[z - \sqrt{(z-a)(z-b)}\right]. \tag{25}$$

Next, from the boundary conditions $(15)_{3,4}$, we obtain

$$\begin{aligned} &B\phi'_+(x_1) + \overline{B}\,\overline{\phi}'_-(x_1) + D\chi'_+(x_1) + \overline{D}\overline{\chi}'_-(x_1) = -p(x_1), \\ &B\phi'_-(x_1) + \overline{B}\,\overline{\phi}'_+(x_1) + D\chi'_-(x_1) + \overline{D}\overline{\chi}'_+(x_1) = -p(x_1), \quad a \leqslant x_1 \leqslant b. \end{aligned} \tag{26}$$

Subtraction of $(26)_2$ from $(26)_1$ yields

$$B\phi'_+(x_1) - \overline{B}\,\overline{\phi}'_+(x_1) + D\chi'_+(x_1) - \overline{D}\overline{\chi}'_+(x_1) = B\phi'_-(x_1) - \overline{B}\,\overline{\phi}'_-(x_1) + D\chi'_-(x_1) - \overline{D}\overline{\chi}'_-(x_1). \tag{27}$$

If define a function

$$\widetilde{\Theta}(z) = B\phi'(z) - \overline{B}\,\overline{\phi}'(z) + D\chi'(z) - \overline{D}\overline{\chi}'(z), \tag{28}$$

which automatically satisfies the condition (27) and is analytical in the plane $x_1 > 0$ and $x_1 < 0$, respectively, then by an argument similar to the one in obtaining Eq. (19), the application of the principle of analytic continuation leads to

$$\widetilde{\Theta}(z) = 0 \quad \text{for all } z. \tag{29}$$

Then, one can have the following conditions:

$$B\phi'_+(x_1) + D\chi'_+(x_1) = \overline{B}\,\overline{\phi}'_+(x_1) + \overline{D}\overline{\chi}'_+(x_1), \quad B\phi'_-(x_1) + D\chi'_-(x_1) = \overline{B}\,\overline{\phi}'_-(x_1) + \overline{D}\overline{\chi}'_-(x_1). \tag{30}$$

Summation of $(26)_1$ and $(26)_2$ and making use of (30) leads to

$$B[\phi'_+(x_1) + \phi'_-(x_1)] + D[\chi'_+(x_1) + \chi'_-(x_1)] = -p(x_1), \quad a \leqslant x_1 \leqslant b. \tag{31}$$

Once the function $\chi'(z)$ is known, Eq. (31) can be solved since B is non-singular as can be seen from Section 2. A solution which vanishes at infinity may be written as

$$\phi'(z) = \frac{X(z)}{2\pi i} \int_a^b \frac{X_+^{-1}(x_1)}{x_1 - z} g(x_1)\,dx_1, \tag{32}$$

where

$$X(z_\alpha) = \left\langle\left\langle \frac{1}{\sqrt{(z_\alpha - a)(z_\alpha - b)}} \right\rangle\right\rangle, \quad g(x_1) = -B^{-1}[p(x_1)] + D[\chi'_+(x_1) + \chi'_-(x_1)], \quad a \leqslant x_1 \leqslant b. \tag{33}$$

If the applied thermal and mechanical loadings $p(x_1) = p_0$ and $h(x_1) = h_0$ are constants, then a closed form expression can be obtained:

$$\phi'(z) = -\frac{1}{2}\Pi(z)B^{-1}p_0 - \Xi(z)B^{-1}D\frac{h_0}{2ki}, \tag{34}$$

where

$$\Pi(z_\alpha) = I - \left\langle\left\langle \frac{z_\alpha - (a+b)/2}{\sqrt{(z_\alpha - a)(z_\alpha - b)}} \right\rangle\right\rangle, \quad \Xi(z_\alpha) = \langle\langle z_\alpha \rangle\rangle - \left\langle\left\langle \frac{z_\alpha^2 - z_\alpha(a+b)/2 - (a-b)^2/8}{\sqrt{(z_\alpha - a)(z_\alpha - b)}} \right\rangle\right\rangle. \tag{35}$$

Integration of (34) gives

$$\begin{aligned} \phi(z) = &-\frac{1}{2}\left\langle\left\langle z_\alpha - \sqrt{(z_\alpha - a)(z_\alpha - b)} \right\rangle\right\rangle B^{-1}p_0 \\ &- \left\langle\left\langle z_\alpha^2 - \left(z_\alpha + \frac{(a+b)}{2}\right)\sqrt{(z_\alpha - a)(z_\alpha - b)} \right\rangle\right\rangle B^{-1}D\frac{h_0}{4ki}. \end{aligned} \tag{36}$$

In general, the heat flux function and the stress functions can be found from Eqs. (21) and (32) provided the boundary conditions are known. In particular, with Eqs. (23), (24) and (34), the explicit solutions for the heat flux and stresses can be determined at any point in the cracked elastic solid under constant combined thermo-mechanical applied loading. For the convenience of sequel derivation, let us introduce a cylindrical coordinate system (r, θ, x_3) and denote

$$\mu_{(\theta)} = \cos\theta + p_\alpha \sin\theta, \quad \lambda_{(\theta)} = \cos\theta + \tau\ \sin\theta. \tag{37}$$

Then

$$z_\alpha = x_1 + p_\alpha x_2 = r\mu_{(\theta)}, \quad z_\tau = x_1 + \tau x_2 = r\lambda_{(\theta)}, \quad r = \sqrt{x_1^2 + x_2^2}. \tag{38}$$

Substitution of Eq. (23) into (12) leads to a heat flux normal to the θ plane as

$$h_\theta^c(r,\theta) = 2k\mathrm{Re}[-\mathrm{i}\lambda_{(\theta)}\chi''(r\lambda_{(\theta)})] = -h_0\,\mathrm{Re}\left[\lambda_{(\theta)}\left(1-\frac{r\lambda_{(\theta)}-(a+b)/2}{\sqrt{(r\lambda_{(\theta)}-a)(r\lambda_{(\theta)}-b)}}\right)\right]. \tag{39}$$

From Eqs. $(24)_1$, (34) and (10), the following expressions can be obtained for the tractions on the θ plane:

$$\begin{aligned}\mathbf{t}_\theta^c(r,\theta) &= \frac{\partial\varphi}{\partial r} = 2\mathrm{Re}[B\phi'\mu_{(\theta)} + D\chi'\lambda_{(\theta)}]\\ &= -\mathrm{Re}[B\Pi(r\mu_{(\theta)})\langle\langle\mu_{(\theta)}\rangle\rangle B^{-1}]p_0 - \frac{h_0}{k}\mathrm{Im}[B\Xi(r\mu_{(\theta)})\langle\langle\mu_{(\theta)}\rangle\rangle B^{-1}D]\\ &\quad + \frac{h_0}{k}\mathrm{Im}\left[\lambda_{(\theta)}D\left(r\lambda_{(\theta)}-\sqrt{(r\lambda_{(\theta)}-a)(r\lambda_{(\theta)}-b)}\right)\right],\end{aligned} \tag{40}$$

where the superscript "c" denotes the corresponding values induced by the main crack.

The stress components on the θ plane can be calculated from the tractions as

$$[\sigma_{\theta\theta},\sigma_{r\theta},\sigma_{3\theta}]^{\mathrm{T}} = \Omega_\theta^{\mathrm{T}}\mathbf{t}_\theta^c(r,\theta), \tag{41}$$

where

$$\Omega_{(\theta)} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{42}$$

4. The thermo-elastic interaction between the crack and a dislocation in an anisotropic medium

The introduction of a dislocation into an elastic medium under thermal loading may cause a discontinuity of temperature across the cut plane of the conventional (or mechanical) dislocation. This temperature discontinuity has been referred to as thermal dislocation [or heat vortex (Dundurs and Comninou, 1979)]. If a crack already exists within the medium, then there will be an interaction between this crack and the dislocation. In this section, a solution for this interaction will be presented in closed form.

For the convenience of derivation, let a new coordinate system (ξ,η,x_3) be introduced by rotating the original coordinate system by an angle ω with respect to the x_3 axis, and another associated cylindrical coordinate system (r,ϑ,x_3) with $r=\sqrt{\xi^2+\eta^2}$ and ϑ measured from $\xi>0$, $\eta=0$. Define

$$\zeta_t = r\widetilde{\lambda}_\vartheta,\quad \widetilde{\lambda}_\vartheta = \cos\vartheta + \widetilde{\tau}\sin\vartheta;\qquad \zeta_\alpha = r\widetilde{\mu}_\vartheta,\quad \widetilde{\mu}_\vartheta = \cos\vartheta + \widetilde{p}_\alpha\sin\vartheta. \tag{43}$$

4.1. A thermo-elastic Green's function for the anisotropic body

The temperature discontinuity can be modelled by the thermal analog of a mechanical dislocation. The Green's functions for the thermo-elastic field (heat vortex) in isotropic media was addressed by Dundurs and Comninou (1979). Sturia and Barber (1988) extended these functions to the case of an anisotropic medium. But all these functions were derived for the cut along the x_1 axis of the (x_1,x_2,x_3) coordinate system, which may restrict the application of these Green's functions. Here, in this section, a new thermo-elastic Green's function form for a more general case will be derived. This function is proved to be more suitable for solving complicated fracture mechanics configurations, as the crack branching problem addressed in this paper.

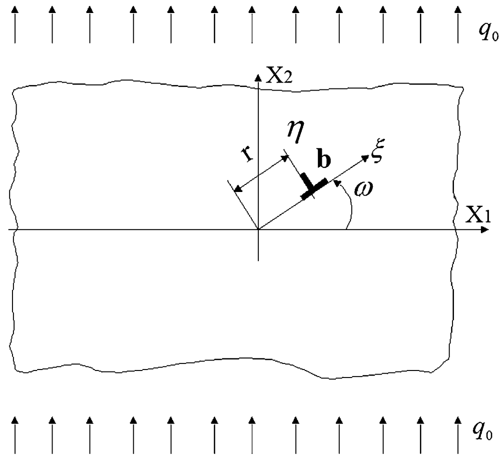

Fig. 2. A thermo-elastic dislocation in an anisotropic medium.

A cut is assumed to be located at $\xi = \xi_0$ along the plane ($\xi < \xi_0, \eta = 0$) (Fig. 2) and a constant temperature discontinuity exists along this cut, i.e.

$$T(\xi, \eta = 0^+) - T(\xi, \eta = 0^-) = T_0, \quad \xi - \xi_0 < 0. \tag{44}$$

A function satisfying this condition may take the form:

$$T(\xi, \eta) = \frac{T_0}{4\pi i}[\log(\zeta - \xi_0) - \log(\overline{\zeta} - \xi_0)], \quad \zeta = \xi + \widetilde{\tau}\eta. \tag{45}$$

Therefore, the heat flux and stress fields induced by the temperature discontinuity on the plane $\eta = 0$ can be written, respectively, as

$$\widetilde{h}_2^t(\xi, 0) = -\frac{k}{2\pi}\frac{T_0}{\xi - \xi_0} \tag{46}$$

and

$$\widetilde{t}_2^t(\xi, 0) = [\sigma_{\xi\eta}, \sigma_{\eta\eta}, \sigma_{3\eta}]^{\mathrm{T}} = \frac{T_0}{2\pi}\mathrm{Im}[\widetilde{D}\log(\xi - \xi_0)], \tag{47}$$

where the superscript "t" denotes the values induced by the heat vortex.

The corresponding heat flux and traction on the ϑ plane can be written, respectively, as

$$\widetilde{h}_\vartheta^t(r, \vartheta) = -\frac{k}{2\pi}\mathrm{Re}\left[\frac{T_0}{r - \xi_0/\widetilde{\lambda}_{(\vartheta)}}\right] \tag{48}$$

and

$$\widetilde{t}_\vartheta^t(r, \vartheta) = \frac{T_0}{2\pi}\mathrm{Im}[\widetilde{D}\widetilde{\lambda}_{(\vartheta)}(\log(r - \xi_0/\widetilde{\lambda}_{(\vartheta)}) + \log(\widetilde{\lambda}_{(\vartheta)}))]. \tag{49}$$

One may also obtain the heat flux and traction on the plane $\vartheta = \pi - \omega$ [i.e., $x_2 = 0$ in the coordinate system (x_1, x_2, x_3)], respectively, as

$$h_2^t(x_1) = -\frac{k}{2\pi}\mathrm{Re}\left[\frac{T_0}{-x_1 - \xi_0/\widetilde{\lambda}_{(\pi-\omega)}}\right], \tag{50a}$$

$$[\sigma_{12}^t(x_1), \sigma_{22}^t(x_1), \sigma_{32}^t(x_1)]^{\mathrm{T}} = \frac{T_0}{2\pi}\Omega_{(\pi-\omega)}^{\mathrm{T}}\mathrm{Im}[\widetilde{D}\widetilde{\lambda}_{(\pi-\omega)}(\log(-x_1-\xi_0/\widetilde{\lambda}_{(\pi-\omega)})+\log\widetilde{\lambda}_{(\pi-\omega)})], \quad x_1<0. \tag{50b}$$

4.2. A dislocation in an anisotropic medium

The properties of a dislocation for anisotropic media in a fixed coordinate system (x_1, x_2, x_3) was first investigated by Eshelby et al. (1953). In this section, we shall give the properties of a dislocation $\mathbf{b}$ cut along the angle $\theta = \omega$ (see Fig. 2). It should be mentioned that the components of the edge dislocation vector thus formulated are $\mathbf{b} = [b_{\theta\theta}, b_{r\theta}, b_{3\theta}]^{\mathrm{T}}$, which are reduced to $\mathbf{b} = [b_1, b_2, b_3]^{\mathrm{T}}$ only if $\omega = 0$.

Let the cut be the same as the one in the formation of the heat vortex, i.e. it is located at $\xi = \xi_0$ on the plane $\xi - \xi_0 < 0$, $\eta = 0$ in the coordinate system (ξ, η, x_3). The displacement and stress functions can be written in the form

$$\mathbf{u}^d = \widetilde{A}\langle\langle\log(\zeta_\alpha - \xi_0)\rangle\rangle q_0 + \overline{\widetilde{A}}\langle\langle\log(\overline{\zeta}_\alpha - \xi_0)\rangle\rangle\bar{q}_0, \tag{51a}$$

$$\phi^d = \widetilde{B}\langle\langle\log(\zeta_\alpha - \xi_0)\rangle\rangle q_0 + \overline{\widetilde{B}}\langle\langle\log(\overline{\zeta}_\alpha - \xi_0)\rangle\rangle\bar{q}_0, \tag{51b}$$

where the superscript "d" denotes the values induced by the mechanical dislocation; the "~" denotes the values in the (ξ, η, x_3) coordinate system.

Substituting the above equations into the conditions,

$$\mathbf{u}^d(\pi) - \mathbf{u}^d(-\pi) = \mathbf{b}, \quad \phi(\pi) - \phi(-\pi) = 0, \tag{52}$$

one can obtain

$$q_0 = -\frac{\mathrm{i}}{2\pi}\widetilde{B}^{\mathrm{T}}\mathbf{b}, \tag{53}$$

where the identity (14) is used.

The traction on the plane $\eta = 0$ can be written as

$$\mathbf{t}_2^d(\xi, 0) = [\sigma_{\xi\eta}, \sigma_{\eta\eta}, \sigma_{3\eta}]^{\mathrm{T}} = \frac{\widetilde{L}}{2\pi}\frac{\mathbf{b}}{\xi - \xi_0}. \tag{54}$$

In turn, the traction on the ϑ plane of the (r, ϑ, x_3) system with respect to the coordinate system (ξ, η, x_3) is

$$\mathbf{t}_\vartheta^d = \frac{1}{2\pi}\mathrm{Re}\left[-2\mathrm{i}\widetilde{B}\left\langle\left\langle\frac{1}{r - \xi_0/\widetilde{\mu}_{(\vartheta)}}\right\rangle\right\rangle\widetilde{B}^{\mathrm{T}}\right]\mathbf{b}, \tag{55}$$

where

$$r = \sqrt{\xi^2 + \eta^2}, \quad \widetilde{\mu}_\vartheta = \cos\vartheta + \widetilde{p}_\alpha\sin\vartheta. \tag{56}$$

In particular, for the plane $x_2 = 0$ of the (x_1, x_2, x_3) coordinate system, i.e. $\vartheta = \pi - \omega$, we obtain

$$[\sigma_{12}^d(x_1), \sigma_{22}^d(x_1), \sigma_{32}^d(x_1)]^{\mathrm{T}} = \frac{\Omega_{(\pi-\omega)}^{\mathrm{T}}}{2\pi}\mathrm{Re}\left[-2\mathrm{i}\widetilde{B}\left\langle\left\langle\frac{1}{-x_1 - \xi_0/\widetilde{\mu}_{(\pi-\omega)}}\right\rangle\right\rangle\widetilde{B}^{\mathrm{T}}\right]\mathbf{b}, \quad x_1 < 0. \tag{57}$$

4.3. The thermo-elastic interaction between the crack and a dislocation

The following useful identities can be easily verified:

$$\widetilde{\lambda}_{(\pi-\omega)} = -\lambda_{(\omega)}^{-1}, \quad \widetilde{\mu}_{(\pi-\omega)} = -\mu_{(\omega)}^{-1}, \tag{58}$$

where

$$\widetilde{\lambda} = \frac{\lambda\cos\omega - \sin\omega}{\cos\omega + \lambda\sin\omega}, \quad \widetilde{\mu} = \frac{\mu\cos\omega - \sin\omega}{\cos\omega + \mu\sin\omega}. \tag{59}$$

Then, Eqs. (50) and (57) can be rewritten as

$$h_2^t(x_1) = \frac{k}{2\pi}\mathrm{Re}\left[\frac{1}{x_1 - \xi_{0t}}\right]T_0, \tag{60a}$$

$$[\sigma_{12}^t(x_1), \sigma_{22}^t(x_1), \sigma_{32}^t(x_1)]^{\mathrm{T}} = -\frac{\Omega_{(\pi-\omega)}^{\mathrm{T}}}{2\pi}\mathrm{Im}[\widetilde{D}\lambda_{(\omega)}^{-1}(\log(x_1 - \xi_{0t}) + \pi - \log\lambda_{(\omega)})]T_0, \tag{60b}$$

$$\begin{aligned}[\sigma_{12}^d(x_1), \sigma_{22}^d(x_1), \sigma_{32}^d(x_1)]^{\mathrm{T}} &= -\frac{\Omega_{(\pi-\omega)}^{\mathrm{T}}}{2\pi}\mathrm{Im}\left[\widetilde{B}\left\langle\left\langle\frac{2}{x_1 - \xi_{0d}}\right\rangle\right\rangle\widetilde{B}^{\mathrm{T}}\right]\mathbf{b} \\ &= -\frac{\Omega_{(\pi-\omega)}^{\mathrm{T}}}{2\pi}\sum_{k=1}^{3}\mathrm{Im}\left[\widetilde{B}I_k\widetilde{B}^{\mathrm{T}}\frac{2}{x_1 - \xi_{0d}^k}\right]\mathbf{b},\end{aligned} \tag{60c}$$

where

$$\xi_{0t} = \xi_0\lambda_{(\omega)}, \quad \xi_{0d} = \xi_0\mu_{(\omega)}, \quad \xi_{0d}^k = \xi_0\mu_{(\omega)}^k, \quad \widetilde{\mu}_{(\vartheta)}^k = \cos\vartheta + \widetilde{p}_k\sin\vartheta \quad (k = 1, 2, 3) \tag{61}$$

and $I_1 = \mathrm{diag}[1, 0, 0]$, $I_2 = \mathrm{diag}[0, 1, 0]$, $I_3 = \mathrm{diag}[0, 0, 1]$.

Substitution of (60a) into (21) and using a contour integral technique (Muskhelishvili, 1953) yields

$$\begin{aligned}\chi_{\mathrm{int}}''(z) = -\frac{T_0}{4\pi\mathrm{i}}\Bigg[&\frac{1}{\sqrt{(z-a)(z-b)}} + \frac{1}{2}\Bigg(\frac{\sqrt{(\xi_{0t}-a)(\xi_{0t}-b)}}{\sqrt{(z-a)(z-b)}}\frac{1}{z-\xi_{0t}} \\ &- \frac{1}{z-\xi_{0t}} + \frac{\sqrt{(\bar{\xi}_{0t}-a)(\bar{\xi}_{0t}-b)}}{\sqrt{(z-a)(z-b)}}\frac{1}{z-\bar{\xi}_{0t}} - \frac{1}{z-\bar{\xi}_{0t}}\Bigg)\Bigg].\end{aligned} \tag{62}$$

The subscript "int" means interaction functions for stress, displacement and temperature distribution fields. Integration of the above Eq. (62) gives

$$\chi_{\mathrm{int}}'(z) = -\frac{T_0}{4\pi\mathrm{i}}B(z, \xi_{0t}), \tag{63}$$

with

$$\begin{aligned}B(z, \xi_{0t}) = \log&\left[z - \frac{(a+b)}{2} + \sqrt{(z-a)(z-b)}\right] \\ &- \frac{1}{2}\Bigg(\log\left[\sqrt{(z-a)(z-b)} + \sqrt{(\xi_{0t}-a)(\xi_{0t}-b)} + \frac{\xi_{0t} - (a+b)/2}{\sqrt{(\xi_{0t}-a)(\xi_{0t}-b)}}(z-\xi_{0t})\right] \\ &+ \log\left[\sqrt{(z-a)(z-b)} + \sqrt{(\bar{\xi}_{0t}-a)(\bar{\xi}_{0t}-b)} + \frac{\bar{\xi}_{0t} - (a+b)/2}{\sqrt{(\bar{\xi}_{0t}-a)(\bar{\xi}_{0t}-b)}}(z-\bar{\xi}_{0t})\right]\Bigg),\end{aligned} \tag{64}$$

where a constant which plays no role in the sequel section was not considered.

Similarly, by substituting Eqs. (60b,c) and (63) into Eqs. (32) and (33), the interaction stress function vectors can be obtained as

$$\phi'_{\rm int}(z) = \sum_{k=1}^{3}\left[\mathbf{Y}(z,\xi^k_{0d})\widetilde{B}I_k\widetilde{B}^{\rm T} - \mathbf{Y}(z,\bar{\xi}^k_{0d})\overline{\widetilde{B}}I_k\overline{\widetilde{B}}^{\rm T}\right]\mathbf{b} + \left[E(z) + \mathbf{y}(z,\xi_{0t})\frac{B^{-1}\Omega^{\rm T}_{(\pi-\omega)}}{8\pi{\rm i}}\widetilde{D}\lambda^{-1}_{(\omega)}\right.$$
$$\left. -\mathbf{y}(z,\bar{\xi}_{0t})\frac{B^{-1}\Omega^{\rm T}_{(\pi-\omega)}}{8\pi{\rm i}}\overline{\widetilde{D}}\bar{\lambda}^{-1}_{(\omega)}\right]T_0 + \left[F(z,\xi_{0t}) + F(z,\bar{\xi}_{0t})\right]\frac{-B^{-1}D}{4\pi{\rm i}}T_0, \tag{65}$$

where

$$\mathbf{Y}(z,\xi^k_{0d}) = \left\langle\left\langle\frac{1}{\sqrt{(z-a)(z-b)}} + \sqrt{\frac{(\xi^k_{0d}-a)(\xi^k_{0d}-b)}{(z-a)(z-b)}}\frac{1}{z-\xi^k_{0d}} - \frac{1}{z-\xi^k_{0d}}\right\rangle\right\rangle\frac{B^{-1}\Omega^{\rm T}_{(\pi-\omega)}{\rm i}}{4\pi}, \tag{66a}$$

$$E(z) = \left\langle\left\langle 1 - \frac{z-(a+b)/2}{\sqrt{(z-a)(z-b)}}\right\rangle\right\rangle\frac{B^{-1}\Omega^{\rm T}_{(\pi-\omega)}}{4\pi}{\rm Im}[\widetilde{D}\lambda^{-1}_{(\omega)}(\pi - \log\lambda_{(\omega)})], \tag{66b}$$

$$\mathbf{y}(z,\xi_{0t}) = \left\langle\left\langle 1 - \frac{z-(a+b)/2 + \sqrt{(\xi_{0t}-a)(\xi_{0t}-b)}}{\sqrt{(z-a)(z-b)}}\right\rangle\right\rangle\log(z-\xi_{0t}), \tag{66c}$$

$$F(z,\xi_{0t}) = \mathbf{y}(z,\xi_{0t}) - \left\langle\left\langle 1 - \frac{z-(a+b)/2}{\sqrt{(z-a)(z-b)}}\right\rangle\right\rangle\log\sqrt{(\xi_{0t}-a)(\xi_{ot}-b)}. \tag{66d}$$

Therefore, the heat flux on the θ plane due to the interaction can be calculated by substituting Eq. (62) into (39), and it reads

$$h^{\rm int}_\theta(r\lambda_{(\theta)}) = 2k{\rm Re}[-{\rm i}\lambda_{(\theta)}\chi''_{\rm int}(r\lambda_{(\theta)})]. \tag{67}$$

Moreover, from Eqs. (65) and (10), the following expression can be written for the traction on this plane:

$$t^{\rm int}_\theta = 2\ {\rm Re}\left[B\phi'_{\rm int}(r,\theta)\frac{\partial z}{\partial r} + D\chi'_{\rm int}(r,\theta)\lambda_{(\theta)}\right]. \tag{68}$$

5. Thermo-elastic crack branching in anisotropic materials

The main crack, located at the $a < x_1 < b$, $x_2 = 0$, is assumed to branch into $x_2 > 0$ (or $x_2 < 0$) at an angle $\theta = \omega$, as shown in Fig. 3. Similar to the conditions for the main crack, the boundary conditions for this branched portion read in the coordinate system (ξ, η, x_3) as

$$h_2(\xi,0^+) = -h(\xi), \quad h_2(\xi,0^-) = -h(\xi), \tag{69a}$$

$$\begin{aligned}&[\sigma_{\xi\eta}(\xi,0^+),\sigma_{\eta\eta}(\xi,0^+),\sigma_{3\eta}(\xi,0^+)]^{\rm T} = -p(\xi),\\ &[\sigma_{\xi\eta}(\xi,0^-),\sigma_{\eta\eta}(\xi,0^-),\sigma_{3\eta}(\xi,0^-)]^{\rm T} = -p(\xi).\end{aligned} \tag{69b}$$

If the applied thermo-mechanical loading at infinity is constant, then

$$h(\xi) = h_0\cos\omega; \quad p(\xi) = \begin{bmatrix}\cos 2\omega & \frac{1}{2}\sin 2\omega & 0\\ -\frac{1}{2}\sin 2\omega & \cos^2\omega & 0\\ 0 & 0 & \cos\omega\end{bmatrix}p_0, \tag{70}$$

where the vector $p_0 = [\sigma_{12}, \sigma_{22}, \sigma_{32}]^{\rm T}$ is a constant applied traction at infinity.

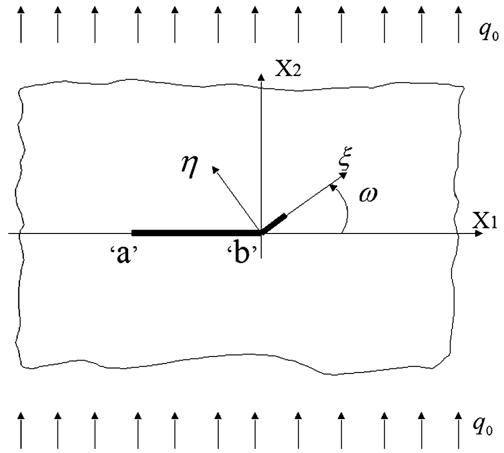

Fig. 3. A branched thermo-elastic crack in an anisotropic medium.

Now let us consider the total heat flux and traction at any point on the plane $\eta = 0$, i.e., $\theta = \omega$ in the (r, θ, x_3) coordinate system; then superposition leads to

$$h_2^{\mathrm{tot}}(\xi, 0) = h_\theta^{\mathrm{c}}(r, \omega) + h_\theta^{\mathrm{int}}(r, \omega) + \tilde{h}_2^{\mathrm{t}}(\xi, 0), \tag{71a}$$

$$\mathbf{t}^{\mathrm{tot}}(\xi, 0) = \Omega_{(\omega)}^{\mathrm{T}} t_\theta^{\mathrm{c}}(r, \omega) + \Omega_{(\omega)}^{\mathrm{T}} t_\theta^{\mathrm{int}}(r, \omega) + \tilde{t}_2^{\mathrm{t}}(\xi, 0) + t_2^{\mathrm{d}}(\xi, 0), \tag{71b}$$

where the superscripts "*c*" and "*t*/*d*" denote that the corresponding fields are induced by the main crack and the heat vortex or mechanical dislocation, respectively; "int" denotes the fields induced by the interaction between the crack and the dislocation; and "tot" is the summation from all contributions.

If the branched portion of the crack is modelled by the continuous distribution of the dislocations with density $T_0(r_0) = -\mathrm{d}T_0(r_0)/\mathrm{d}r_0$ and $\mathbf{b}(r_0) = -\mathrm{d}\mathbf{b}(r_0)/\mathrm{d}r_0$, then the boundary condition ((69) and Eq. (71)) lead to a system of singular integral equations as

$$\frac{k}{2\pi}\int_b^c \frac{A_t(r, r_0)}{r - r_0} T_0 \,\mathrm{d}r_0 + \frac{k}{2\pi}\int_b^c K_t(r, r_0) T_0 \,\mathrm{d}r_0 = -h_0\cos(\omega) - h_\theta^c(r, \omega), \tag{72}$$

where

$$A_t(r, r_0) = 1 + \frac{1}{2}\mathrm{Re}\left[1 - \sqrt{\frac{(r_0\lambda_{(\omega)} - a)(r_0\lambda_{(\omega)} - b)}{(r\lambda_{(\omega)} - a)(r\lambda_{(\omega)} - b)}}\right], \tag{73a}$$

$$K_t(r, r_0) = \mathrm{Re}\left[\frac{1}{2}\left(1 - \sqrt{\frac{(r_0\bar{\lambda}_{(\omega)} - a)(r_0\bar{\lambda}_{(\omega)} - b)}{(r\lambda_{(\omega)} - a)(r\lambda_{(\omega)} - b)}}\right)\frac{\lambda_{(\omega)}}{r\lambda_{(\omega)} - r_0\bar{\lambda}_{(\omega)}} - \frac{\lambda_{(\omega)}}{\sqrt{(r\lambda_{(\omega)} - a)(r\lambda_{(\omega)} - b)}}\right], \tag{73b}$$

$$h_\theta^c(r, \omega) = -h_0\mathrm{Re}\left[\lambda_{(\omega)}\left(1 - \frac{r\lambda_{(\omega)} - (a+b)/2}{\sqrt{(r\lambda_{(\omega)} - a)(r\lambda_{(\omega)} - b)}}\right)\right] \tag{73c}$$

and

$$\frac{1}{2\pi}\int_b^c \frac{A_b(r, r_0)}{r - r_0}\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{2\pi}\int_b^c K_b(r, r_0)\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{2\pi}\int_b^c K_{bt}(r, r_0) T_0 \,\mathrm{d}r_0 == -p(r) - \Omega_{(\omega)}^{\mathrm{T}}\mathbf{t}_\theta^c(r, \omega) \tag{74}$$

with

$$A_b(r,r_0) = -\widetilde{L} - \Omega^{\mathrm{T}}_{(\omega)}\mathrm{Im}\left[\sum_{k=1}^{3} BI_k\left\langle\left\langle 1 - \sqrt{\frac{(r_0\mu^k_{(\omega)}-a)(r_0\mu^k_{(\omega)}-b)}{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}\right\rangle\right\rangle B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}\widetilde{B}I_k\widetilde{B}^{\mathrm{T}}\right], \tag{75a}$$

$$\begin{aligned}K_b(r,r_0) = -\Omega^{\mathrm{T}}_{(\omega)}\sum_{k=1}^{3}\mathrm{Im}\Bigg\{ B\Bigg[&\left\langle\left\langle\left(1-\sqrt{\frac{(r_0\mu^k_{(\omega)}-a)(r_0\mu^k_{(\omega)}-b)}{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}\right)\frac{\mu_{(\omega)}}{r\mu_{(\omega)}-r_0\mu^k_{(\omega)}}\right\rangle\right\rangle\widetilde{I}_k \\
&-\left\langle\left\langle\frac{\mu_{(\omega)}}{\sqrt{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}\right\rangle\right\rangle\Bigg]B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}\widetilde{B}I_k\widetilde{B}^{\mathrm{T}} \\
&+B\left\langle\left\langle\frac{\mu_{(\omega)}}{\sqrt{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}+\left(\sqrt{\frac{(r_0\overline{\mu}^k_{(\omega)}-a)(r_0\overline{\mu}^k_{(\omega)}-b)}{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}-1\right)\frac{\mu_{(\omega)}}{r\mu_{(\omega)}-r_0\overline{\mu}^k_{(\omega)}}\right\rangle\right\rangle \\
&\times B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}\overline{\widetilde{B}}I_k\overline{\widetilde{B}}^{\mathrm{T}}\Bigg\},\end{aligned} \tag{75b}$$

$$\begin{aligned}K_{bt}(r,r_0) = &-\Omega^{\mathrm{T}}_{(\omega)}\mathrm{Re}[B[4\pi E(r\lambda_{(\omega)})\lambda_{(\omega)}] \\
&-\Omega^{\mathrm{T}}_{(\omega)}\mathrm{Im}\left[\mathbf{y}(r\lambda_{(\omega)},r_0\lambda_{(\omega)})\frac{B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}}{2}\widetilde{D}-\mathbf{y}(r\lambda_{(\omega)},r_0\overline{\lambda}_{(\omega)})\frac{B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}}{2}\overline{\widetilde{D}}\overline{\lambda}^{-1}_{(\omega)}\lambda_{(\omega)}\right] \\
&+\Omega^{\mathrm{T}}_{(\omega)}\mathrm{Im}\left\{B[F(r\lambda_{(\omega)},r_0\lambda_{(\omega)})+F(r\lambda_{(\omega)},r_0\overline{\lambda}_{(\omega)})]\frac{B^{-1}D}{2}\lambda_{(\omega)}\right\} \\
&+\Omega^{\mathrm{T}}_{(\omega)}\mathrm{Im}[DB(r\lambda_{(\omega)},r_0\lambda_{(\omega)})\lambda_{(\omega)}]-\mathrm{Im}[\widetilde{D}\log(r-r_0)],\end{aligned} \tag{75c}$$

where $\widetilde{I}_1 = \mathrm{diag}[0,1,1]$, $\widetilde{I}_2 = \mathrm{diag}[1,0,1]$, $\widetilde{I}_3 = \mathrm{diag}[1,1,0]$.

It can be seen that the coefficients $A_t(r,r_0)$ and $A_b(r,r_0)$ in Eqs. (72) and (74) are functions of both r and r_0. Using the technique in Muskhelishvili (1992), these equations can be rewritten as

$$\frac{k}{2\pi}\int_b^c \frac{T_0}{r-r_0}\,\mathrm{d}r_0 + \frac{k}{2\pi}\int_b^c K_T(r,r_0)T_0\,\mathrm{d}r_0 = -h_0\cos(\omega) - h^c_\theta(r,\omega), \tag{76a}$$

$$\frac{1}{2\pi}\int_b^c \frac{-\widetilde{L}}{r-r_0}\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{2\pi}\int_b^c K_B(r,r_0)\mathbf{b}\,\mathrm{d}r_0 + \frac{1}{2\pi}\int_b^c K_{bt}(r,r_0)T_0\,\mathrm{d}r_0 = -p(r) - \Omega^{\mathrm{T}}_{(\omega)}\mathbf{t}^c_\theta(r,\omega), \tag{76b}$$

where

$$\begin{aligned}K_T(r,r_0) = \mathrm{Re}\Bigg[&\frac{1}{2}\left(1-\sqrt{\frac{(r_0\lambda_{(\omega)}-a)(r_0\lambda_{(\omega)}-b)}{(r\lambda_{(\omega)}-a)(r\lambda_{(\omega)}-b)}}\right)\frac{1}{r-r_0} \\
&+\frac{1}{2}\left(1-\sqrt{\frac{(r_0\overline{\lambda}_{(\omega)}-a)(r_0\overline{\lambda}_{(\omega)}-b)}{(r\lambda_{(\omega)}-a)(r\lambda_{(\omega)}-b)}}\right)\frac{\lambda_{(\omega)}}{r\lambda_{(\omega)}-r_0\overline{\lambda}_{(\omega)}}-\frac{\lambda_{(\omega)}}{\sqrt{(r\lambda_{(\omega)}-a)(r\lambda_{(\omega)}-b)}}\Bigg],\end{aligned} \tag{77a}$$

$$K_B(r,r_0) = -\Omega^{\mathrm{T}}_{(\omega)}\sum_{k=1}^{3}\mathrm{Im}\Bigg\{B\Bigg[\Big\langle\Big\langle\Bigg(1-\sqrt{\frac{(r_0\mu^k_{(\omega)}-a)(r_0\mu^k_{(\omega)}-b)}{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}\Bigg)\frac{\mu_{(\omega)}}{r\mu_{(\omega)}-r_0\mu^k_{(\omega)}}\Big\rangle\Big\rangle$$
$$-\Big\langle\Big\langle\frac{\mu_{(\omega)}}{\sqrt{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}\Big\rangle\Big\rangle\Bigg]B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}\widetilde{B}I_k\widetilde{B}^{\mathrm{T}}$$
$$+B\Big\langle\Big\langle\frac{\mu_{(\omega)}}{\sqrt{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}+\Bigg(\sqrt{\frac{(r_0\overline{\mu}^k_{(\omega)}-a)(r_0\overline{\mu}^k_{(\omega)}-b)}{(r\mu_{(\omega)}-a)(r\mu_{(\omega)}-b)}}-1\Bigg)\frac{\mu_{(\omega)}}{r\mu_{(\omega)}-r_0\overline{\mu}^k_{(\omega)}}\Big\rangle\Big\rangle$$
$$\times B^{-1}\Omega^{\mathrm{T}}_{(\pi-\omega)}\overline{\widetilde{B}}I_k\overline{\widetilde{B}}^{\mathrm{T}}\Bigg\}. \tag{77b}$$

The other coefficients are the same as those specified in Eqs. (73) and (75).

Let

$$r=\frac{(1+x)l}{2},\quad r_0=\frac{(1+t)l}{2},\quad l=c-b, \tag{78}$$

where $|x|\leqslant 1$ and $|t|\leqslant 1$; then Eq. (76) can be rewritten as

$$\frac{k}{2\pi}\int_{-1}^{1}\frac{T_0}{x-t}\,\mathrm{d}t+\frac{k}{2\pi}\int_{-1}^{1}\widetilde{K}_T(x,t)T_0\,\mathrm{d}t=-h_0\cos(\omega)-h^c_\theta(x,\omega), \tag{79a}$$

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{-\widetilde{L}}{x-t}\mathbf{b}\,\mathrm{d}t+\frac{1}{2\pi}\int_{-1}^{1}\widetilde{K}_B(x,t)\mathbf{b}\,\mathrm{d}t+\frac{1}{2\pi}\int_{-1}^{1}\widetilde{K}_{bt}(x,t)T_0\,\mathrm{d}t=-p(x)-\Omega^{\mathrm{T}}_{(\omega)}\mathbf{t}^c_\theta(x,\omega), \tag{79b}$$

where $\widetilde{K}_T(x,t), \widetilde{K}_B(x,t)$ and $\widetilde{K}_{bt}(x,t)$ are obtained by substituting (78) in $K_T(r,r_0)$, $K_B(r,r_0)$ and $K_{bt}(r,r_0)$, respectively. This system of singular equations involves two unknowns, namely T_0 and $\mathbf{b}$ which are coupled through the term $\widetilde{K}_{bt}$ in $(79)_2$. One can let (Erdogan et al., 1973)

$$T_0=\left[(1+t)^{-s_1}(1-t)^{1/2}\right]T(t);\quad \mathbf{b}(t)=\left[(1+t)^{-s_2}/(1-t)^{1/2}\right]b(t). \tag{80}$$

Next is the discussion of the schemes for numerically solving the integral equations (79). The principle of these numerical schemes can be found in Erdogan et al. (1973). One may assume that the heat vortex density at both ends of the crack branched portion is bounded, then s_1 can be equal to $-1/2$ and Eq. $(79)_1$ can be solved by Gauss–Chebvshev integration. The numerical version of $(79)_1$ can be expressed as

$$\sum_{i=1}^{n}\frac{1-t_i^2}{n+1}T(t_i)\left[\frac{1}{t_i-x_k}-\widetilde{K}_T(t_i,x_k)\right]=\frac{2}{k}[h_0\cos\omega+h^c_\theta(x_k,\omega)],\quad t_i=\cos\left(\frac{i\pi}{n+1}\right)\ (i=1,\dots,n),$$
$$x_k=\cos\left(\frac{\pi}{2}\frac{2k-1}{n+1}\right)\ (k=1,\dots,n+1). \tag{81}$$

Eq. (81) is satisfied for $n+1$ values of $T(t_i)$, but only n unknowns of $T(t_i)$ are needed in (81). Therefore, one of these $T(t_i)$ should be omitted (Erdogan et al., 1973). In order to ensure that the singularity at the intersection point of the main crack and the branched crack should be less than that of the branched crack tip, the $T(-1)$ was set to zero. Then Eq. (81) can be solved uniquely.

Once the solution for T_0 is obtained, one can move to solve $(79)_2$. As specified in the literature by some authors such as by Keer and Miller (1982), the singularity at the intersection point of the main crack and the branched crack should be of order no larger than 1/2. This assertion can also be obtained by a procedure similar to the one in deriving Eqs. (83) in this paper. Therefore, one then may set $s_2 = 1/2$ and $(79)_2$ can be rewritten in numerical form as

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{1}{n} b(t_i)\left[\frac{\widetilde{L}}{t_i - x_k} - \widetilde{K}_B(t_i, x_k)\right] = 2\left[p(x_k) + \Omega^{\mathrm{T}}_{(\omega)} \mathbf{t}^c_\theta(x_k, \omega) + \frac{1}{2\pi}\int_{-1}^{1} \widetilde{K}_{bt}(x_k, t) T_0 \,\mathrm{d}t\right],\\
&\sum_{i}^{n} \frac{\pi}{n} b(t_i) = 0,\\
&t_i = \cos\left(\pi \frac{2\mathrm{i} - 1}{2n}\right), \quad (i = 1, \ldots n);\\
&x_k = \cos\left(\frac{\pi k}{n}\right) \quad (i = 1, \ldots, n-1),
\end{aligned} \tag{82}$$

where the second equation i.e. $(82)_2$ comes from the condition $\int_{-1}^{1} \mathbf{b}(t)\,\mathrm{d}t = 0$, which satisfies the condition of single-valuedness of displacement around the crack. The integration of the third term on the right hand side of $(82)_1$ was performed by using Simpson's rule. Since the nodes used in (81) and (82) are different, the polynomial interpolations were also used to obtain the values of $\widetilde{K}_{bt}(x, t)$ and $T_0(t)$ from the nodes in (81) for those values needed for the nodes in $(82)_1$.

The stress intensity factors at the branched crack tip can be numerically calculated by employing the technique given by Muskhelishvili (1953)

$$\begin{aligned}
K = [K_{\mathrm{II}}, K_I, K_{\mathrm{III}}]^{\mathrm{T}} &= \lim_{r \to l^+} \sqrt{2\pi(r - l)} \frac{1}{\pi} \int_{-1}^{1} \frac{\widetilde{M} b(t)}{t - x} w(t)\,\mathrm{d}t\\
&= \lim_{r \to l^+} \sqrt{2\pi(r - l)} \frac{1}{\pi} \int_{-1}^{1} \frac{\widetilde{M} b(t)(1 + t)^s}{(t - x)(1 - t)1/2}\,\mathrm{d}t = \sqrt{\frac{\pi l}{2}} \widetilde{L} b(1),
\end{aligned} \tag{83}$$

where (78) was used. Moreover, the energy release rate of the crack branch can be computed by using the following expression as pointed out in (Barnett and Asaro, 1972)

$$G = \frac{1}{2} K^{\mathrm{T}}\ \mathrm{Re}[\mathrm{i}\widetilde{A}\widetilde{B}^{-1}]\ K, \tag{84}$$

in which,

$$\mathrm{i}\widetilde{A}\ \widetilde{B}^{-1} = \widetilde{L}^{-1} - \mathrm{i}\widetilde{S}\widetilde{L}^{-1}. \tag{85}$$

Therefore,

$$G(\omega) = \frac{1}{2} K^{\mathrm{T}} \widetilde{L}^{-1} K, \tag{86}$$

where $\widetilde{L} = \Omega^{\mathrm{T}}(\omega) L \Omega(\omega)$, and $\Omega(\omega)$ is defined in (43).

6. Numerical results and discussion

In this section, we shall investigate the influence of thermal conductivity on the crack branching in composite materials. The thermo-elastic properties are chosen for a general orthotropic materials as $v_{12} = -S_{12}/S_{11} = 0.25$; $S_{66} = 2(S_{11} - S_{12})$; $k_{11} = 42.1$ W/m/K; $k_{22} = k_{33}$; $\alpha_{11} = 0.025 \times 10^{-6}$ m/m/K; $\alpha_{22} = \alpha_{33} = 32.4 \times 10^{-6}$ m/m/K, where v_{12} is the Poisson's ratio and s_{ij} $(i,j = 1, 2, 3)$ are material compliance coefficients. The degree of anisotropy is defined as S_{11}/S_{22}. The unit heat flux h_0 or/and pure unit tension σ_{22} were the specified applied loading in the numerical calculations.

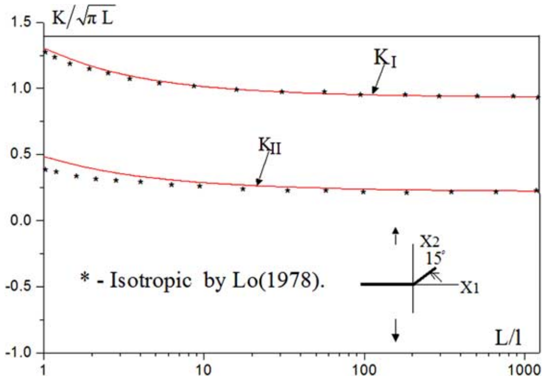

Fig. 4. Stress intensity factors vs. L/l for a branching angle $\omega = 15°$ (unit tension applied loading).

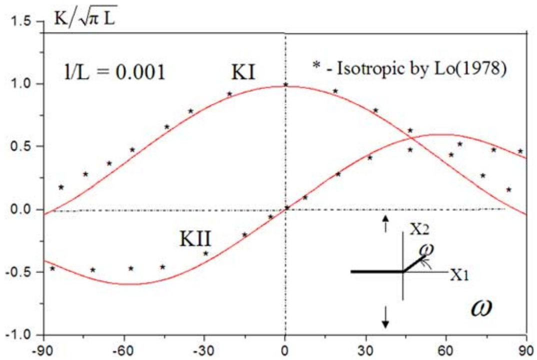

Fig. 5. Stress intensity factors vs. branching angles for the isotropic case (unit tension applied loading).

The results for the special case of nearly isotropic medium are plotted in Figs. 4 and 5, in which the applied loading is pure unit tension σ_{22}. The number of nodes, $n = 120$, was used in all the computations. In Fig. 4, we plot the Modes I and II stress intensity factors with respect to the ratio L/l (where "l" denotes the length of the branched portion of the crack and "L" the half-length of the main crack), under the assumed branching angle $\omega = 15°$. This plot shows that the results converge when the ratio $L/l > 50$. The onset of a crack branching usually is of primary interest. Therefore, the "infinitesimal" crack branch is assumed to be $l/L = 0.001$ in the sequel computation. Fig. 5 presents the variation of Modes I and II stress intensity factors of the branched crack tip vs. the branching angles. The results by Lo (1978) for an isotropic medium with same geometric and loading conditions are also plotted in Figs. 4 and 5. It can be seen that these two sets of results are remarkably closed to each other, especially for the infinitesimal branched crack tip. This would verify that the assumption $s_2 = 1/2$ in Section 5 is reasonable and the method in the present paper may be suitable in dealing with crack branching problems.

Figs. 6 and 7 show, respectively, the stress intensity factors and energy release rate for a nearly isotropic material, i.e. $S_{22} = 1.01S_{11}$; in Figs. 8 and 9 are the stress intensity factors and energy release rate for an anisotropic material, the degree of anisotropy is $S_{22} = 2.50S_{11}$. In these cases the ratio k_{22}/k_{11} of the heat conduction coefficients is assumed to be 0.01. From the results in Figs. 6 and 7, it can be seen that the branching angle at which the K_{I} attains its maximum value (K_{II} reaching its minimum value) coincide with the angle which makes the energy release rate attain its maximum value; while those angles in Figs. 8 and 9 are different. This observation shows that the K-based criteria are still valid for the thermo-elastic problem of isotropic materials. But for the thermo-elastic problem of anisotropic media, the G-based criteria should

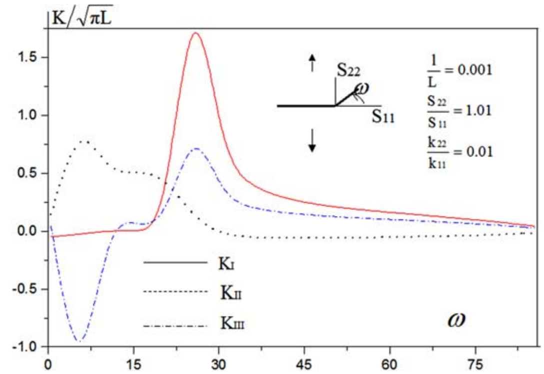

Fig. 6. Stress intensity factors vs. branching angles for a nearly isotropic material.

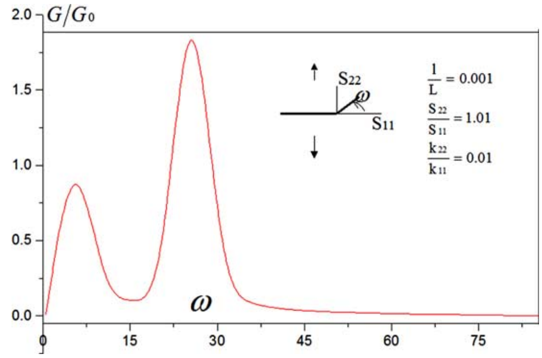

Fig. 7. Energy release rate vs. branching angles for a nearly isotropic material.

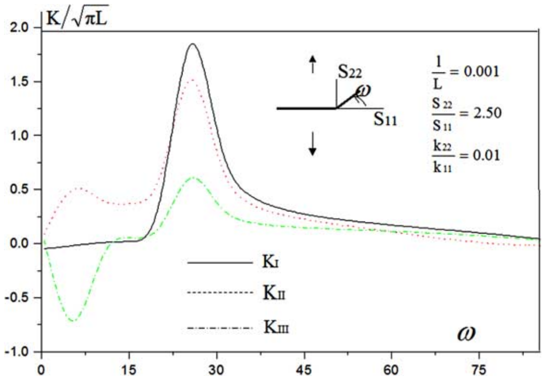

Fig. 8. Stress intensity factors vs. branching angles for an anisotropic material.

be used. It can also be seen that the curves K_{II} and K_{III} in Figs. 6 and 8 have two local maxima in absolute value. This makes the curves uneven. If no thermal loading applied, the K_{II} and K_{III} curves for general anisotropic medium usually have two local maximum values of opposite sign and symmetrically distributed as shown in the literature (e.g. Obata et al., 1989). But the thermal loading shifts and mixes these two locals, thus the curves look uneven. The unevenness of energy release rate curves follows that of K_{II} and K_{III}.

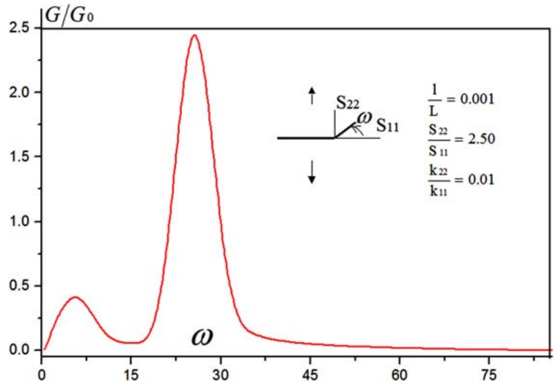

Fig. 9. Energy release rate vs. branching angles for an anisotropic material.

Notice that a non-zero K_{III} exists in Figs. 6 and 8, although the material is orthotropic and the only applied load is the σ_{22}; also the K_I approaches small negative values (i.e. suggesting that the surfaces around the crack tip are not open) when the branching angles approach to zero. These two interesting phenomena are due to the thermal loading effects. As we know, when the branching angles $\omega \approx 0$, this problem degenerates to a straight crack of anisotropic medium under thermo-mechanically loading, in which the contact phenomena (negative K_I) may happen under some combined loading as pointed out by some authors such as Sturia and Barber (1988). If this overlapping needs further study, the contact model may be an alternative approach. But it can be seen that the above observation may further confirm that the current method is a good approximation for investigating thermo-elastic crack branching problems.

The influence of the degree of anisotropy is also illustrated in these cases. In particular, when $S_{22} = 1.01S_{11}$, the G_{max}/G_0 is 1.875 with a corresponding branching angle $\omega_{max} = 22.5°$ (see Fig. 7); while when $S_{22} = 2.5S_{11}$, the $G_{max}/G_0 = 2.925$ with a corresponding branching angle $\omega_{max} = 27.25°$ (see Fig. 9). Here, G_0 is the value without branching.

Presented in Fig. 10 is the combined influence of thermal conduction properties and the degree of anisotropy on the branching angles. It can be observed that the branching angles increase as the degree of anisotropy (S_{22}/S_{11}) increases when $0.01 \leqslant k_{22}/k_{11} \leqslant 0.275$; while this result is reversed when $0.275 \leqslant k_{22}/k_{11} \leqslant 0.425$. But when $0.425 \leqslant k_{22}/k_{11} \leqslant 0.5$, the tendency is mixed. Such plots can provide a guideline for the selecting the thermal properties of anisotropic materials.

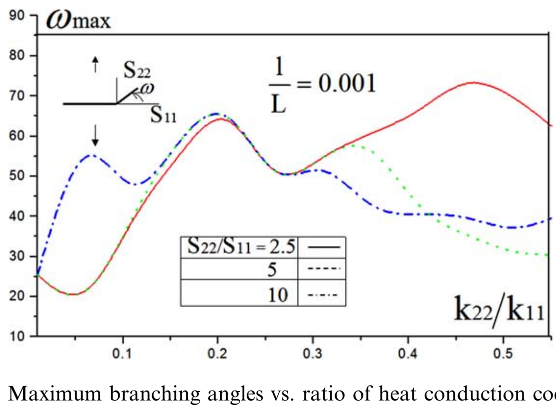

Fig. 10. Maximum branching angles vs. ratio of heat conduction coefficients.

7. Conclusion

Thermo-elastic crack branching of anisotropic material was investigated and the influences of thermal properties and the degree of anisotropy on the onset of crack branching was addressed. A closed form solution to the interaction between the heat dislocation and a crack was also presented. The following conclusions can be drawn from the results obtained this paper: (1) A G-based criterion seems more reasonable than a K-based in predicting the onset of branching in the thermo-elastic problem. (2) A K_{III} exists for the orthotropic material under normal loading σ_{22} due to thermal effects. (3) Negative K_{I} (overlapping of the crack faces around the crack tip) is possible under certain mechanical loading due to the thermal effects. (4) The coefficients of heat conduction and the degree of anisotropy of the composite material have strong combined effects on the crack branching.

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