The Mode III Interface Crack in Piezo-Electro-Magneto-Elastic Dissimilar Bimaterials

1 Introduction

One class of contemporary materials, widely used in engineering in devices (in sensor, transducer, actuator components, etc.), are the piezoelectric and piezomagnetic composite materials. Due to their exceptional functions, such as flat frequency response [1–4] and transformation of energy from one form to the other (mechanical, electric, and magnetic energy, or thermal energy) [5,6], this type of composite exhibiting piezoelectric and piezomagnetic properties has found increasing applications in microwave electronic, optoelectronic, and electronic instruments. Like in conventional composites, defects or flaws may usually be introduced during the manufacturing process or during service by impact loading. These defects would often deteriorate the performance of the devices being made of piezo-electro-magneto-elastic media.

Recently, more and more attention has been directed towards the problems of cracks in the electro-magneto-elastic solids [7–12]. Dissimilar bimaterials or layered composites are often incorporated into a variety of components, such as smart structure sensors, actuators, and broadband magnetic probes. Having been recognized as one of the common failure modes of general dissimilar bimaterial media, the interface cracks could also be developed in the piezo-electro-magneto-elastic structures and thus affect the features of the electro-magneto-elastic apparatus. Though these interface cracks may severely diminish the performance of this type of structure, one may see that little attention has been given to the study of the magneto-electric coupling effects on the interface crack propagation behavior in piezo-electro-magneto-elastic bimaterials.

This paper is organized as follows: (1) In Sec. 2 is a summary of some basic equations for piezo-electro-magneto-elasticity in Strohs formalism. (2) A compact form solution to the interface crack is formulated in Sec. 3. The expressions for the ECOD, ESIF, and the energy release rate are derived in closed form. The “energy method” is also proposed in this section and used to obtain the solution to the magneto-elastic field inside the interface crack. One may interestingly find that this method could be extended to more complicated problems in piezo-electro-magneto-elastic solids. (3) The numerical results in Sec. 4 show the influence of the property mismatches between the two constituents on the interface crack propagation. An interesting result one may find is that the applied external electro-magnetic field may slow the growth of mode III interface cracks in piezo-electro-magneto-elastic bimaterial solids. Since all the formulas in this paper are obtained in explicit expressions, and are thus easily trackable, this study may serve as a benchmark for further investigations in piezo-electro-magneto-elastic media.
2 Basic Equations

In a fixed Cartesian coordinate system \((x_1, x_2, x_3)\), the generalized Hookes law for an elastic material with both piezoelectric and piezomagnetic fields is of the following form [1]:

\[
\sigma_{ij} = c_{ijkl} e_{kl} + e_{ikl} \phi^k + \mathcal{Q}_{ijkl} \phi^l, \\
D_i = e_{ijkl} e_{kl} - e_{ikl} \phi^k - \alpha_{ijkl} \phi^l, \\
B_i = q_{ijkl} e_{kl} - \alpha_{ijkl} \phi^k - \mu_{ijkl} \phi^l
\]  
(1)

where \(i,j,k,l\) range in \([1,2,3]\) and the repeated indices imply summation, the comma stands for differentiation with respect to corresponding coordinate variables; \(\sigma_{ij}\) is the elastic stress, \(u_i\) is the elastic displacement, and \(c_{ijkl}\) is the elastic moduli tensor; \(D_i\) is the electric displacement, \(\phi^k\) is the electrostatic potential, and \(e_{ikl}\) is the electric permittivity; \(B_i\) is the magnetic induction (magnetic fluxes), \(\phi^k\) is the magnetic scalar potential, and \(\mu_{ijkl}\) the magnetic permeability; \(e_{ikl}\) are the piezoelectric, piezomagnetic, and magnetoelectric coefficients, respectively. For the material constants, the following relationships hold:

\[
c_{ijkl} = c_{jikl} = c_{ijlk}, \quad e_{ijkl} = e_{ikjl}, \quad \mathcal{Q}_{ijkl} = \mathcal{Q}_{jkil}, \\
\alpha_{ijkl} = \alpha_{ijlk}, \quad e_{ikl} = e_{jkl}, \quad \mu_{ijkl} = \mu_{jikl}
\]  
(2)

The equilibrium equations in the absence of body forces read

\[
\sigma_{ij} = 0, \quad D_i = 0, \quad B_i = 0
\]  
(3)

For two-dimensional antiplane deformation of a transversely isotropic solid, we have

\[
\begin{align*}
\sigma_{ij} & = 0, \quad D_i = 0, \quad B_i = 0, \\
u_1 & = 0, \quad u_2 = 0, \quad u_3 = u_3(x_1, x_2), \\
\phi^k & = \phi^k(x_1, x_2), \quad \phi^l = \phi^l(x_1, x_2)
\end{align*}
\]  
(4)

One may define the extended displacement as

\[
u = [u_3, \phi^k, \phi^l]^T
\]  
(5)

For a plane system, a nontrivial solution to Eq. (3) may then take the following form:

\[
u = Af(z_0) + \overline{A}f(\overline{z}_0), \quad \psi = Bf(z_0) + \overline{B}f(\overline{z}_0), \quad z_0 = x_1 + p_0 x_2
\]  
(6)

where \(\psi\) is the stress function vector and \(f(z_0)\) are functions to be determined by boundary conditions.

If one defines the extended stress fields as

\[
t = [\sigma_{32}, D_2, B_2]^T, \quad s = [\sigma_{31}, D_1, B_1]^T
\]  
(7)

then these stresses can be written in terms of the stress functions as

\[
s = \left[-\frac{\partial \psi}{\partial x_2}\right]^T, \quad t = \left[\frac{\partial \psi}{\partial x_1}\right]^T = \psi'
\]  
(8)

Substituting Eq. (6) back into the equation (3), one readily obtains

\[
A = I = \text{diag}[1, 1, 1], \quad B = \begin{pmatrix}
e_{44} & e_{15} & 0 \\
e_{15} & e_{11} & -\alpha_{11} \\
0 & -\alpha_{11} & -\mu_{11}
\end{pmatrix}, \quad p_0 = i
\]  
(9)

where \(i^2 = -1\).

If we define a matrix \(M\) as

\[
M = iAB^{-1}
\]  
(10)

then

\[
M = \begin{pmatrix}
e_{11} - \mu_{11} & e_{15} - \alpha_{11} & 0 \\
e_{15} & e_{11} - \alpha_{11} & -\alpha_{11} \\
0 & -\alpha_{11} & -\mu_{11}
\end{pmatrix}
\]  
(11)

where

\[
\Delta = c_{44} e_{11} + e_{15}^2 + e_{15} - 2\alpha_{11} e_{15} - c_{44} \alpha_{11}
\]  
(12)

The matrix \(M\) is real and symmetric.

3 A Solution to Mode III Interface Crack

Let the medium \("I"\) occupy the upper half-space (donated by \(L\)) and medium \("II"\) be in the lower half-space (donated by \(R\)) (Fig. 1). Then from Eqs. (6) and (9), one has the following expression for this bimedia:

\[
u' = \phi_I(z) + \overline{\phi_I}(\overline{z}), \quad \psi' = B_I \phi_I(z) + \overline{B_I} \overline{\phi_I}(\overline{z})
\]  
(13)

where, \(\nu', \psi'\) are displacement and stress functions for \(z \in L\), and

\[
u'' = \phi_{II}(z) + \overline{\phi_{II}}(\overline{z}), \quad \psi'' = B_{II} \phi_{II}(z) + \overline{B_{II}} \overline{\phi_{II}}(\overline{z})
\]  
(14)

where \(\nu'', \psi''\) are displacement and stress functions for \(z \in R\).

For convenience, the symbols \("I"\) and \("II"\), denoting the quantities for medium \("L"\) and \("R"\), respectively, may be put as subscripts or superscripts.

Let the interface crack be located in the region \(a < x_1 < b, -\infty < x_3 < \infty\) of the plane \(x_2 = 0\). The \(p_0' = \text{diag}[1, 1, 1]^T\) is applied at infinity (Fig. 1). Inside the crack often is air or vacuum, and the electro-magnetic field usually is considered constant under uniform remote applied load \([11, 12, \text{etc.}]\). These unknown components for the electro-magnetic field are denoted as \(D_{0a}^0, B_{0a}^0, E_{0a}^0, \) and \(H_{0a}^0\), which, respectively, observe the relationships

\[
E_{0a}^0 = \frac{D_{0a}^0}{\epsilon_0}, \quad H_{0a}^0 = \frac{B_{0a}^0}{\mu_0}, \quad \alpha = [1, 2]
\]  
(15)

Employing the superposition principle leads the original boundary value problem to an equivalent problem with the loading
\[
p_0 = [\sigma_{32}^+ \Delta D_2^0, \Delta B_2^0]_T \tag{16}\]
being applied on the two surfaces of the interface crack, where, in Eq. (16),
\[
\Delta D_2^0 = D_2^+ - D_2^-, \quad \Delta B_2^0 = B_2^+ - B_2^-\tag{17}.
\]
The displacement continuity along the bonded interface gives
\[
\phi_{\nu}(x_1) + \bar{\phi}_{\nu}(x_1) = \phi_{\nu}(x_1) + \bar{\phi}_{\nu}(x_1)
\]
or
\[
\phi_{\nu}(x_1) - \bar{\phi}_{\nu}(x_1) = \phi_{\nu}(x_1) - \bar{\phi}_{\nu}(x_1)
\]  
(18)

A function can be defined being analytical on the whole plane, except the cut along the interface crack, as follows:
\[
\Phi(z) = \begin{cases} 
\phi(z) - \bar{\phi}(z), & z \in L \\
\phi(z) - \bar{\phi}(z), & z \in R
\end{cases}
\]  
(19)

Then, this function automatically satisfies the condition (18). Here, a convention
\[
\phi(z) = \phi_0(x_1), \quad x_2 \to 0^a
\]  
(20)
is employed and will be used in the following sections. Differentiation of Eq. (19) with respect to \( z \) yields
\[
\Phi'(z) = \begin{cases} 
\phi'(z) - \bar{\phi}'(z), & z \in L \\
\phi'(z) - \bar{\phi}'(z), & z \in R
\end{cases}
\]  
(21)
The stress continuity on the bonded interface leads to
\[
B_I \phi'_\nu(x_1) + \bar{B}_I \bar{\phi}'_\nu(x_1) = B_I \phi'_\nu(x_1) + \bar{B}_I \bar{\phi}'_\nu(x_1)
\]  
(22)
Similarly, we can define a function, which automatically satisfies the condition (22) and is analytical on the whole plane except the cut along the interface crack, as
\[
\omega(z) = \begin{cases} 
B_I \phi'(z) - \bar{B}_I \bar{\phi}'(z), & z \in L \\
B_I \phi'(z) - \bar{B}_I \bar{\phi}'(z), & z \in R
\end{cases}
\]  
(23)
From Eqs. (21) and (23), we obtain
\[
B_I \phi'_\nu(z) = N[I + \Phi'(z) + \bar{M}_I \omega(z)], \quad \tag{24a}
\]
\[
\bar{B}_I \bar{\phi}'_\nu(z) = B_I \phi'(z) - \omega(z), \quad z \in L; \quad \tag{24b}
\]
and
\[
B_I \phi'_\nu(z) = \bar{N}[I + \Phi'(z) + \bar{M}_I \omega(z)], \quad \tag{25a}
\]
\[
\bar{B}_I \bar{\phi}'_\nu(z) = B_I \phi'(z) - \omega(z), \quad z \in R. \quad \tag{25b}
\]
In the above equations, the following matrix was used:
\[
N^{-1} = M_I + \bar{M}_I = M_I + M_H \tag{26}
\]
Since \( M_I \) and \( M_H \) are real symmetric, so is \( N \). Furthermore, define
\[
H = M_I + M_H \tag{27}
\]
Therefore, the boundary traction conditions along the interface crack surface give
\[
B_I \phi'_\nu(x_1) + \bar{B}_I \bar{\phi}'_\nu(x_1) - \omega(x_1) = -p_0(x_1), \quad \tag{28a}
\]
Subtraction of Eq. (28b) from (28a) yields
\[
\omega(x_1) - \omega_{\nu}(x_1) = 0 \tag{29}
\]
which implies that the \( \omega(z) \) is continuous on the whole interface.

By the analytical continuation principle [15], the function \( \omega(z) \) is analytical on the whole plane. But according to Liouville’s theorem [15], this \( \omega(z) \) must be a constant function in the whole domain. However, the condition that this function should vanish at infinity means this constant function must be identically zero in the whole plane, i.e.,
\[
\omega(z) = 0, \quad \text{for all } z \tag{30}
\]
Either Eq. (28a) or (28b) leads to a general Hilbert equation in matrix notation:
\[
\Phi'(x_1) + \Phi'(x_1) = i H p_0(x_1), \quad a < x_1 < b \tag{31}
\]
The homogenous equation corresponding to the general Hilbert equation (31) can be written as
\[
X_k(x_1) + X_k(x_1) = 0, \quad a < x_1 < b \tag{32}
\]
where
\[
X_k(x_1) = \frac{1}{\sqrt{(z - a)(z - b)}} \text{diag}[1, 1, 1] \tag{33}
\]
A solution which vanishes at infinite could be [16]
\[
\Phi'(z) = \frac{1}{2\pi i} X(z) \int \frac{X(x_1)_1}{x_1 - z} [i p_0(x_1)] dx_1 \tag{34}
\]
Specifically, for constant applied loading, one has (see the Appendix)
\[
\Phi'(z) = \text{diag}\left[1 - \frac{z - (a + b)/2}{\sqrt{(z - a)(z - b)}}\right]H^2/2(i p_0) \tag{35}
\]
Integrating Eq. (35) results in
\[
\Phi(z) = \text{diag}[z - \sqrt{(z - a)(z - b)}]H^2/2(i p_0) \tag{36}
\]
where the constant contributing rigid body motion is omitted.

Next, let us consider some fracture characterizing parameters such as the crack tip field intensity factors, extended displacement discontinuities near the crack tips, and the energy release rate. From the equations (24a) and (25b), the extended traction along the interface could be expressed as
\[
t(x_1) = N \cdot i \Phi_{\nu}(x_1) + \bar{N} \cdot i \Phi_{\nu}(x_1) = H^{-1}[i \Phi_{\nu}(x_1) + i \Phi_{\nu}(x_1)] \tag{37}
\]
We shall show that the right-hand side of Eq. (37) is real, as required.

Substituting the stress function (34) to traction expression (37) leads to
\[
t(x_1) = -p + [X_k(x_1) + X_k(x_1)] \left(x_1 - \frac{a + b}{2}\right) p_0/2 \tag{38}
\]
When Eq. (32) is employed, the traction along the interface reads:
\[
t(x_1) = \begin{cases} 
-p_0 + [(x_1 - a)(x_1 - b)]^{1/2} \text{diag}\left(x_1 - \frac{a + b}{2}\right) p_0, & x_1 < a \text{ and } b < x_1 \\
-p_0, & a < x_1 < b
\end{cases} \tag{39}
\]
which is a real vector as expected.

Then the extended tractions at a distance \( r \) ahead of the crack tip such as \( b \) (Fig. 1) can be expressed in the form of

\[
\mathbf{t}(r) = (2\pi r)^{-1/2} \sqrt{\frac{\pi(b-a)}{2}} \mathbf{p}_0 = (2\pi r)^{-1/2}[K_\sigma, K_D, K_B]^T
\]

(40)

where \( K \)'s are real numbers and defined as

\[
K_\sigma = \sqrt{\frac{\pi(b-a)}{2}} \sigma_{32}, \quad K_D = \sqrt{\frac{\pi(b-a)}{2}} \Delta D_2, \\
K_B = \sqrt{\frac{\pi(b-a)}{2}} \Delta B_2
\]

These \( K \)'s may be called the extended stress intensity factors (ESIFs). If we let

\[
K = [K_\sigma, K_D, K_B]^T
\]

(42)

then the expression (42) becomes

\[
K = \sqrt{\frac{\pi(b-a)}{2}} \mathbf{p}_0
\]

(43)

with \( \mathbf{p}_0 \) defined in (16).

One may also extend the conventional crack open displacement (COD) to piezo-magneto-electric materials. From Eqs. (13), (14), and (19), this extended crack open displacement (ECOD) may readily be evaluated by

\[
\Delta \mathbf{u}(x) = \mathbf{u}_e(x) - \mathbf{u}_0(x) = \mathbf{F}_e(x) - \mathbf{F}_0(x) = \left[ \begin{array}{c} \{0, -1, 0, 0\} \end{array} \right], \quad a < x_3 < b \\
= \left[ \begin{array}{c} x_3 - (a - b)x_3 \end{array} \right] \mathbf{H} \mathbf{p}_0, \quad 0 < x_3 < a \text{ or } b < x_3;
\]

(44)

Then the ECOD at a small distance \( r \) behind the tip of the interface crack may read

\[
\Delta \mathbf{u}(r) = \sqrt{\frac{r}{2\pi}} \mathbf{H}(2K)
\]

(45)

also an expected real vector.

Now, the energy release rate, \( G \), can be computed and it reads

\[
G = \lim_{\delta \to 0} \frac{1}{2} \int_0^\delta (\mathbf{t}(r) \Delta \mathbf{u}(\delta - r) dr) = \frac{1}{2} K^2 HK
\]

(46)

One may realize that all the expressions derived so far include the unknown components \( \Delta D_2 \) and \( \Delta B_2 \) of the electro-magnetic field inside the crack. There are two approaches to determine these unknowns. The first method views the crack as a degenerated hole, using the continuous conditions on the hole surface to determine the electro-magnetic fields. This method may work well for monolithic material as shown in literature such as in [12], because of the convenient affine mapping function. But it is hard to extend this method derived for monolithic materials to the bimaterial media because of the differences in the material properties between the two constituents of a bimaterial system. To offset this difficulty, here, another approach, called the "energy method," is proposed. As one may know, when a remote load starts to apply, an electric-magnetic field begins to build up inside the interface crack. This newly built field causes reactions to fields induced by the applied loading inside the whole material system. One may see that the energy release rate, \( G \), is a saddle surface with respect to variables \( \Delta D_2 \) and \( \Delta B_2 \), the electric-magnetic field inside the interface crack. This means for each value of \( G \), there exist many corresponding sets of \( \Delta D_2 \) and \( \Delta B_2 \) except at the stationary point, in which only a unique \( \Delta D_2 \) and \( \Delta B_2 \) corresponds to a unique value of \( G \).

Therefore, the value of \( \Delta D_2 \) and \( \Delta B_2 \) at the stationary point could be the final competition result of the above-mentioned interaction. Then one would have following equations:

\[
\frac{\partial G}{\partial \Delta D_2} = \frac{H_{121} \sigma_{32}^e + H_{122} \Delta D_2 + H_{132} \Delta B_2}{G} = 0,
\]

(47a)

\[
\frac{\partial G}{\partial \Delta B_2} = \frac{H_{131} \sigma_{32}^e + H_{132} \Delta D_2 + H_{332} \Delta B_2}{G} = 0
\]

(47b)

which leads to

\[
\Delta D_2^0 = \Delta D_2 - \Delta D_2^0 = \Delta D_2 - (H_{121} H_{31} - H_{122} H_{33}/(H_{122} H_{33} - H_{331} \sigma_{32}^e), \\
\Delta B_2^0 = \Delta B_2 - \Delta B_2^0 = \Delta B_2 - (H_{132} H_{32} - H_{123} H_{33}/(H_{123} H_{33} - H_{332} \sigma_{32}^e)
\]

(48)

where \( H_{ij} \) \( (i,j=1,2,3) \) are elements of the bimaterial matrix \( H \) defined in (27). The result of (48) can be shown the same as those in literature if the two media of this current bimaterial are identical. This agreement justifies the above energy method. From the result in (48), one may see that the electric-magnetic field inside the interface crack is a function of the bimaterial property under given remote applied loading.

One may also observe from (47a) that if one wants \( \Delta D_2 \to 0 \) without magnetic field, then \( H_{22} \) needs to approach a very big value. This is called electrically impermeable. The parameter \( \lambda_e \), introduced by McMeeking [17], is used to characterize the electric permeability. A similar parameter, \( \lambda_m \), could be defined from the observation made on (47b), in which if \( \Delta B_2 \to 0 \) without electric field, then \( H_{33} \) has to approach a very big value, a phenomenon called magnetically impermeable. These two parameters \( \lambda_e \) and \( \lambda_m \) have the relationship of \( \lambda_m / \lambda_e = (\mu_0 / \mu_0) / (H_{122} / H_{33}) \).

Therefore, for an impermeable interface crack, \( \Delta D_2 = \Delta B_2 = 0 \) and the ESIF can be expressed as

\[
K = \sqrt{\frac{\pi(b-a)}{2}} \left[ \sigma_{32}^e, \Delta D_2, \Delta B_2 \right]^T
\]

(49)

The energy release rate for this interface crack reads

\[
G_{imp} = \frac{1}{4} K^2 HK = \frac{\pi(b-a)}{8} \left[ H_{121} \sigma_{32}^e + H_{122} \Delta D_2^0 + H_{132} \Delta B_2^0 \\
+ 2H_{211} \sigma_{32}^e \Delta D_2^0 + 2H_{121} \sigma_{32}^e \Delta B_2^0 + 2H_{332} \sigma_{32}^e \Delta B_2^0 \right]
\]

(50)

For a permeable interface crack, \( \lambda_m = \lambda_e = 0 \), the \( \Delta D_2 \) and \( \Delta B_2 \) are given by Eq. (48), and the ESIF can be expressed as

\[
K = \sqrt{\frac{\pi(b-a)}{2}} \left[ \sigma_{32}^e, \Delta D_2, \Delta B_2 \right]^T
\]

(51)

The corresponding energy release rate reads

\[
G_{perm} = \frac{\pi(b-a)}{8} \text{det}(\hat{H})(\sigma_{32}^e)^2
\]

(52)

where the matrix \( \hat{H} \), a principal submatrix of \( H \), is

\[
\hat{H} = \left( \begin{array}{cc} H_{22} & H_{23} \\ H_{32} & H_{33} \end{array} \right)
\]

(53)

and \( \text{det}(\cdot) \) is the determinant of a square matrix.

One interesting observation from Eq. (52) is that, though the energy release rate, \( G \), is independent of the applied electric-magnetic load, it is affected by electric-magnetic properties of the two constituents of the bimaterial media.

4 Numerical Results

In this section, the influence of the material property mismatches between the two constituents of the bimedia and the effects from magneto-electric coupling on the interface crack
growth behavior will be demonstrated by some numerical results. The basic data for the material properties selected here are similar to those in [6]. These constants read as \( c_{44}^{II} = 43.7 \text{ GPa}; \) \( e_{15}^{I} = 8.12 \text{ C/m}^2; \) \( e_{11}^{II} = 7.86 \times 10^{-9} \text{ C/Vm}; \) \( \alpha_{11}^{II} = 0.0; \) \( Q_{15}^{II} = 165.0 \text{ N/Am}; \mu_{11}^{II} = 180.5 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \) for the upper medium (medium “I”); and \( c_{44}^{II} = 44.6 \text{ GPa}; \) \( e_{15}^{II} = 3.48 \text{ C/m}^2; \) \( e_{11}^{II} = 3.42 \times 10^{-9} \text{ C/Vm}; \) \( \alpha_{11}^{II} = 0.0; \) \( Q_{15}^{I} = 385.0 \text{ N/Am}; \mu_{11}^{II} = 414.5 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \) for the lower medium (medium “II”).

Figures 2 and 3 present the influences of the bimaterial property mismatches \( c_{44}^{I}/c_{44}^{II}, e_{11}^{I}/e_{11}^{II}, \) and \( \mu_{11}^{II}/\mu_{11}^{I} \) on \( \Delta D_2^0 \) and \( \Delta B_2^0, \) which relate to the electric-magnetio field, \( D_0^I \) and \( B_0^I, \) inside the interface crack by Eq. (17). One may easily see from Fig. 2 that the electric displacement \( \Delta D_2^0 \) decreases as the degree of anisotropy of these two constituents of the bimedia, defined by \( c_{44}^{I}/c_{44}^{II}; \) increases, while it increases as the electric permittivity ratio, \( e_{11}^{I}/e_{11}^{II}; \) increases. But it practically does not change as the magnetic permeability ratio, \( \mu_{11}^{II}/\mu_{11}^{I}; \) increases. The magnetic induction field \( \Delta B_2^0 \) decreases as \( e_{11}^{I}/e_{11}^{II}; \) and \( \mu_{11}^{II}/\mu_{11}^{I}; \) increases, while it increase as \( e_{11}^{I}/e_{11}^{II}; \) increases, as shown in Fig. 3. One can also see the \( \Delta D_2^0 \) and \( \Delta B_2^0 \) do not vary with the increase of \( c_{44}^{I}/c_{44}^{II}, e_{11}^{I}/e_{11}^{II}, \) and \( \mu_{11}^{II}/\mu_{11}^{I}; \) after they reach some value.

Figure 4 shows the influence on the energy release rate, \( G, \) of the mismatch of the degree of anisotropy for the bimedia under pure mechanical tension. The \( G \) decreases as the \( c_{44}^{I}/c_{44}^{II} \) increase, both for permeable and impermeable interface cracks. It can also be seen that when \( c_{44}^{I}/c_{44}^{II} \) reaches some value (around 12.5 for this bimedia), the \( G \) almost does not vary with the increase in the mismatch on \( e_{11}^{I}; \) between the two constituents of the bimedia. Another interesting result observed from this figure is that for a given \( \sigma_{23}, G_{perm} \) is larger than \( G_{imp} \). This observation shows that the electric-magnetic field inside the interface crack may have an interaction with the stress field inside the bimaterial system, thus it has an influence on the propagation behavior of the interface crack. This observation may also suggest that the design of a piezo-electro-magneto-elastic bimaterial system based on a permeable assumption is more conservative than based on impermeable assumption.

Figures 5–7 show the influences on \( G \) from the directions of applied \( D_2^I \) and \( B_2^I, \) respectively. Figure 5 shows the results for loading \( D_2^I \) and \( \sigma_{32}, \) Fig. 6 for \( B_2^I \) and \( \sigma_{32}, \) while Fig. 7 is for combined loading \( D_2^I, B_2^I \), and \( \sigma_{32}. \) In these figures, a negative \( G \) can be observed under certain mechanically applied load, namely \( \sigma_{32}^{rd}, \) for a given \( D_2^I \) and/or \( B_2^I. \) These negative values on \( G \) may mean that the applied electric-magnetic loading would retard the propagation of an interface crack in piezo-electro-magnetic bimaterials, a result which was also found in Ref. [12] for cracks in monolithic piezo-electro-magnetic materials. The \( \sigma_{32}^{rd} \) varies as the direction of \( D_2^I \) or \( B_2^I \) reverses. One can also observe that there exists a direction in which the combined loading applied would make \( \sigma_{32}^{rd} \) reach its maximum and minimum value.

Figures 8 and 9 more clearly show the retarding effects, respectively, of \( e_{11}^{I}/e_{11}^{II} \) and \( \mu_{11}^{II}/\mu_{11}^{I} \) on the energy release rate \( G \) under pure loading \( D_2^I \) or \( B_2^I. \) In these two pictures, the value of \( G \) is always negative since the applied mechanical loading \( \sigma_{32} \) is zero. The \( G \) increases as \( e_{11}^{I}/e_{11}^{II} \) or \( \mu_{11}^{II}/\mu_{11}^{I} \) increases, a result consist with the observation in Figs. 11 and 12.

Plotted in Figs. 10–12 are, correspondingly, the influences of \( c_{44}^{I}/c_{44}^{II}, e_{11}^{I}/e_{11}^{II}, \) and \( \mu_{11}^{II}/\mu_{11}^{I} \) on the energy release rate, \( G, \) under combined electric, magnetic, and mechanical loading for an impermeable interface crack. The plotting in solid line is for the
positive direction in $D_2$ and $B_2$, the dashed line is for revised direction in $D_2$ and $B_2$. The $G$ decreases as the $c_{44}^{II}/c_{44}^{I}$ increases and keeps almost unchanged when $c_{44}^{II}/c_{44}^{I}$ reaches a certain value for both applied loading directions, as shown in Fig. 10. On the contrary, the $G$ increases as the $\varepsilon_{11}^{II}/\varepsilon_{11}^{I}$ and $\mu_{11}^{II}/\mu_{11}^{I}$ increase, respectively. The observations in these figures may suggest that a reasonable selection in the mechanical and electric-magnetic properties for the two constituents of a bimaterial media may lower the energy release rate, making this bimedia much safer with regard to propagation of cracks.

Finally, it should be mentioned that the important contribution of our paper is the novel procedure, which has been developed to solve for the electric-magnetic fields inside an interface crack in a general bimaterial. The exact agreement of the results from this method with the results from the mapping method for the special case of homogeneous material (i.e., no bimaterial) in the literature, which, again, is the only case solved in the literature, provides validity for our “energy method” approach. It should be noted at this point that the contribution of the electric-magnetic fields inside a crack is very important for the devices being made of piezo-magneto-electro-elastic materials, since these fields may interfere with the desirable signals of electric-magnetic fields, like in broadband detecting devices. The results of our study could offer tentative guidelines for the damage-tolerant design of the devices.

5 Conclusions

In the present paper, the mode III interface crack in dissimilar piezo-magneto-electro-elastic bimaterial media is investigated in Stroh’s formulism. In this study, the electric-magnetic field inside the interface crack is also considered and an “energy method” is proposed for obtaining the solution to this electric-magnetic field. Two types of interface cracks, namely permeable and impermeable cracks, are addressed. All the solutions are derived in closed form. The following conclusions can be reached from the results in this study:

1. The “energy method” is a very effective way to derive a solution to the electric-magnetic field inside a crack, thus
solving the whole interface crack problem when the electric-magnetic field inside a crack is taken into account.

2. The mismatches of \( c_{44}, e_{11}, \) and \( \mu_{11} \) between the two constituents of a bimaterial media have strong effects on the potential propagation of a mode III interface crack. There exists an optimal selection on \( c_{44}, e_{11}, \) and \( \mu_{11} \) that would minimize the energy release rate for this mode III interface crack.

3. The directions of the applied loading \( D_{2}^{e} \) and \( B_{2}^{m} \) also have an effect on the possible growth of the interface crack in a piezo-electro-magneto-elastic bimaterial media.

4. The applied electric and/or magnetic loading \( D_{2}^{e} \) and \( B_{2}^{m} \) usually retard the propagation of the mode III interface crack.

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**Appendix: Contour Integral for** \( \Phi(z)’ \)

The method used here can be viewed as the generalization of the technique in \([16, 110, and 70]\) which is for a single equation. Let \( \gamma \) be a contour which includes the arc \( ab \), and let this contour shrink into the arc \( ab \). Then for \( q(x_{1}) \) constant

\[
\int_{\gamma} \frac{X(x_{1})}{x_{1} - z} \frac{d}{dx_{1}} N_{1}^{\gamma} dx_{1} = \int_{ab} \frac{X(x_{1})}{x_{1} - z} \frac{d}{dx_{1}} 
\]

From Eq. (32), one could have

\[
X_{\gamma}(x_{1}) = -N_{1}^{\gamma}N_{2}(x_{1}), \quad a < x_{1} < b
\]

Substituting Eq. (A2) into (A1) leads

**Fig. 9** Energy release rate, \( G \), versus the magnetic permeability ratio, \( \mu_{11}^{\mu}/\mu_{11}^{\mu} \), for an impermeable interface crack under pure \( B_{0}^{m} \) loading

**Fig. 10** Energy release rate, \( G \), versus the stiffness ratio, \( c_{44}^{II}/c_{44}^{I} \), for an impermeable interface crack under combined applied loading

**Fig. 11** Energy release rate, \( G \), versus the electric permittivity ratio, \( e_{11}^{\epsilon}/e_{11}^{\epsilon} \), for an impermeable interface crack under combined applied loading

**Fig. 12** Energy release rate, \( G \), versus the magnetic permeability ratio, \( \mu_{11}^{\mu}/\mu_{11}^{\mu} \), for an impermeable interface crack under combined applied loading
\[ \int_{\gamma} \frac{[X(\xi)]^{-1} N^{-1}}{\xi - z} \, d\xi = \int_{\gamma} \frac{[X(x_1)]^{-1} N^{-1}[I + \tilde{N} N^{-1}]}{x_1 - z} \, dx_1 \quad (A3) \]

Then,

\[ \int_{\gamma} \frac{[X(x_1)]^{-1} N^{-1}}{x_1 - z} \, dx_1 = \int_{\gamma} \frac{[X(\xi)]^{-1} N^{-1}[I + \tilde{N} N^{-1}]^{-1}}{\xi - z} \, d\xi = \int_{\gamma} \frac{[X(\xi)]^{-1} [N + \tilde{N}]^{-1}}{\xi - z} \, d\xi \quad (A4) \]

Since

\[ N = \tilde{N} = H^{-1} \quad (A5) \]

then,

\[ \Phi'(z) = \frac{1}{2\pi i} X(z) \int_{\gamma} \frac{[X(x_1)]^{-1} N^{-1}[ip]}{x_1 - z} \, dx_1 = \text{diag}\left( 1 - \frac{z - (a + b)/2}{\sqrt{(z - a)(z - b)}} \right) H(ip) \quad (A6) \]

References


