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Nonlinear High-Order Core Theory for Sandwich Plates with Orthotropic Phases

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A new nonlinear high-order theory for orthotropic elastic sandwich plates is formulated. In this theory, in which the compressibility of the soft core in the transverse direction is considered, the transverse displacement in the core is of fourth order in the transverse coordinate and the in-plane displacements are of fifth order in the transverse coordinate. The theory is derived so that all core/face-sheet displacement continuity conditions are fulfilled. The nonlinear governing equations, as well as the boundary conditions for sandwich plates with orthotropic phases, are derived via a variational principle. The solution procedure is outlined and numerical results for the simply supported case of transversely loaded plates are produced for several typical sandwich configurations. These results are compared with the corresponding ones from the elasticity solution. Furthermore, the results using the classical sandwich model with and without shear are also presented. The comparison among these numerical results shows that the solution from the current theory is very close to that of the elasticity in terms of both the displacements and the transverse stress through the core. Observations in the current work suggest that this new high-order theory could have significant applications in studying the structural and failure behavior of sandwich plates.

Nomenclature

a	=	plate length
b	=	plate width
c	=	core half-thickness
f	=	face-sheet thickness
h_{tot}	=	total plate thickness
q_0	=	peak value of the applied sinusoidally distributed transverse load
u	=	in-plane x displacement
v	=	in-plane y displacement
w	=	transverse displacement (along z)
x	=	in-plane coordinate along the length
y	=	in-plane coordinate along the width
z	=	thicknesswise coordinate

Subscript

0	=	middle surface

Superscripts

b	=	bottom face sheet
c	=	core
t	=	top face sheet

I. Introduction

IT IS the core and its layout in a sandwich construction, consisting of two metallic or composite thin face sheets separated by a thick core, that give this type of structure superior properties: namely, high stiffness and strength with little resultant weight penalty and high energy-absorption capability with regard to impact loading. With the increasing interest in the application of sandwich structures to aerospace vehicles, marine vessels, and civil infrastructure, the last decade has seen extensive efforts being devoted to the study of the static and dynamic behavior, failure mechanisms, design, and manufacturing improvement of such structures [1–18]. In spite of this fact, a detailed review of the literature reveals that most of the researchers have based their work on the noncompressible core model, in which the rigidity of the core in the thickness direction is assumed infinite; that is, it is transversely incompressible. Depending on whether or not the shear effects are considered, this model is also referred to as the classical model or the first-order shear model, respectively. This analytical model was proposed in the 1960s [1,2] and is simple for practical use. Its assumption is acceptable for most cases in which the sandwich core is stiff and the loading is static.

However, when the core is soft compared with the faces of the sandwich structure, the incompressible model may not be appropriate. As a matter of fact, recent numerical simulations [3] and experimental results [4] show that the core is under significant nonlinear deformation when a sandwich structure is subject to underwater blast loading. Because the classical or first-order shear models do not account for the core transverse deformation, the information regarding the transverse stress distribution through the thickness cannot be properly obtained. The absence of such knowledge may lead to inaccurate prediction of the failure modes, such as the debonding between the faces and the core, or of the energy-absorption capacity. To fill this gap, Hohe et al. [5] proposed an approximate model with constant transverse strain in the core. Furthermore, Frostig et al. [6] proposed a model with linear transverse strain through the thickness of the core, but this model was formulated as a one-dimensional model (i.e., for a beam). However, when a sandwich structure is subject to severe loading such as that induced from blast loading or impact, the core could undergo highly nonlinear transverse deformation [3,4]. Therefore, a new analytical but practical (i.e., easy to be implemented) high-order compressible core model that can take this nonlinearity into account would have significance in the study and design of sandwich structures.

The accuracy of any of these models can be readily assessed because an elasticity solution exists. Indeed, Pagano [7] presented the three-dimensional elasticity solution for rectangular laminates and sandwich plates for the cases of 1) a phase with negative discriminant of the cubic characteristic equation, which is formed from the orthotropic material constants, and 2) an isotropic phase, which results in a zero discriminant. The roots in this case are all real and unequal (negative discriminant) or all real and equal (isotropic case). In a recent paper, Kardomateas [8] presented the corresponding solution for the case of positive discriminant, in which case two of the roots are complex conjugates. This is actually a

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case frequently encountered in sandwich construction, in which the orthotropic core is stiffer in the transverse than the in-plane directions.

In this paper, we present an advanced new high-order core model in which the transverse displacement within the core is no longer a constant, but it varies as a fourth-order function of the transverse coordinate. The in-plane displacements of the core are fifth-order functions of the transverse coordinate. The transverse compressibility and shear deformation of the core are incorporated into the constitutive law. To verify the current model, a study of a typical sandwich plate with elastic orthotropic phases subject to transverse loading is performed. The results are obtained using the elasticity solution [7,8], the classical model [1,2,9], and the first-order shear model [1,2,9]. The comparison among these results is made to illustrate the validity of the current model. We should note that the linear core strain model in [6] was presented for beamlike sandwich panels and no formulas for plates are available; furthermore, the constant-core-strain model assumptions were outlined in [5], but no explicit equations that can be readily solved for the displacements were presented; therefore, a comparison with the linear model in [5] was also not possible at this point.

The sandwich structures considered here are composed of two composite/metallic thin face sheets of high mechanical properties separated by a thick soft core. Transverse compressibility of the core will be taken into account. In the development of the high-order core theory, the following assumptions will be adopted:

1) The face sheets satisfy the Kirchhoff–Love assumption, and their thicknesses are small compared with the overall thickness of the sandwich section. In the current study, the two face sheets are further assumed to be identical.

2) The core is compressible in the transverse direction: that is, its thickness may change.

3) The bonding between the face sheets and the core is assumed to be perfect.

The paper is organized as follows: In Sec. II we develop the new high-order compressible core theory. In this theory, the transverse displacement of the initial midplane is considered as an unknown function of the in-plane coordinates (x, y). The in-plane and transverse displacements in the core are then expressed as functions in terms of the displacements of the two face sheets and the displacement of the core initial midplane. In the derivation, the displacement continuity conditions at the interfaces between face sheets and the core are employed. In Sec. III we present the derivation of the governing equations and associated boundary conditions for the sandwich plate. Subsequently, the equations for orthotropic sandwich plates are presented in detail. In Sec. IV we outline the solution procedure to solve the nonlinear equations by applying this theory to a simply supported sandwich plate under transverse loading. In Sec. V we present the numerical results for a typical sandwich-plate configuration. The comparison of the results with those from the elasticity solution, the classical model, and the first-order shear model is also given in this section. In Sec. VI we give some conclusions and suggestions on future work. For completeness, the governing equations for the classical model and the first-order shear model are given in Appendices A and B.

II. Derivation of the New High-Order Shear Theory

In the following, we consider a sandwich plate with two identical face sheets of thickness f and a core of thickness $2c$ and let a Cartesian coordinate system (x, y, z) be on the middle plane of the core, as shown in Fig. 1. The corresponding displacements are denoted by (u, v, w). We further use the superscripts t, b, and c to refer to the top face sheet, bottom face sheet, or core, respectively, and the subscript 0 refers to the middle surface of the corresponding phase.

A. Displacement and Strain Representation for the Face Sheets

The face sheets are assumed to satisfy the Kirchhoff–Love assumptions, and their thicknesses are small compared with the overall thickness of the sandwich section. Therefore, the displacements for the top face sheet are expressed as

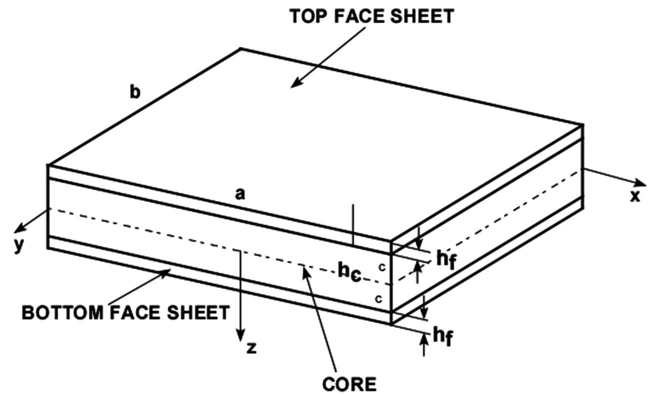

Fig. 1 Definition of the geometrical configuration for the sandwich plate.

$$u^t(x, y, z) = u_0^t(x, y) - \left(z + c + \frac{f}{2}\right) w^t_{,x}(x, y) \tag{1a}$$

$$v^t(x, y, z) = v_0^t(x, y) - \left(z + c + \frac{f}{2}\right) w^t_{,y}(x, y) \tag{1b}$$

$$w^t(x, y, z) = w^t(x, y), \qquad -(c + f) \le z \le -c \tag{1c}$$

and for the bottom face sheet,

$$u^b(x, y, z) = u_0^b(x, y) - \left(z - c - \frac{f}{2}\right) w^b_{,x}(x, y) \tag{2a}$$

$$v^b(x, y, z) = v_0^b(x, y) - \left(z - c - \frac{f}{2}\right) w^b_{,y}(x, y) \tag{2b}$$

$$w^b(x, y, z) = w^b(x, y), \qquad c \le z \le (c + f) \tag{2c}$$

Omitting the superscripts t and b, the nonlinear strain displacement relations for the face sheets can take the following form:

$$[\epsilon] = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = [\epsilon_0] + \zeta[k] = \begin{bmatrix} \epsilon_{0x} + \zeta k_x \\ \epsilon_{0y} + \zeta k_y \\ \gamma_{0xy} + \zeta k_{xy} \end{bmatrix}, \qquad \zeta = z \pm \left(c + \frac{f}{2}\right) \tag{3a}$$

in which the $\pm$ sign in the variable ζ corresponds to the top and bottom face sheets, respectively, and $[\epsilon_0]$ is the middle surface strain given by

$$[\epsilon_0] = \begin{bmatrix} \epsilon_{0x} \\ \epsilon_{0y} \\ \gamma_{0xy} \end{bmatrix} = \begin{bmatrix} u_{0,x} + \frac{1}{2}w_{,x}^2 \\ v_{0,y} + \frac{1}{2}w_{,y}^2 \\ u_{0,y} + v_{0,x} + w_{,x}w_{,y} \end{bmatrix} \tag{3b}$$

Moreover, $[k]$ is the curvature

$$[k] = \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} = \begin{bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{bmatrix} \tag{3c}$$

B. Displacements and Strains for the High-Order Core Theory

In practice, the core can be deformed in the transverse direction when a sandwich structure is loaded. As a first-order approximation, this deformation may be neglected, as is the case in the classical sandwich model. But in many instances it is necessary to include the transverse deformation. For example, the transverse deformation may be crucial in the energy-absorption capability of a structure subject to extreme loading such as blast loading. To capture the core transverse compressibility, we can use a higher-order series expansion in terms of the transverse coordinate to represent the in-plane and out-of plane displacements. In the current study, the transverse displacement in the core, w^c, is of fourth order in the transverse direction z:

$$w^c(x,y,z) = \left[\beta_0 - \beta_2\frac{z^2}{(2c)^2} - \beta_4\frac{z^4}{(2c)^4}\right]w_o^c(x,y) + \left[\beta_2\frac{z^2}{(2c)^2} + \beta_4\frac{z^4}{(2c)^4}\right]\bar{w}(x,y) - \left[\beta_1\frac{z}{2c} + \beta_3\frac{z^3}{(2c)^3}\right]\hat{w}(x,y), \qquad -c \le z \le c \tag{4a}$$

where $w_0^c(x,y)$ is the transverse displacement of the middle surface of the core, $[\beta_0 \cdots \beta_4]$ are constants to be determined, and $\bar{w}(x,y)$ and $\hat{w}(x,y)$ are, respectively, the average and difference of the middle surface transverse displacements for the two face sheets:

$$\bar{w}(x,y) = \tfrac{1}{2}[w^t(x,y) + w^b(x,y)], \quad \hat{w}(x,y) = \tfrac{1}{2}[w^t(x,y) - w^b(x,y)] \tag{4b}$$

The in-plane displacements in the core, u^c and v^c, are of fifth order in z, expressed as follows:

$$u^c(x,y,z) = \bar{u}(x,y) + \beta_5\frac{z}{2c}\hat{u}(x,y) + z\frac{f}{2c}w^c_{,x}(x,y,z) \tag{4c}$$

$$v^c(x,y,z) = \bar{v}(x,y) + \beta_6\frac{z}{2c}\hat{v}(x,y) + z\frac{f}{2c}w^c_{,y}(x,y,z) \tag{4d}$$

where $\bar{u}(x,y,t)$, $\hat{u}(x,y,t)$, $\bar{v}(x,y,t)$, and $\hat{v}(x,y,t)$ are, respectively, the average and difference of the middle surface in-plane displacements for the two face sheets:

$$\bar{u}(x,y) = \tfrac{1}{2}[u_0^t(x,y) + u_0^b(x,y)], \quad \hat{u}(x,y) = \tfrac{1}{2}[u_0^t(x,y) - u_0^b(x,y)] \tag{4e}$$

$$\bar{v}(x,y) = \tfrac{1}{2}[v_0^t(x,y) + v_0^b(x,y)], \quad \hat{v}(x,y) = \tfrac{1}{2}[v_0^t(x,y) - v_0^b(x,y)] \tag{4f}$$

Therefore, there are seven constants β_i $(i = 0, 6)$ to be determined from displacement continuity, as follows:

For the top-face-sheet/core interface, $z = -c$,

$$u^c(x,y,z)|_{z=-c} = u_0^t(x,y) - \frac{f}{2}w^t_{,x}(x,y) \tag{5a}$$

$$v^c(x,y,z)|_{z=-c} = v_0^t(x,y) - \frac{f}{2}w^t_{,y}(x,y) \tag{5b}$$

$$w^c(x,y,z)|_{z=-c} = w^t(x,y) \tag{5c}$$

For the bottom-face-sheet/core interface, $z = c$,

$$u^c(x,y,z)|_{z=c} = u_0^b(x,y) + \frac{f}{2}w^b_{,x}(x,y) \tag{5d}$$

$$v^c(x,y,z)|_{z=c} = v_0^b(x,y) + \frac{f}{2}w^b_{,y}(x,y) \tag{5e}$$

$$w^c(x,y,z)|_{z=c} = w^b(x,y) \tag{5f}$$

Also, at the midsurface of the core, $z = 0$,

$$w^c(x,y,z)|_{z=0} = w_0^c(x,y) \tag{5g}$$

Substitution of Eqs. (4) into the seven continuity conditions (5) leads to

$$\beta_0 = \beta_1 = 1, \quad \beta_2 = -2, \quad \beta_3 = -4, \quad \beta_4 = -8, \quad \beta_5 = \beta_6 = -\tfrac{1}{2} \tag{6}$$

Thus, the transverse displacement in the core in this new high-order core theory can be expressed as follows (fourth order in z):

$$w^c(x,y,z) = \left(1 - \frac{z^2}{2c^2} - \frac{z^4}{2c^4}\right)w_0^c(x,y) + \left(\frac{z^2}{2c^2} + \frac{z^4}{2c^4}\right)\bar{w}(x,y) - \left(\frac{z}{2c} + \frac{z^3}{2c^3}\right)\hat{w}(x,y), \qquad -c \le z \le c \tag{7a}$$

and the in-plane displacements in the core are (fifth order in z)

$$u^c(x,y,z) = \bar{u}(x,y) - \frac{z}{c}\hat{u}(x,y) + z\frac{f}{2c}w^c_{,x}(x,y,z) \tag{7b}$$

$$v^c(x,y,z) = \bar{v}(x,y) - \frac{z}{c}\hat{v}(x,y) + z\frac{f}{2c}w^c_{,y}(x,y,z) \tag{7c}$$

where $w_0^c(x,y,t)$ is the transverse displacement of the middle surface of the core.

This leads to the following strain displacement relations for the core:

$$\epsilon^c_{zz} = \left(-\frac{1}{4c} + \frac{z}{2c^2} - \frac{3z^2}{4c^3} + \frac{z^3}{c^4}\right)w^t(x,y) - \left(\frac{z}{c^2} + \frac{2z^3}{c^4}\right)w_o^c(x,y) + \left(\frac{1}{4c} + \frac{z}{2c^2} + \frac{3z^2}{4c^3} + \frac{z^3}{c^4}\right)w^b(x,y) \tag{8a}$$

$$\gamma^c_{xz} = -\frac{1}{c}\hat{u}(x,y) + \eta_1(z)w^t_{,x}(x,y) + \eta_2(z)w^c_{o,x}(x,y) + \eta_3(z)w^b_{,x}(x,y) \tag{8b}$$

$$\gamma^c_{yz} = -\frac{1}{c}\hat{v}(x,y) + \eta_1(z)w^t_{,y}(x,y) + \eta_2(z)w^c_{o,y}(x,y) + \eta_3(z)w^b_{,y}(x,y) \tag{8c}$$

in which

$$\eta_1(z) = -\left(1 + \frac{f}{c}\right)\frac{z}{4c} + \left(1 + \frac{3f}{2c}\right)\frac{z^2}{4c^2} - \left(1 + \frac{2f}{c}\right)\frac{z^3}{4c^3} + \left(1 + \frac{5f}{2c}\right)\frac{z^4}{4c^4} \tag{8d}$$

$$\eta_2(z) = \left(1 + \frac{f}{2c}\right) - \left(1 + \frac{3f}{2c}\right)\frac{z^2}{2c^2} - \left(1 + \frac{5f}{2c}\right)\frac{z^4}{2c^4} \tag{8e}$$

$$\eta_3(z) = \left(1 + \frac{f}{c}\right)\frac{z}{4c} + \left(1 + \frac{3f}{2c}\right)\frac{z^2}{4c^2} + \left(1 + \frac{2f}{c}\right)\frac{z^3}{4c^3} + \left(1 + \frac{5f}{2c}\right)\frac{z^4}{4c^4} \tag{8f}$$

It should be noted that the core is considered to be undergoing large rotation with a small displacement; therefore, the in-plane strains can be neglected.

C. Constitutive Relations

The equations developed so far can be applied to general materials. In the following sections, we shall assume that the face sheets are orthotropic laminated composites and that the core is also orthotropic. The general stress–strain relationship for any layer of the face sheets reads as

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix}\quad\text{or}\quad [\sigma]=[C][\epsilon] \tag{9a}$$

where C_{ij} $(i,j=1,\ 2,\ 6)$ are the plane-stress reduced-stiffness coefficients. With Eqs. (3), (8), and (9a) and ζ defined from Eq. (3a) as the local coordinate from the midplane of each face sheet, one can find the resultants for the top and bottom face sheets of a sandwich plate:

$$[N^{t,b}]=\begin{bmatrix}N_x^{t,b}\\ N_y^{t,b}\\ N_{xy}^{t,b}\end{bmatrix}=\int_{-f/2}^{f/2}[\sigma^{t,b}]\,\mathrm{d}\varsigma=\int_{-f/2}^{f/2}[C^{t,b}][\epsilon^{t,b}]\,\mathrm{d}\varsigma$$
$$=[A^{t,b}][\epsilon_0^{t,b}]+[B^{t,b}][k^{t,b}] \tag{9b}$$

$$[M^{t,b}]=\begin{bmatrix}M_x^{t,b}\\ M_y^{t,b}\\ M_{xy}^{t,b}\end{bmatrix}=\int_{-f/2}^{f/2}[\sigma^{t,b}]\varsigma\,\mathrm{d}\varsigma=[B^{t,b}][\epsilon_o^{t,b}]+[D^{t,b}][k^{t,b}] \tag{9c}$$

in which the stiffness coefficients are

$$[A_{ij}^{t,b},B_{ij}^{t,b},D_{ij}^{t,b}]=\int_{-f/2}^{f/2}C_{ij}\times\{1,\varsigma,\varsigma^2\}\,\mathrm{d}\varsigma,\qquad i,j=1,2,6 \tag{9d}$$

Applying a similar procedure, one can obtain the following resultant expressions for the bottom face sheet:

$$[N^b]=[A^b][\epsilon_0^b]+[B^b][k^b] \tag{9e}$$

$$[M^b]=[B^b][\epsilon_0^b]+[D^b][k^b] \tag{9f}$$

with the stiffness coefficients reading as

$$[A_{ij}^b,B_{ij}^b,D_{ij}^b]=\int_c^{c+f}C_{ij}\times\{1,z,z^2\}\,\mathrm{d}z,\qquad i,j=1,2,6 \tag{9g}$$

The stress–strain relations for an orthotropic core can be written as

$$\sigma_{zz}^c=E^c\epsilon_{zz}^c,\qquad \tau_{xz}^c=G_{xz}^c\gamma_{xz}^c,\qquad \tau_{yz}^c=G_{yz}^c\gamma_{yz}^c \tag{10}$$

III. Governing Equations and Associated Boundary Conditions

The governing equations and appropriate boundary conditions can be derived using the variational principle. The sandwich plate is assumed to be subject to a transverse loading $q(x,y)$ on the top face sheet. Let the strain energy be denoted by U and the external work by W, then the variational principle states

$$\delta(U-W)=0 \tag{11a}$$

in which

$$\delta U=\int_0^b\int_0^a\bigg[\int_{-c-f}^{-c}(\sigma_{xx}^t\delta\epsilon_{xx}^t+\sigma_{yy}^t\delta\epsilon_{yy}^t+\tau_{xy}^t\delta\gamma_{xy}^t)\,\mathrm{d}z$$
$$+\int_{-c}^{c}(\sigma_{zz}^c\delta\epsilon_{zz}^c+\tau_{xz}^c\delta\gamma_{xz}^c+\tau_{yz}^c\delta\gamma_{yz}^c)\,\mathrm{d}z$$
$$+\int_c^{c+f}(\sigma_{xx}^b\delta\epsilon_{xx}^b+\sigma_{yy}^b\delta\epsilon_{yy}^b+\tau_{xy}^b\delta\gamma_{xy}^b)\,\mathrm{d}z\bigg]\mathrm{d}x\,\mathrm{d}y \tag{11b}$$

$$\delta W=\int_0^b\int_0^a q(x,y)\delta w^t\,\mathrm{d}x\,\mathrm{d}y \tag{11c}$$

The governing differential equations and the boundary conditions can be obtained by substituting the stress–strain relations (9) and (10), strain displacement relations, and displacement profile relations (7) and then employing integration by parts. This results in seven equations: three for each face sheet and one for the core. There are seven unknowns: u_0^t, v_0^t, w^t, w_0^c, u_0^b, v_0^b, and w^b.

The equations for the top face sheet can be written as

$$\delta u_0^t\!:\ N_{x,x}^t+N_{xy,y}^t+G_{xz}^c\left[-\frac{(u_0^t-u_0^b)}{2c}+\frac{11}{15}w_{o,x}^c+\alpha_0(w_{,x}^t+w_{,x}^b)\right]$$
$$=0 \tag{12a}$$

$$\delta v_0^t\!:\ N_{xy,x}^t+N_{y,y}^t+G_{yz}^c\left[-\frac{(v_o^t-v_o^b)}{2c}+\frac{11}{15}w_{o,y}^c+\alpha_0(w_{,y}^t+w_{,y}^b)\right]$$
$$=0 \tag{12b}$$

$$\delta w_0^t\!:\ M_{x,xx}^t+2M_{xy,xy}^t+M_{y,yy}^t+(N_x^tw_{,x}^t)_{,x}+(N_{xy}^tw_{,x}^t)_{,y}$$
$$+(N_{yx}^tw_{,y}^t)_{,x}+(N_y^tw_{,y}^t)_{,y}+2\alpha_1c(G_{xz}^cw_{,xx}^t+G_{yz}^cw_{,yy}^t)$$
$$+2\alpha_2c(G_{xz}^cw_{0,xx}^c+G_{yz}^cw_{0,yy}^c)-2\alpha_3c(G_{xz}^cw_{,xx}^b+G_{yz}^cw_{,yy}^b)$$
$$-\alpha_0[G_{xz}^c(u_{0,x}^t-u_{0,x}^b)+G_{yz}^c(v_{0,y}^t-v_{0,y}^b)]$$
$$-\frac{E^c}{2c}\left(\frac{61}{21}w^t-\frac{358}{105}w_0^c+\frac{53}{105}w^b\right)+q(x,y)=0 \tag{12c}$$

For the core,

$$\delta w_0^c\!:\ \frac{179E^c}{105c}(2w_0^c-w^t-w^b)+\frac{11}{15}G_{xz}^c(u_{0,x}^t-u_{0,x}^b)$$
$$+\frac{11}{15}G_{yz}^c(v_{0,y}^t-v_{0,y}^b)-2\alpha_2c[G_{xz}^c(w_{,xx}^t+w_{,xx}^b)$$
$$+G_{yz}^c(w_{,yy}^t+w_{,yy}^b)]-2\alpha_4c(G_{xz}^cw_{0,xx}^c+G_{yz}^cw_{0,yy}^c)=0 \tag{13}$$

For the bottom face sheet,

$$\delta u_0^b\!:\ N_{x,x}^b+N_{xy,y}^b-G_{xz}^c\left[\frac{(-u_0^t+u_0^b)}{2c}+\frac{11}{15}w_{0,x}^c+\alpha_0(w_{,x}^t+w_{,x}^b)\right]$$
$$=0 \tag{14a}$$

$$\delta v_0^b\!:\ N_{xy,x}^b+N_{y,y}^b-G_{yz}^c\left[\frac{(-v_0^t+v_0^b)}{2c}+\frac{11}{15}w_{0,y}^c+\alpha_0(w_{,y}^t+w_{,y}^b)\right]$$
$$=0 \tag{14b}$$

$$\delta w_0^b\!:\ M_{x,xx}^b+2M_{xy,xy}^b+M_{y,yy}^b+(N_x^bw_{,x}^b)_{,x}+(N_{xy}^bw_{,x}^b)_{,y}$$
$$+(N_{yx}^bw_{,y}^b)_{,x}+(N_y^bw_{,y}^b)_{,y}+2\alpha_1c(G_{xz}^cw_{,xx}^b+G_{yz}^cw_{,yy}^b)$$
$$+2\alpha_2c(G_{xz}^cw_{o,xx}^c+G_{yz}^cw_{o,yy}^c)-2\alpha_3c(G_{xz}^cw_{,xx}^t+G_{yz}^cw_{,yy}^b)$$
$$-\alpha_0[G_{xz}^c(u_{o,x}^t-u_{o,x}^b)+G_{yz}^c(v_{o,y}^t-v_{o,y}^b)]$$
$$-\frac{E^c}{2c}\left(\frac{61}{21}w^b-\frac{358}{105}w_o^c+\frac{53}{105}w^t\right)=0 \tag{14c}$$

The constants α_i $(i=0,\ldots,4)$ in the preceding equations relate to the ratio of face thickness and core thickness and are defined as follows:

$$\alpha_0=\frac{2}{15}+\frac{f}{4c},\qquad \alpha_1=\frac{29}{315}+\frac{373}{630}\left(\frac{f}{2c}\right)+\frac{247}{252}\left(\frac{f}{2c}\right)^2 \tag{15a}$$

$$\alpha_2 = \frac{37}{630} + \frac{37}{630}\left(\frac{f}{2c}\right) - \frac{383}{630}\left(\frac{f}{2c}\right)^2 \tag{15b}$$

$$\alpha_3 = \frac{11}{630} + \frac{11}{630}\left(\frac{f}{2c}\right) - \frac{23}{180}\left(\frac{f}{2c}\right)^2 \tag{15c}$$

$$\alpha_4 = \frac{194}{315} + \frac{194}{315}\left(\frac{f}{2c}\right) + \frac{383}{315}\left(\frac{f}{2c}\right)^2 \tag{15d}$$

The corresponding boundary conditions at $x = 0$, a, read as follows:

For the top face sheet,

$$u_0^t = \tilde{u}^t \quad \text{or} \quad N_x^t = \tilde{N}_x^t \tag{16a}$$

$$v_0^t = \tilde{v}^t \quad \text{or} \quad N_{xy}^t = \tilde{N}_{xy}^t \tag{16b}$$

$$w^t = \tilde{w}^t \quad \text{or} \quad N_x^t w_{,x}^t + M_{x,x}^t + N_{xy}^t w_{,y}^t + 2M_{xy,x}^t + G_{xz}^c[\alpha_0(u_0^b - u_0^t) + 2\alpha_1 c w_{,x}^t + 2\alpha_2 c w_{0,x}^c - 2\alpha_3 c w_{,x}^b] = \tilde{Q}_x^t \tag{16c}$$

where $\tilde{Q}_x^t$ is the resultant top-face-sheet shear, defined as the integral of τ_{xz} over the top face sheet, and

$$w_{,x}^t = \tilde{w}_{,x}^t \quad \text{or} \quad M_x^t = \tilde{M}_x^t \tag{16d}$$

For the core,

$$w_0^c = \tilde{w}_0^c \quad \text{or} \quad \frac{11}{15}(u_0^b - u_0^t) + 2\alpha_2 c w_{,x}^t + 2\alpha_4 c w_{0,x}^c + 2\alpha_2 c w_{,x}^b = \tilde{Q}_c \tag{17}$$

where $\tilde{Q}_c$ is the resultant core shear divided by the core shear modulus; that is, $\tilde{Q}_c$ is defined as the integral of τ_{xz}/G_c over the core.

For the bottom face sheet,

$$u_o^b = \tilde{u}^b \quad \text{or} \quad N_x^b = \tilde{N}_x^b \tag{18a}$$

$$v_o^b = \tilde{v}^b \quad \text{or} \quad N_{xy}^b = \tilde{N}_{xy}^b \tag{18b}$$

$$w^b = \tilde{w}^b \quad \text{or} \quad N_x^b w_{,x}^b + M_{x,x}^b + N_{xy}^b w_{,y}^b + 2M_{xy,x}^b + G_{xz}^c[\alpha_0(u_0^b - u_0^t) - 2\alpha_3 c w_{,x}^t + 2\alpha_2 c w_{0,x}^c + 2\alpha_1 c w_{,x}^b] = \tilde{Q}_x^b \tag{18c}$$

where, again, $\tilde{Q}_x^b$ is the resultant bottom-face-sheet shear, defined as the integral of τ_{xz} over the bottom face sheet, and

$$w_{,x}^b = \tilde{w}_{,x}^b \quad \text{or} \quad M_x^b = \tilde{M}_x^b \tag{18d}$$

The tilde accent denotes the known external boundary values. Similar equations can be written for $y = 0$, b. For the sandwich plates made out of orthotropic materials, one can rewrite the set of the nonlinear governing equations by substituting the stiffness constants.

For the top face sheet,

$$\left[A_{11}^t \frac{\partial^2}{\partial x^2} + A_{66}^t \frac{\partial^2}{\partial y^2} - \frac{G_{xz}^c}{2c}\right]u_0^t + (A_{12}^t + A_{66}^t)\frac{\partial^2}{\partial x \partial y} v_0^t + G_{xz}^c \alpha_0 w_{,x}^t + G_{xz}^c \frac{11}{15} w_{0,x}^c + G_{xz}^c \frac{u_0^b}{2c} + G_{xz}^c \alpha_0 w_{,x}^b = \hat{F}_1^t \tag{19a}$$

$$(A_{21}^t + A_{66}^t)\frac{\partial^2}{\partial x \partial y} u_0^t + \left[A_{66}^t \frac{\partial^2}{\partial x^2} + A_{22}^t \frac{\partial^2}{\partial y^2} - \frac{G_{yz}^c}{2c}\right]v_0^t + G_{yz}^c \alpha_0 w_{,y}^t + G_{yz}^c \frac{11}{15} w_{0,y}^c + G_{yz}^c \frac{v_0^b}{2c} + G_{yz}^c \alpha_0 w_{,y}^b = \hat{F}_2^t \tag{19b}$$

$$\begin{aligned}
&\left[D_{11}^t \frac{\partial^4}{\partial x^4} + 2(D_{12}^t + 2D_{66}^t)\frac{\partial^4}{\partial x^2 y^2} + D_{22}^t \frac{\partial^4}{\partial y^4} + \frac{61E^c}{42c} - 2\alpha_1 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^t \\
&- \left[\frac{179E^c}{105c} + 2\alpha_2 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w_0^c \\
&+ \left[\frac{53E^c}{210c} + 2\alpha_3 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^b \\
&+ \alpha_0 G_{xz}^c \frac{\partial}{\partial x}(u_0^t - u_0^b) + \alpha_0 G_{yz}^c \frac{\partial}{\partial y}(v_0^t - v_0^b) \\
&= q(x, y) + \hat{F}_3^t
\end{aligned} \tag{19c}$$

in which the right-hand sides are nonlinear terms defined as

$$\hat{F}_1^t = -A_{11}^t w_{,x}^t w_{,xx}^t - (A_{12}^t + A_{66}^t) w_{,y}^t w_{,xy}^t - A_{66}^t w_{,x}^t w_{,yy}^t \tag{19d}$$

$$\hat{F}_2^t = -(A_{21}^t + A_{66}^t) w_{,x}^t w_{,xy}^t - A_{66}^t w_{,xx}^t w_{,y}^t - A_{22}^t w_{,y}^t w_{,yy}^t \tag{19e}$$

$$\hat{F}_3^t = (N_x^t w_{,x}^t)_{,x} + (N_{xy}^t w_{,x}^t)_{,y} + (N_{yx}^t w_{,y}^t)_{,x} + (N_y^t w_{,y}^t)_{,y} \tag{19f}$$

For the core,

$$\begin{aligned}
&\frac{11}{15} G_{xz}^c \frac{\partial}{\partial x}(u_0^t - u_0^b) + \frac{11}{15} G_{yz}^c \frac{\partial}{\partial y}(v_0^t - v_0^b) \\
&- \left[\frac{179E^c}{105c} + 2\alpha_2 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^t \\
&+ \left[\frac{358E^c}{105c} - 2\alpha_4 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w_0^c \\
&- \left[\frac{179E^c}{105c} + 2\alpha_2 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^b = 0
\end{aligned} \tag{20}$$

For the bottom face sheet,

$$\left[A_{11}^b \frac{\partial^2}{\partial x^2} + A_{66}^b \frac{\partial^2}{\partial y^2} - \frac{G_{yz}^c}{2c}\right]u_0^b + (A_{12}^b + A_{66}^b)\frac{\partial^2}{\partial x \partial y} v_0^b - G_{xz}^c \alpha_0 w_{,x}^b - G_{xz}^c \frac{11}{15} w_{0,x}^c + \frac{G_{xz}^c}{2c} u_0^t - G_{xz}^c \alpha_0 w_{,x}^t = \hat{F}_1^b \tag{21a}$$

$$(A_{21}^b + A_{66}^b)\frac{\partial^2}{\partial x \partial y} u_0^b + \left[A_{66}^b \frac{\partial^2}{\partial x^2} + A_{22}^b \frac{\partial^2}{\partial y^2} - \frac{G_{yz}^c}{2c}\right]v_0^b - \alpha_0 G_{yz}^c w_{,y}^b - G_{yz}^c \frac{11}{15} w_{0,y}^c + \frac{G_{yz}^c}{2c} v_0^t - \alpha_0 G_{yz}^c w_{,y}^t = \hat{F}_2^b \tag{21b}$$

$$\begin{aligned}
&\left[D_{11}^b \frac{\partial^4}{\partial x^4} + 2(D_{12}^b + 2D_{66}^b)\frac{\partial^4}{\partial x^2 y^2} + D_{22}^b \frac{\partial^4}{\partial y^4} + \frac{61E^c}{42c} - 2\alpha_1 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^b \\
&- \left[\frac{179E^c}{105c} + 2\alpha_2 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w_0^c \\
&+ \left[\frac{53E^c}{210c} + 2\alpha_3 c\left(G_{xz}^c \frac{\partial^2}{\partial x^2} + G_{yz}^c \frac{\partial^2}{\partial y^2}\right)\right]w^t \\
&+ \alpha_0 G_{xz}^c \frac{\partial}{\partial x}(u_0^t - u_0^b) + \alpha_0 G_{yz}^c \frac{\partial}{\partial y}(v_0^t - v_0^b) = \hat{F}_3^b
\end{aligned} \tag{21c}$$

in which

$$\hat{F}_1^b = -A_{11}^b w_{,x}^b w_{,xx}^b - (A_{12}^b + A_{66}^b) w_{,y}^b w_{,xy}^b - A_{66}^b w_{,x}^b w_{,yy}^b \quad (21d)$$

$$\hat{F}_2^b = -(A_{21}^b + A_{66}^b) w_{,x}^b w_{,xy}^b - A_{66}^b w_{,xx}^b w_{,y}^b - A_{22}^b w_{,y}^b w_{,yy}^b \quad (21e)$$

$$\hat{F}_3^b = (N_x^b w_{,x}^b)_{,x} + (N_{xy}^b w_{,x}^b)_{,y} + (N_{yx}^b w_{,y}^b)_{,x} + (N_y^b w_{,y}^b)_{,y} \quad (21f)$$

IV. Application of the High-Order Theory to a Simply Supported Sandwich Plate

In this section, the solution procedure for the response of sandwich plates will be demonstrated through the study of the simply supported case under transversely applied loading. In this case, the boundary conditions are as follows:

For $x = 0$, a, and $y = 0$, b (Fig. 1),

$$u_0^t = u_0^b = v_0^t = v_0^b = 0, \qquad w^t = w^b = 0, \qquad w_0^c = 0 \quad (22a)$$

and

$$M_{xx}^t = M_{xx}^b = 0 \quad \text{for } x = 0, a \quad (22b)$$

$$M_{yy}^t = M_{yy}^b = 0 \quad \text{for } y = 0, b \quad (22c)$$

The displacements can be written in the form

$$u_0^t = \sum_{m,n} U_{mn}^t \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad v_0^t = \sum_{m,n} V_{mn}^t \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad (23a)$$

$$u_0^b = \sum_{m,n} U_{mn}^b \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad v_0^b = \sum_{m,n} V_{mn}^b \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad (23b)$$

$$w^t = \sum_{m,n} W_{mn}^t \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad w^b = \sum_{m,n} W_{mn}^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (23c)$$

$$w_0^c = \sum_{m,n} W_{mn}^c \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (23d)$$

where U_{mn}^t, V_{mn}^t, U_{mn}^b, V_{mn}^b, W_{mn}^t, W_{mn}^b, and W_{mn}^c are unknown constants.

Substituting Eq. (23) into Eqs. (19–21) results in $\hat{F}_i^t$, $\hat{F}_i^b$ ($i = 1, 2, 3$), and $q(x, y)$ being expressed in the following form:

$$\hat{F}_1^t = \sum_{m,n} \hat{F}_{1mn}^t \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \hat{F}_1^b = \sum_{m,n} \hat{F}_{1mn}^b \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (24a)$$

$$\hat{F}_2^t = \sum_{m,n} \hat{F}_{2mn}^t \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \qquad \hat{F}_2^b = \sum_{m,n} \hat{F}_{2mn}^b \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \quad (24b)$$

$$\hat{F}_3^t = \sum_{m,n} \hat{F}_{3mn}^t \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \hat{F}_3^b = \sum_{m,n} \hat{F}_{3mn}^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (24c)$$

$$q(x, y) = \sum_{m,n} \hat{Q}_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (24d)$$

Thus, one can obtain the following set of nonlinear equations in matrix form:

$$\kappa_{mn} U_{mn} = F_{mn} \quad (25)$$

where the displacement vector U_{mn} is defined as

$$U_{mn} = [U_{mn}^t, V_{mn}^t, W_{mn}^t, W_{mn}^c, U_{mn}^b, V_{mn}^b, W_{mn}^b]^T$$

and the loading vector F_{mn} is defined as

$$[\hat{F}_{1mn}^t, \hat{F}_{2mn}^t, \hat{F}_{3mn}^t, 0.0, \hat{F}_{1mn}^b, \hat{F}_{2mn}^b, \hat{F}_{3mn}^b]^T$$

The solution in terms of the displacements to Eqs. (25) can be obtained without much difficulty if the loading coefficients $\{\hat{F}_{jmn}^t, \hat{F}_{jmn}^b\}$ ($j = 1, 2, 3$) are constants. But these loading coefficients were derived from the expressions (19d–19f) and (21d–21f), which are nonlinear functions of the displacements. Therefore, the right-hand side of Eq. (25), F_{mn} is a nonlinear function of U_{mn}. However, if the displacements U_{mn} are known with the known applied loading $\hat{Q}_{mn}$, making use of Eqs. (23), (19d–19f), and (21d–21f), one can determine the functions $\hat{F}_1^t$, $\hat{F}_2^t$, $\hat{F}_3^t$, $\hat{F}_1^b$, $\hat{F}_2^b$, and $\hat{F}_3^b$. The next approximate values of displacements are found by solving Eq. (25) with the updated loading vector. This procedure is continued until a series of approximate solutions for the in-plane and transverse displacements are determined by the nth iteration with a convergence tolerance ϵ applied on the displacements normalized by the total height of the sandwich section, such that $\epsilon \leq 10^{-5}$ between two consecutive solutions.

V. Numerical Results and Discussions

In this section, we shall present the numerical results for several typical sandwich-plate configurations with orthotropic phases. The results using the elasticity solution, the classical model, and the first-order shear model will also be presented. The results from the current high-order theory will be compared with the results obtained from these models. It should be noted that a comparison is not made with the Frostig et al. [6] theory, because this theory was developed for sandwich beams; that is, it is a one-dimensional theory, unlike the present theory, which is two-dimensional (for sandwich plates). A comparison is also not made with the linear core strain theory [5], because although the basic assumptions of the theory were given in [5], the explicit equations in terms of displacements were not given and therefore they were not available for direct application.

Let us assume a loading of the form

$$q(x, y) = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \qquad 0 \leq x \leq a, \qquad 0 \leq y \leq b \quad (26a)$$

applied on the top face sheets of the sandwich plates. From Eq. (24d), one can obtain the following loading in the transformed space:

$$\hat{Q}_{mn} = \delta_{m1}\delta_{n1} q_0 \quad (26b)$$

where δ_{mn} is the Kronecker delta.

Let us consider a sandwich configuration consisting of unidirectional graphite/epoxy faces with moduli (in gigapascal) of $E_1^f = 181.0$, $E_2^f = E_3^f = 10.3$, $G_{12}^f = G_{31}^f = 7.17$, and $G_{23}^f = 5.96$ and Poisson's ratios of $\nu_{12}^f = \nu_{13}^f = 0.277$ and $\nu_{32}^f = 0.400$. The core

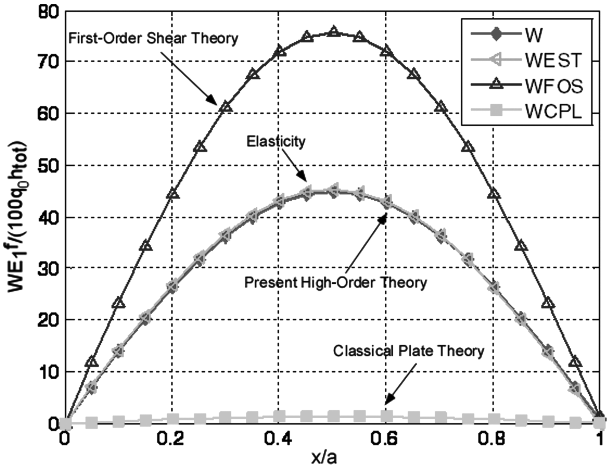

Fig. 2 Transverse displacement at the top face sheet as a function of x for $y = b/2$; case of $a = b = 5h_{\text{tot}}$.

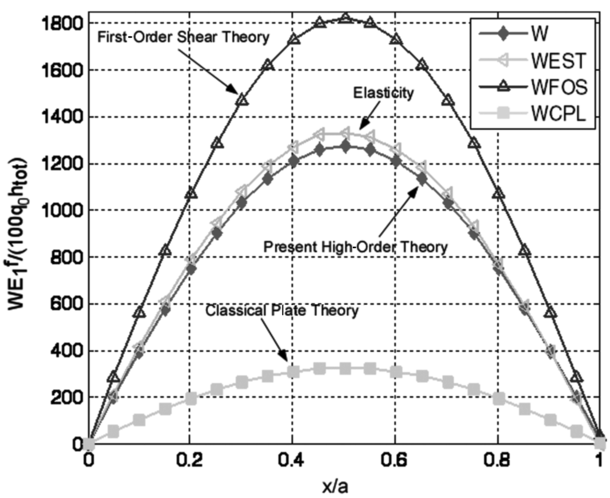

Fig. 3 Transverse displacement at the top face sheet as a function of x for $y = b/2$; case of $a = b = 20h_{\text{tot}}$.

is made of hexagonal glass/phenolic honeycomb with moduli (in gigapascal) of $E_1^c = E_2^c = 0.032$, $E_3^c = E_z^c = 0.300$, $G_{12}^c = 0.013$, and $G_{31}^c = G_{23}^c = 0.048$ and Poisson's ratios of $\nu_{12}^c = \nu_{31}^c = \nu_{32}^c = 0.25$.

The two face sheets are assumed to be identical with a thickness of $f = 2$ mm. The core thickness is $2c = 16$ mm. The total thickness of the plate is defined as $h_{\text{tot}} = 2f + 2c$. In the following results, the displacements are normalized with

$$100h_{\text{tot}} \frac{q_0}{E_1^f}$$

and the normal stresses with

$$q_0 \frac{a^2}{h_{\text{tot}}^2}$$

Plotted in Figs. 2 and 3 are the normalized displacements at the top face sheets as a function of x at $y = b/2$ for two plates with $a = b = 5h_{\text{tot}}$ and $20h_{\text{tot}}$, respectively. Both the classical and first-order shear (where shear is assumed to be carried exclusively by the core) seem to be inadequate, the classical theory being too nonconservative and the first-order shear theory being too conservative; this demonstrates the need for higher-order theories in dealing with sandwich structures. The present high-order theory gives a displacement profile that is practically identical to the elasticity solution for $a = b = 5h_{\text{tot}}$ and it is very close to the elasticity solution for $a = b = 20h_{\text{tot}}$.

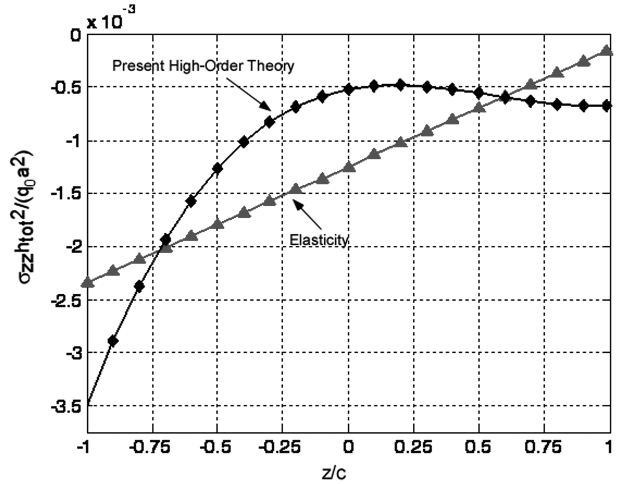

Fig. 4 Through-thickness distribution in the core of the transverse normal stress σ_{zz} at $z = a/2$ and $y = b/2$; case of $a = b = 20h_{\text{tot}}$.

One may further see that the accuracy of the first-order shear model and the classical model increases as the plate span increases from $a = b = 5h_{\text{tot}}$ to $20h_{\text{tot}}$, in the sense that they get closer to the elasticity solution. The reason for this observation is that the plate of $a = b = 5h_{\text{tot}}$ is a very short/thick plate (it is essentially a three-dimensional structure, rather than a plate); that is, the shear and transverse compressibility, as well as their coupling in the sandwich plate, play a big role in its deformation. Because both the first-order shear and classical models assume an infinite transverse rigidity, they can only capture a portion of the total deformation of the plate. But when the span becomes larger, such as $a = b = 20h_{\text{tot}}$, the three-dimensional effects such as the coupling between shear effects and core compressibility become less pronounced.

The distribution of the transverse stress in the core, σ_{zz}, as a function of z at $x = a/2$ and $y = b/2$ for a sandwich plate of $a = b = 20h_{\text{tot}}$ is plotted in Fig. 4, in which only the values using elasticity and present high-order model are presented, because the first-order shear model and the classical model consider the core incompressible. The minus sign means that the stress is compressive. Of interest is the surface at $z/c = -1$, which is the face-sheet/core interface at the side on which the loading is applied and shows the highest value of stress. One can see that the nonlinear model gives a conservative prediction on this peeling stress (the stress that separates the core and the face along their interface, where a debond failure often initiates) at $z/c = -1$. It should also be noted that a linear high-order model [5], which would imply a constant transverse normal strain, would subsequently give a nearly constant normal stress, and therefore it would not be able to capture the large variation of normal stress through the core thickness, as observed in Fig. 4. These observations suggest that using the nonlinear model could help in the safety design of sandwich structures.

VI. Conclusions

In this paper, a new high-order theory for sandwich plates is presented and explored in detail. In the derivation of the governing equations and boundary conditions, both the core compressibility and the core shear are considered. A procedure to solve the nonlinear equations is also outlined. Numerical results from this theory are presented for a typical sandwich-plate configuration with orthotropic phases. These results are compared with those obtained using the elasticity solution, the first-order shear, and the classical models to justify the merits of current theory. Observations from the current work suggest following conclusions:

1) In terms of the displacement profile, the current new theory gives a prediction that is very close to the exact elasticity value.

2) Current new theory gives a very adequate solution to the distribution of the transverse normal stress in the core of a sandwich plate.

Though the transverse normal stress in the core could contribute to several failure modes, such as debonding at the interface between faces and the core, local wrinkling, and core crushing, there have not been plate-theory-based analytical models that could properly describe this transverse stress profile in the core of a sandwich plate. Therefore, this new theory could have wide applications in the investigation of the behavior of sandwich structures.

Appendix A: Classical Sandwich Model (Without Shear Effects)

In the classical sandwich model, the core is assumed to be incompressible in the transverse direction, and the transverse displacements of the face sheets and the core are considered to be the same. As such, the governing equation for a plate subject to transverse loading $q(x, y)$ of the top face sheet reads as [1,2,9]

$$D_{11}\frac{\partial^4 w(x,y)}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w(x,y)}{\partial y^4} = q(x,y) \tag{A1}$$

where the stiffness matrix is defined as

$$D_{ij} = C_{ij}(2fc^2 + 2f^2c + \tfrac{2}{3}f^3) \quad \text{or} \quad D_{ij} = 2C_{ij}fc^2 \quad \text{if } f \ll c \tag{A2}$$

For a simply supported rectangular plate, the transverse displacements can be expressed as

$$w(x,y) = \sum_{m,n} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{A3}$$

and the load can be expressed in the same manner as in Eq. (24d). Substituting the preceding expression into Eq. (A1) leads to

$$W_{mn} = \hat{Q}_{mn} \Bigg/ \left[D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \right] \tag{A4}$$

Appendix B: First-Order Shear Model

If we let $\bar{\alpha}_x$ be the shear deformation in the x direction and $\bar{\alpha}_y$ be the shear deformation in the y direction, then the governing equations with shear effects can be written as [1,2,9]

$$D_{11}\frac{\partial^2 \bar{\alpha}_x}{\partial x^2} + D_{66}\frac{\partial^2 \bar{\alpha}_x}{\partial y^2} + (D_{12}+D_{66})\frac{\partial^2 \bar{\alpha}_y}{\partial x \partial y} - \kappa D_{55}\left(\bar{\alpha}_x + \frac{\partial w}{\partial x}\right) = 0 \tag{B1a}$$

$$(D_{12}+D_{66})\frac{\partial^2 \bar{\alpha}_x}{\partial x \partial y} + D_{66}\frac{\partial^2 \bar{\alpha}_y}{\partial x^2} + D_{22}\frac{\partial^2 \bar{\alpha}_y}{\partial y^2} - \kappa D_{44}\left(\bar{\alpha}_y + \frac{\partial w}{\partial y}\right) = 0 \tag{B1b}$$

$$\kappa D_{55}\left(\frac{\partial \bar{\alpha}_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + \kappa D_{44}\left(\frac{\partial \bar{\alpha}_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + q(x,y) = 0 \tag{B1c}$$

where $\kappa = \pi^2/12$ or $\kappa = 5/6$ is the transverse shear factor, the bending stiffness matrix is defined as in Eq. (A2), and D_{44}, D_{55} are the shear stiffness constants, defined as

$$D_{44} = 2G^c_{xz}c, \qquad D_{55} = 2G^c_{yz}c \tag{B2}$$

Notice in the preceding equation that we have assumed that the shear is carried exclusively by the core. For a simply supported rectangular plate, the solution to Eqs. (B1) can be set in the following form:

$$w(x,y) = \sum_{m,n} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{B3a}$$

$$\bar{\alpha}_x(x,y) = \sum_{m,n} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{B3b}$$

$$\bar{\alpha}_y(x,y) = \sum_{m,n} B_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \tag{B3c}$$

and the load can be expressed in the same manner as in Eq. (24d). If we set

$$\lambda_m = \frac{m\pi}{a}, \qquad \lambda_n = \frac{n\pi}{b} \tag{B4a}$$

$$L_{11} = D_{11}\lambda_m^2 + D_{66}\lambda_n^2 + \kappa D_{55}, \quad L_{12} = (D_{12}+D_{66})\lambda_m\lambda_n \tag{B4b}$$

$$L_{22} = D_{66}\lambda_m^2 + D_{22}\lambda_n^2 + \kappa D_{44}, \qquad L_{13} = \kappa D_{55}\lambda_m \tag{B4c}$$

$$L_{33} = \kappa D_{55}\lambda_m^2 + \kappa D_{44}\lambda_n^2, \qquad L_{23} = \kappa D_{44}\lambda_n \tag{B4d}$$

then substituting Eq. (B3) into Eq. (B1) leads to

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{Q}_{mn} \end{bmatrix} \tag{B5}$$

which yields the solution

$$A_{mm} = (L_{12}L_{23} - L_{22}L_{13})\frac{\hat{Q}_{mn}}{\Delta} \tag{B6a}$$

$$B_{mn} = (L_{12}L_{23} - L_{11}L_{23})\frac{\hat{Q}_{mn}}{\Delta} \tag{B6b}$$

$$W_{mn} = (L_{11}L_{22} - L_{12}^2)\frac{\hat{Q}_{mn}}{\Delta} \tag{B6c}$$

where Δ is the determinant of matrix $[L]$. Note that this shear model is similar to the Timoshenko shear beam model.

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