AMD-Vol. 162

MECHANICS OF THICK COMPOSITES

PRESENTED AT
THE 1st JOINT MECHANICS MEETING OF
ASME • ASCE • SES — MEET'N'93
CHARLOTTESVILLE, VIRGINIA
JUNE 6–9, 1993

SPONSORED BY
THE APPLIED MECHANICS DIVISION, ASME

EDITED BY
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THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 East 47th Street ▪ United Engineering Center ▪ New York, N.Y. 10017

AMD-Vol. 162, Mechanics of Thick Composites
ASME 1993

BUCKLING OF A THICK ORTHOTROPIC CYLINDRICAL SHELL UNDER EXTERNAL PRESSURE INCLUDING HYGROSCOPIC EFFECTS

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ABSTRACT

The stability of equilibrium of an orthotropic thick cylindrical shell subjected to external pressure in a hygroscopic environment is investigated. In this approach, the structure is considered a three-dimensional elastic body rather than a shell. First, a fundamental analysis that formulates the basic buckling equations with the appropriate boundary conditions in the elasticity context is performed. Subsequently, the critical loads for pure mechanical loading (external pressure) are derived. Following this benchmark analysis, the particular emphasis is placed on examining the effect of the boundary-layer transient hygroscopic stress field on the critical load. Constant moisture concentrations on the inner and outer surfaces are imposed in addition to the external pressure. Since the moisture diffusion process is relatively slow, the hygroscopic stresses are confined for practical time values to a boundary-layer region near the surfaces. The analysis first uses the series expansion for the Bessel functions for small arguments and then the Hankel asymptotic expansions since an increasing number of terms is found to be needed. Compared to the classical shell theory approach, the results of this research show that the shell theory predictions on the critical load can be highly non-conservative when moderately thick construction is involved. However, the hygroscopic boundary-layer has a negligible influence on the critical load.

INTRODUCTION

A class of important structural applications of fiber-reinforced composite materials involves the configuration of laminated shells. Although thin plate construction has been the thrust of the initial applications, much attention is now being paid to configurations classified as moderately thick shell structures. Such designs can be used in components in the aircraft and automobile industries, as well as in the marine industry. Moreover, composite laminates have been considered in space vehicles in the form of circular cylindrical shells as a primary load carrying structure.

In these light-weight shell structures, loss of stability is of primary concern. This subject has been researched to-date through the application of the cylindrical shell theory (e.g. Simitses, Shaw and Sheinman, 1985). However, previous work (e.g. Pagano and Whitney, 1970) has shown that considerable care must be exercised in applying thin shell theory formulations to predict the response of composite cylinders. Besides the anisotropy, composite shells have one other important distinguishing feature, namely extensional-to-shear modulus ratio much larger than that of their metal counterparts.

In order to more accurately account for the above mentioned effects, various modifications in the classical theory of laminated shells have generally been performed (e.g. Whitney and Sun, 1974; Librescu, 1975; Reddy and Liu, 1985). These higher order shell theories can be applied to buckling problems with the potential of improved predictions for the critical load.

Towards the objective of producing an solution based on three-dimensional elasticity to the problem of buckling of composite shell structures, against which results from various shell theories could be compared, this work presents an elasticity solution to the problem of buckling of composite cylindrical orthotropic shells subjected to external pressure. Numerical results for an example case of a fiber reinforced hollow cylinder under external pressure are derived and compared with shell theory predictions.

Another important requirement for the confident application of these structural designs in severe hygroscopic environments is the adequate understanding of the influence of the environment, and in particular the effect of the stresses induced by moisture. It is well-known that a polymeric resin absorbs moisture from its environment. In this context, Wang and Choi (1982) suggested that an unanticipated failure of a composite structure, frequently initiated at the edges, can be a result of hygroscopic stresses near the edges.

Although the majority of hygroelastic analyses have been performed in plate structures, some studies have also been reported in thin shell geometries. In particular, Doxee and Springer (1989) analyzed hygrothermal stresses and strains in an axisymmetric composite shell according to their higher order shell theory. It should be emphasized that just as the classical lamination plate theory cannot predict the boundary-layer hygroscopic stress field in plate geometries, the classical shell theory cannot capture these highly localized stresses in shell geometries.

Concerning the influence of hygroscopic fields on the point of stability loss, Snead and Palazotto (1983) used the finite element method to investigate the moisture and temperature effects on the instability of cylindrical, thin composite panels subjected to axial loads. They concluded that the bifurcation load will be degraded as moisture concentations and temperatures increase and also these are influenced by the panel's ply orientations. Lee and Yen (1988) did work with the similar problem to Snead and Palazotto's (1983). They included the effects of the transverse shear deformations to the classical shell theory. On the basis of their finite element analysis, their results are consistent to Snead's and Palazotto's (1983).

In this work, the influence of transient hygroscopic stresses on the critical point of a hollow orthotropic circular cylinder loaded by external pressure is examined. It is assumed that both the inner and outer surfaces are at constant (but different) concentrations of moisture. The material properties are assumed independent of the concentration of moisture. The elasticity solution produced in this paper provides accurate results for certain simple configurations, but, more importantly, forms a basis for comparing various shell theories that could be potentially used for more complex geometries.

FORMULATION

Let us consider the equations of equilibrium in terms of the second Piola-Kirchhoff stress tensor Σ in the form

$$\text{div}\,(\Sigma . \mathbf{F}^T) = 0 \,, \qquad (1a)$$

where $\mathbf{F}$ is the deformation gradient defined by

$$\mathbf{F} = \mathbf{I} + \mathrm{grad}\vec{V} \ , \tag{1b}$$

where $\vec{V}$ is the displacement vector and $\mathbf{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}.\mathbf{F} - \mathbf{I}\right) \ . \tag{1c}$$

More specifically, in terms of the linear strains:

$$e_{rr} = \frac{\partial u}{\partial r} \ , \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \ , \qquad e_{zz} = \frac{\partial w}{\partial z} \ , \tag{2a}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \ , \qquad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \ , \qquad e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \ , \tag{2b}$$

and the linear rotations:

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z} \ , \qquad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \ , \qquad 2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta} \ . \tag{2c}$$

the deformation gradient $\mathbf{F}$ is

$$\mathbf{F} = \begin{bmatrix} 1 + e_{rr} & \frac{1}{2}e_{r\theta} - \omega_z & \frac{1}{2}e_{rz} + \omega_\theta \\ \frac{1}{2}e_{r\theta} + \omega_z & 1 + e_{\theta\theta} & \frac{1}{2}e_{\theta z} - \omega_r \\ \frac{1}{2}e_{rz} - \omega_\theta & \frac{1}{2}e_{\theta z} + \omega_r & 1 + e_{zz} \end{bmatrix} \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, w_0 the r, θ and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1 \ ; \qquad v = v_0 + \alpha v_1 \ ; \qquad w = w_0 + \alpha w_1 \ , \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r,\theta,z)$, $\alpha v_1(r,\theta,z)$, $\alpha w_1(r,\theta,z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r,\theta,z)$, $v_1(r,\theta,z)$, $w_1(r,\theta,z)$ are assumed finite and α is an infinitesimally small quantity independent of r, θ, z.

Following Kardomateas (1993), we obtain the following buckling equations:

$$\frac{\partial}{\partial r}\left(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta\right) +$$
$$+\frac{\partial}{\partial z}\left(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta\right) + \frac{1}{r}\left(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z\right) = 0 \ , \tag{5a}$$

$$\frac{\partial}{\partial r}\left(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r\right) +$$
$$+\frac{\partial}{\partial z}\left(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r\right) + \frac{1}{r}\left(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r\right) = 0 \ , \tag{5b}$$

$$\frac{\partial}{\partial r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r\right) +$$
$$+\frac{\partial}{\partial z}\left(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r\right) + \frac{1}{r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) = 0 \ . \tag{5c}$$

In the previous equations, σ_{ij}^0 and ω_j^0 are the values of σ_{ij} and ω_j at the initial equilibrium position, i.e. for $u = u_0$, $v = v_0$ and $w = w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position, i.e. for $u = u_1$, $v = v_1$ and $w = w_1$.

The boundary conditions associated with (1a) can be expressed as:

$$(\mathbf{F}.\Sigma^T).\hat{N} = \vec{t}(\vec{V}) \, , \tag{6}$$

where $\vec{t}$ is the traction vector on the surface which has outward unit normal $\hat{N} = (\hat{l}, \hat{m}, \hat{n})$ before any deformation. The traction vector $\vec{t}$ depends on the displacement field $\vec{V} = (u, v, w)$. Again, following Kardomateas (1993), we obtain for the bounding surfaces:

$$(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta)\,\hat{l} + (\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta)\,\hat{m} +$$

$$+ (\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta)\,\hat{n} = p(\omega'_z\hat{m} - \omega'_\theta\hat{n}) \, , \tag{7a}$$

$$(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r)\,\hat{l} + (\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r)\,\hat{m} +$$

$$+ (\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r)\,\hat{n} = -p\left(\omega'_z\hat{l} - \omega'_r\hat{n}\right) \, , \tag{7b}$$

$$(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta)\,\hat{l} + (\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta)\,\hat{m} +$$

$$+ (\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta)\,\hat{n} = p\left(\omega'_\theta\hat{l} - \omega'_r\hat{m}\right) \, . \tag{7c}$$

Since for the lateral surfaces $\hat{m} = \hat{n} = 0$ and $\hat{l} = 1$,

$$\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta = 0 \, , \tag{8a}$$

$$\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r = -p\omega'_z \, , \tag{8b}$$

$$\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta = p\omega'_\theta \, , \tag{8c}$$

Pre-buckling State. The problem at hands is that of a hollow cylinder rigidly fixed at its ends and deformed by uniformly distributed external pressure p (Fig. 1). The cylinder has an inner radius, r_1 and an outer radius, r_2. The radial, circumferential and axial coordinates are denoted by r, θ and z, respectively. It is assumed that the initial concentration (at $t = 0$) is C_0. For $t > 0$, the boundaries $r = r_1$ and $r = r_2$ are kept at constant concentrations C_1 and C_2, respectively. The reference concentration is taken as zero. The moisture problem is solved by the Fickian diffusion equation:

$$\frac{\partial C(r,t)}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) \qquad r_1 \le r \le r_2 \, , \tag{9a}$$

where $C(r,t)$ is the moisture concentration and D is the moisture diffusivity of the composite in the r direction. The initial and boundary conditions are

$$C(r, t = 0) = C_0 \qquad r_1 \le r \le r_2 \, , \tag{9b}$$

$$C(r_1, t) = C_1 \quad \text{and} \quad C(r_2, t) = C_2 \qquad t > 0 \, , \tag{9c}$$

where C_0, C_1 and C_2 are constants. Crank (1975) gives the general solution for the distribution of the concentration of moisture $C(r,t)$ to Eq. (9) in terms of the Bessel functions of the first and second kind J_n and Y_n, as follows:

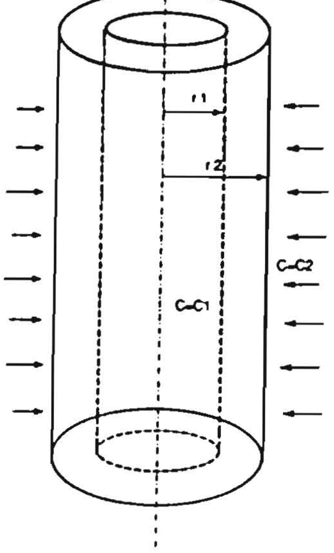

Fig. 1. Hollow cylinder under external pressure.

$$C(r,t) = b_1 \ln(r/r_1) + b_2 \ln(r_2/r) + \sum_{n=1}^{\infty} [c_n J_0(r\alpha_n) + d_n Y_0(r\alpha_n)]\, e^{-D\alpha_n^2 t} , \tag{10a}$$

where

$$b_1 = \frac{C_2}{\ln(r_2/r_1)} ; \quad b_2 = \frac{C_1}{\ln(r_2/r_1)} , \tag{10b}$$

$$c_n = \frac{\pi J_0(r_1\alpha_n) Y_0(r_2\alpha_n)}{J_0(r_1\alpha_n) + J_0(r_2\alpha_n)} \left(C_0 - \frac{C_2 J_0(r_1\alpha_n) - C_1 J_0(r_2\alpha_n)}{J_0(r_1\alpha_n) - J_0(r_2\alpha_n)} \right), \tag{10c}$$

$$d_n = -\frac{J_0(r_2\alpha_n)}{Y_0(r_2\alpha_n)} c_n , \tag{10d}$$

and α_n are the positive roots of:

$$J_0(r_1\alpha_n) Y_0(r_2\alpha_n) - J_0(r_2\alpha_n) Y_0(r_1\alpha_n) = 0 . \tag{10e}$$

The hygroscopic stress-strain relations for the orthotropic body are

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} - \beta_r \Delta C \\ \epsilon_{\theta\theta} - \beta_\theta \Delta C \\ \epsilon_{zz} - \beta_z \Delta C \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} , \tag{11}$$

where c_{ij} are the elastic constants and β_i the swelling coefficients (1, 2 and 3 represent r, θ and z, respectively). The geometry (Fig. 1) is axisymmetric. Since the moisture concentration is assumed to depend only on the r direction, the stresses are independent of θ and z and the hoop displacement is zero. In addition to the constitutive equation (11), the equilibrium equations have to be satisfied; since $\tau_{r\theta}=\tau_{rz}=\tau_{\theta z}=0$, only one equilibrium equation remains:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 . \tag{12}$$

In this work the displacement field derived by Lekhnitskii (1981) for time-independent problems and modified by Kardomateas (1989) for time-dependent thermal stress problems (which are analogous to the time-dependent moisture-induced stress problems) is used:

$$u_r = U(r,t) + z(w_y \cos\theta - w_x \sin\theta) + u_0 \cos\theta + v_0 \sin\theta ,$$

$$u_\theta = -z(w_y \sin\theta + w_x \cos\theta) + w_z r - u_0 \sin\theta + v_0 \cos\theta , \tag{13}$$

$$u_z = z f(t) - r(w_y \cos\theta - w_x \sin\theta) + w_0 ,$$

where the function $U(r,t)$ represents the radial displacement accompanied by deformation. The constants u_0, v_0 and w_0 denote the rigid body translation along the x, y and z directions in the Cartesian coordinate system, respectively, and w_x, w_y and w_z denote the rigid body rotation in the x, y and z directions (these may also be functions of time, but since they do not appear in the strain expressions, such a dependence would not affect the expressions that follow in this section).

The parameter $f(t)$ is obtained from boundary conditions. The strains are expressed in terms of the displacements as follows:

$$\epsilon_{rr} = \frac{\partial U(r,t)}{\partial r}, \qquad \epsilon_{\theta\theta} = \frac{U(r,t)}{r}, \qquad \epsilon_{zz} = f(t) ,$$

$$\gamma_{\theta z} = \gamma_{zr} = \gamma_{r\theta} = 0 . \tag{14}$$

Since the structure is assumed, for simplicity, fixed at the ends, $f(t) = 0$. Notice that this is a difference with the boundary condition of axial force developed due to the pressure at the ends, which was assumed in the boundary-layer hygroscopic stress field study by Kardomateas and Chung (1993).

Substituting Eqs. (14) and (11) into the equilibrium Eq. (12) gives the following differential equation for $U(r,t)$:

$$c_{11}\left[\frac{\partial^2 U(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,t)}{\partial r}\right] - \frac{c_{22}}{r^2}U(r,t) = q_1\frac{\partial C(r,t)}{\partial r} + q_2\frac{C(r,t)}{r} , \tag{15}$$

where

$$q_1 = c_{11}\beta_r + c_{12}\beta_\theta + c_{13}\beta_z , \tag{16a}$$

$$q_2 = (c_{11} - c_{12})\beta_r + (c_{12} - c_{22})\beta_\theta + (c_{13} - c_{23})\beta_z . \tag{16b}$$

To solve Eq. (15), set

$$U(r,t) = U_0(r) + \sum_{n=1}^{\infty} R_n(r) e^{-D\alpha_n^2 t} . \tag{17}$$

Substituting Eqs. (10), (16) and (17) into Eq. (15) yields the following equations to be satisfied for U_0, and R_n for $n = 1, 2, \cdots, \infty$:

$$c_{11}U_0''(r) + \frac{c_{11}}{r}U_0'(r) - \frac{c_{22}}{r^2}U_0(r) = q_1\frac{b_1 - b_2}{r} + q_2 b_1\frac{\ln(r/r_1)}{r} + q_2 b_2\frac{\ln(r_2/r)}{r} , \tag{18a}$$

$$c_{11}R_n''(r) + \frac{c_{11}}{r}R_n'(r) - \frac{c_{22}}{r^2}R_n(r) = c_n\left[q_2\frac{J_0(r\alpha_n)}{r} - q_1\alpha_n J_1(r\alpha_n)\right] +$$

$$+d_n\left[q_2\frac{Y_0(r\alpha_n)}{r}-q_1\alpha_nY_1(r\alpha_n)\right]\qquad n=1,\ldots\infty\,. \tag{18b}$$

For each of the previous equations, the solution is the sum of a homogeneous solution and a particular one. The solution of the homogeneous equation is in the form $G_1(t)r^{\lambda_1}$ $+G_2(t)r^{\lambda_2}$ with

$$\lambda_{1,2}=\pm\sqrt{c_{22}/c_{11}}\,. \tag{18c}$$

Now set $G_i(t)$ in the form: $G_i(t)=G_{i0}+\sum G_{in}e^{-D\alpha_n^2 t}$, $i=1,2$.

Since the constants G_{ij} are yet unknown, we shall indicate the places where they enter in the expressions that follow (these constants are found later from the boundary conditions). For $c_{11}\neq c_{22}$ the solution of (18a) for $U_0(r)$ is

$$U_0(r)=G_{10}r^{\lambda_1}+G_{20}r^{\lambda_2}+U_0^{*}(r)\,, \tag{19a}$$

where

$$U_0^{*}(r)=\frac{q_2b_1}{c_{11}-c_{22}}r\ln(r/r_1)+\frac{q_2b_2}{c_{11}-c_{22}}r\ln(r_2/r)+\frac{[q_1(c_{11}-c_{22})-2q_2c_{11}]}{(c_{11}-c_{22})^2}(b_1-b_2)r\,. \tag{19b}$$

For $c_{11}=c_{22}$ the corresponding solution of (18a) is given in the expanded paper by Kardomateas and Chung (1993) with $f_0=0$.

To solve (18b) we use the series expansions of the Bessel functions to obtain a series expansion of the right-hand side, as given in Appendix I of Kardomateas and Chung (1993). In the following, γ stands for the Euler's constant ($\simeq 0.577215\ldots$).

For $c_{11}\neq c_{22}$, the solution is:

$$R_n(r)=G_{1n}r^{\lambda_1}+G_{2n}r^{\lambda_2}+R_n^{*}(r)\,, \tag{20a}$$

$$R_n^{*}(r)=B_{0n}r+\frac{2q_2d_n}{\pi(c_{11}-c_{22})}r\ln(r\alpha_n/2)+\sum_{k=0}^{\infty}B_{1nk}r^{2k+3}\ln(r\alpha_n/2)+B_{2nk}r^{2k+3}\,, \tag{20b}$$

where B_{0n}, B_{1nk} and B_{2nk} are given explicitly in Kardomateas and Chung (1993).

The series expansion for the Bessel functions cannot be used for large arguments; hence, the requirement of including an increasing number of terms and therefore large arguments, necessitates finding a particular solution for the "large arguments" domain. This is achieved by using the Hankel asymptotic expansions of the Bessel functions of the first and second kind. Employing the substitution

$$\rho=r\alpha_n\,;\quad R_n^{**}(\rho)=R_n^{*}(r)\,, \tag{21a}$$

gives the following equation for $R_n^{**}(\rho)$

$$c_{11}\alpha_n^2\left(R_n^{**\prime\prime}(\rho)+\frac{R_n^{**\prime}(\rho)}{\rho}\right)-c_{22}\alpha_n^2\frac{R_n^{**}(\rho)}{\rho^2}=\sum_{k=0}^{\infty}\frac{(-1)^k\alpha_n\psi_1(k)}{(2k)!(8\rho)^{2k}\rho\sqrt{\pi\rho}}$$

$$\times\ \{(c_n+d_n)(q_2\sin\rho-a_{1,k}\rho\cos\rho+a_{2,k}\rho^2\sin\rho)+$$

$$+\ (c_n-d_n)(q_2\cos\rho+a_{1,k}\rho\sin\rho+a_{2,k}\rho^2\cos\rho)\}\,. \tag{21b}$$

The solution of the above equation for the function $R_n^{**}(\rho)$ is found to be

$$R_n^{**}(\rho)=\sum_{k=0}^{\infty}p_{k,1}^{n}\rho^{-2k-1/2}\cos\rho+s_{k,1}^{n}\rho^{-2k-1/2}\sin\rho+$$

$$+p_{k,2}^{n}\rho^{-2k-3/2}\cos\rho + s_{k,2}^{n}\rho^{-2k-3/2}\sin\rho \,. \qquad (22)$$

The coefficients $p_{k,1}^n$, $s_{k,1}^n$, $p_{k,2}^n$, $s_{k,2}^n$ are determined by considering the terms in the sum that contribute to the terms $\rho^{-2k-1/2}\cos\rho$, $\rho^{-2k-1/2}\sin\rho$, $\rho^{-2k-3/2}\cos\rho$, $\rho^{-2k-3/2}\sin\rho$ in the right hand side of (21b). Recursive formulas for $p_{k,1}^n$, $s_{k,1}^n$, and for $p_{k,2}^n$, $s_{k,2}^n$ are thus obtained, and these are given in Kardomateas and Chung (1993) (along with the definitions of $a_{1,k}$, $a_{2,k}$ and $\psi_1(k)$ in the previous formulas).

Notice that since both these solutions for the different domains (i.e. series expansion or Hankel asymptotic expansion) are particular solutions a homogeneous solution term is added at the transition point as is discussed in the latter reference.

Thus, the expression for $U(r,t)$ satisfying the equlibrium equations is obtained with the unknown coefficients G_{10}, and G_{20}; G_{1n} and G_{2n} for $n = 1, 2 \ldots$. These coefficients are determined from the following boundary conditions:

$$\sigma_{rr}(r_1,t) = 0\,, \quad \sigma_{rr}(r_2,t) = -p\,; \quad \tau_{r\theta}(r_i,t) = \tau_{rz}(r_i,t) = 0, \quad i = 1,2\,. \qquad (23)$$

where p is the external pressure. Only those for the stress σ_{rr} are not identically satisfied. The stress σ_{rr} on the boundaries is written in terms of the displacement field:

$$\sigma_{rr}(r_i,t) = c_{11}U_{,r}(r_i,t) + c_{12}\frac{U(r_i,t)}{r} - q_1 C(r_i,t), \quad i = 1,2\,, \qquad (24)$$

Substituting Eqs. (10), (16), (17) into (24) for $U_0(r)$ gives the following two linear equations for G_{10} and G_{20}:

$$(c_{11}\lambda_1 + c_{12})r_i^{\lambda_1-1}G_{10} + (c_{11}\lambda_2 + c_{12})r_i^{\lambda_2-1}G_{20} =$$
$$= -c_{11}U_0^{*\prime}(r_i) - c_{12}\frac{U_0^*(r_i)}{r_i} + q_1\left[b_1\ln(r_i/r_1) + b_2\ln(r_2/r_i)\right] + p_i \quad i = 1,2\,, \qquad (25a)$$

where $p_i = 0$, at $i = 1$; $p_i = -p$ at $i = 2$.

In a similar fashion, by substituting the expressions for $R_n(r)$, there correspond two linear equations for G_{1n}, G_{2n} for $n = 1, \ldots \infty$, as follows,

$$(c_{11}\lambda_1 + c_{12})r_i^{\lambda_1-1}G_{1n} + (c_{11}\lambda_2 + c_{12})r_i^{\lambda_2-1}G_{2n} =$$

$$= -c_{11}R_n^{*\prime}(r_i) - c_{12}\frac{R_n^*(r_i)}{r_i} + q_1\left[c_n J_0(r_i\alpha_n) + d_n Y_0(r_i\alpha_n)\right]\,; \quad i = 1,2\,. \qquad (25b)$$

Therefore the constants G_{ij} and hence the displacement U can be found by solving (25). After obtaining the displacement field, the stresses can be found by substituting in (14) and (11).

Perturbed State. In the petrurbed position we seek plane equilibrium modes as follows:

$$u_1(r,\theta) = A_n(r)\cos n\theta\,; \quad v_1(r,\theta) = B_n(r)\sin n\theta\,; \quad w_1(r,\theta) = 0\,. \qquad (26)$$

As discussed in Kardomateas (1993), we can use for the first order strains the simple linear strain expressions. Therefore,

$$\epsilon_{rr}' = A_n'(r)\cos n\theta\,, \qquad (27a)$$

$$\epsilon_{\theta\theta}' = \frac{A_n(r) + nB_n(r)}{r}\cos n\theta\,, \qquad (27b)$$

$$\gamma_{r\theta}' = \left[B_n'(r) - \frac{B_n(r) + nA_n(r)}{r}\right]\sin n\theta\,, \qquad (27c)$$

$$\epsilon'_{zz} = \gamma'_{\theta z} = \gamma'_{rz} = 0 \,, \tag{27d}$$

and the first order rotations are

$$2\omega'_z = \left[B'_n(r) + \frac{B_n(r) + nA_n(r)}{r}\right] \sin n\theta \,, \tag{27e}$$

$$\omega'_\theta = \omega'_r = 0 \,. \tag{27f}$$

Denote $A_n^{(i)}(r)$, $B_n^{(i)}(r)$ the i-th derivative of $A_n(r)$, $B_n(r)$ respectively, with the notation $A_n^{(0)}(r) = A_n(r)$ and $B_n^{(0)}(r) = B_n(r)$. Substituting into the buckling equations (5) and using the constitutive relations, we obtain the following two linear homogeneous ordinary differential equations of the second order for $A_n(r)$, $B_n(r)$:

$$\sum_{i=0}^{2} A_n^{(i)}(r) r^{i-2} \left[d_{i0} + d_{i1}\sigma^0_{\theta\theta}(r,t;p)\right] +$$

$$+\sum_{i=0}^{1} B_n^{(i)} r^{i-2} \left[q_{i0} + q_{i1}\sigma^0_{\theta\theta}(r,t;p)\right] = 0 \,, \qquad r_1 \le r \le r_2 \tag{28a}$$

$$\sum_{i=0}^{2} B_n^{(i)} r^{i-2} \left[b_{i0} + b_{i1}\sigma^0_{rr}(r,t;p)\right] +$$

$$+\sum_{i=0}^{1} A_n^{(i)} r^{i-2} \left[f_{i0} + f_{i1}\sigma^0_{rr}(r,t;p)\right] = 0 \,, \qquad r_1 \le r \le r_2 \tag{28b}$$

where

$$\sigma^0_{rr}(r,t;p) = c_{11}\frac{\partial U(r,t;p)}{\partial r} + c_{12}\frac{U(r,t;p)}{r} - q_1 C(r,t), \tag{29a}$$

$$\sigma^0_{\theta\theta}(r,t;p) = c_{12}\frac{\partial U(r,t;p)}{\partial r} + c_{22}\frac{U(r,t;p)}{r} - (q_1 - q_2)C(r,t), \tag{29b}$$

In terms of $c = r_1/r_2$, the expression of $U(r,t;p)$ is in the form

$$U(r,t;p) = (U_{c1} + pU_{a1})r^{\lambda_1} + (U_{c2} + pU_{a2})r^{-\lambda_1} +$$

$$+U_0^*(r) + \sum_{n=1}^{\infty} e^{-D\alpha_n^2 t}\left[G_{1n}r^{\lambda_1} + G_{2n}r^{-\lambda_1} + R_n^*(r)\right] \,, \tag{29c}$$

Notice that $U_0^*(r)$, $R_n^*(r)$, G_{1n} and G_{2n} are not dependent on p, and in fact the relationship (29c) gives whatever dependence on p exists (which is linear). This is important because in the numerical implementation that will be described shortly, derivatives of σ^0_{rr} and $\sigma^0_{\theta\theta}$ with respect to p are needed, and these are found directly from (29) without a need for a numerical differentiation.

The boundary conditions (8) are written as follows:

$$A'_n(r_j)c_{11} + \left[A_n(r_j) + nB_n(r_j)\right]\frac{c_{12}}{r_j} = 0 \,, \qquad j = 1,2 \tag{30a}$$

$$B'_n(r_j)\left[\left(c_{66} + \frac{p_j}{2}\right) + \frac{1}{2}\sigma^0_{rr}(r_j)\right] +$$

$$+\left[B_n(r_j) + nA_n(r_j)\right]\left[\left(-c_{66} + \frac{p_j}{2}\right)\frac{1}{r_j} + \frac{1}{2r_j}\sigma^0_{rr}(r_j)\right] = 0 \,, \qquad j = 1,2 \tag{30b}$$

where $p_j = p$ for $j = 2$ i.e. $r = r_2$ (outside boundary), and $p_j = 0$ for $j = 1$ i.e. $r = r_1$ (inside boundary).

The constants d_{ij}, q_{ij}, b_{ij}, f_{ij} in the above equations are given in Appendix I of Kardomateas (1993) and depend on the material stiffness coefficients c_{ij} and the constant n.

For a fixed time, equations (28) and (30) constitute an eigenvalue problem for ordinary second order linear differential equations in the r variable, with the applied external pressure p the parameter. This is essentially a standard two point boundary value problem. The relaxation method was used (Press et al, 1989) which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the shell. For this purpose, an equally spaced mesh of 121 points was employed and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the shell theory solution was used. The minimum eigenvalue is obtained for $n = 2$. An investigation of the convergence showed that essentially the same results were produced with twice as many mesh points.

RESULTS AND DISCUSSION

As an illustrative example, the critical pressure was determined for a composite circular cylinder of inner radius $r_1 = 1$m. The moduli in GN/m^2 and Poisson's ratios used (typical for a glass/epoxy material) are listed below, where 1 is the radial (r), 2 is the circumferential (θ), and 3 the axial (z) direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$.

Figure 2 shows the critical pressure as a function of the ratio of outside vs. inside radius r_2/r_1 (pure mechanical loading). The elasticity solution is compared with the predictions of classical shell theory (e.g. Ambartsumyan, 1961).

The direct expression for the critical pressure from classical shell theory is:

$$p_{cr,sh} = \frac{E_2}{(1 - \nu_{23}\nu_{32})}(n^2 - 1)\frac{h^3}{12R^3} . \qquad (31)$$

where $R = (r_1 + r_2)/2$ is the mid-surface radius, and $h = r_2 - r_1$ is the shell thickness.

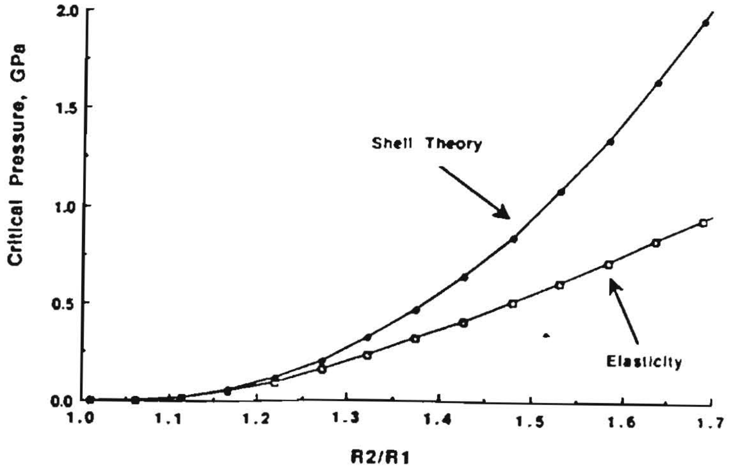

Fig. 2. Critical pressure, p_{cr} vs ratio of outside/inside radius, R_2/R_1. Comparison of the three-dimensional elasticity and the shell theory predictions.

The previous value can be found by using the Donnell nonlinear shell theory equations (Brush and Almroth, 1975) and seeking the buckled shapes in the form (26) where $A_n(r) = A_n$, i.e. it is now a constant instead of function of r, and $B_n(r) = B_n + (r - R)\beta$ with B_n being a constant, i.e. it admits a linear variation through the thickness. Since $\beta = (v_1 - u_{1,\theta})/R$, the latter can also be written in the form $B_n(r) = B_n + (r - R)(B_n + n)/R$. Notice that the classical shell theory would be unable to capture the effect of the axisymmetric moisture stress field because of the assumption of constant radial displacement, $U(r)$.

For a more specific comparison of the results for a range of radii ratios that would probably constitute practically moderately thick to thick shell construction, Table 1 shows the values of the critical load, as derived by the present elasticity formulation and the shell theory predictions for the orthotropic material. A similar comparison is performed for the isotropic case in Kardomateas (1993).

Concerning the effect of the hygroscopic boundary-layer stress field on the critical load, the same geometry and material is used. The typical values of hygroscopic expansion coefficients (e.g., Hahn, 1976) are: $\beta_r = \beta_z = 6.67\text{x}10^{-3}/\text{wt}\%$, $\beta_\theta = 0$. For this material, the moisture diffusivity in the radial direction is D = $2.145\text{x}10^{-13}$ m^2/sec. This value was obtained by substituting a temperature of 50°C to the equation for the temperature-dependent moisture diffusivity in Hahn (1976).

Table 2 shows the critical pressure with the effect of the hygroscopic effects included at a normalized time $\bar{t} = Dt/(r_2 - r_1)^2 = 3.7\text{x}10^{-5}$. The initial concentration (at $t = 0$) is taken $C_0 = 0.005$, whereas the concentrations at the ends for $t > 0$ are: $C_1 = 0.005$ and $C_2 = 0.05$. In all cases, inclusion of the hygroscopic effects leads to a very slight reduction of the critical pressure relative to the benchmark pure mechanical loading (external pressure) case. Hence, the local (contained near the boundaries) hygroscopic boundary layer does not influence the global buckling behavior. This hygroscopic field may have other effects, however, such as the initiation of local failure.

The hygroscopic effect on the bifurcation load, that was found in Snead and Palazotto's (1983) and Lee and Yen's (1988) studies, is due to their assumption of a degradation of the material stiffness due to moisture and temperature. On the contrary, in this paper we have assumed moisture-independent properties. Hence, this study emphasizes the fact that reductions in the critical load due to moisture are attributed to the reduction in elastic moduli and not to the moisture-induced stress fields.

Finally, Figure 3 shows the effect of material constants by presenting a comparison of the critical load for the orthotropic case with the previously given moduli and Poisson's ratios, and the corresponding one by assuming isotropic material with modulus $E = E_2$, i.e. the modulus along the periphery, and Poisson's ratio $\nu = 0.3$. It is seen that the orthotropy results in significantly lower critical load with increased thickness.

Table 1

Critical Pressure, $p_{cr}R_2^3/(E_2h^3)$

Orthotropic, moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$

Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.400$

r_2/r_1	**Elasticity**	**Shell**	**Percentage Increase**
1.20	0.2784	0.3308	18.8%
1.25	0.2780	0.3495	25.7%
1.30	0.2762	0.3681	33.3%
1.35	0.2733	0.3864	41.4%
1.40	0.2696	0.4046	50.1%

From the results presented previously, it can be concluded that predictions of stability loss in composite thick structures can be quite non-conservative if classical approaches are used. The present formulation and solution provide a means of accurately assessing the limitations of shell theories in predicting stability loss when the applications involve orthotropy and moderately thick construction. Furthermore, the present work includes a formulation for including the effect of the transient boundary-layer hygroscopic stress field on the critical load. Further work is needed to assess the accuracy of improved, higher order shell theories predictions on the critical load in comparison to the elasticity ones.

Acknowledgement The financial support of the Office of Naval Research, Mechanics Division, Grant N00014-91-J-1892, and the interest and encouragement of the Project Monitor, Dr. Y. Rajapakse, are both gratefully acknowledged.

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Table 2
Hygroscopic Effects on the Critical Pressure, p_{cr}, GN/m^2

Orthotropic, moduli in GN/m^2: $E_2 = 57$, $E_1 = E_3 = 14$, $G_{31} = 5.0$, $G_{12} = G_{23} = 5.7$
Poisson's ratios: $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, $\nu_{31} = 0.4$; $R_1 = 1.0$m

Moisture Concentrations: initial, at $t = 0$: $C_0 = 0.005$
at $t > 0$: C_1 (at r_1) = 0.005; C_2 (at r_2) = 0.05

r_2/r_1	With Moisture Effects	Without Moisture Effects
1.10	0.011679	0.011680
1.20	0.07313	0.07346
1.30	0.19342	0.19345
1.40	0.35842	0.35847

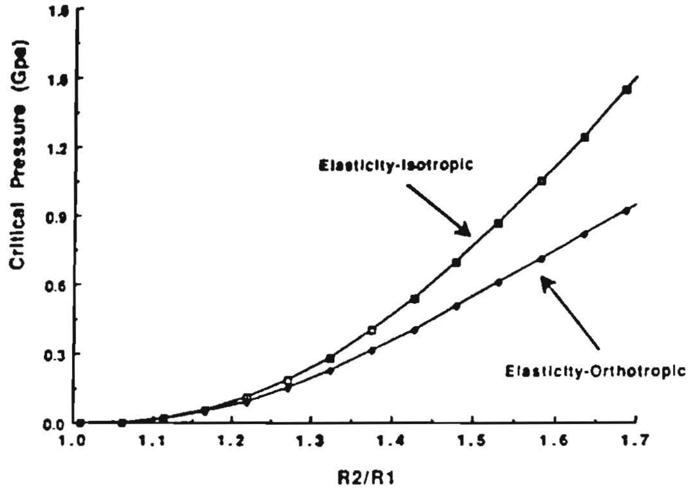

Fig. 3. Critical pressure, p_{cr} vs ratio of outside/inside radius, R_2/R_1, for the orthotropic case, and the isotropic one with $E = E_2$, i.e. the modulus along the periphery and Poisson's ratio $\nu = 0.3$.

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