The Effect of Transverse Shear on the Postbuckling and Growth Characteristics of Delaminations in Composites

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A geometrically nonlinear formulation for the behavior of composite delaminated beams of arbitrary stacking sequence, and with the effects of transverse shear deformation included, is presented. The formulation is based on a first-order shear deformation kinematic model, which incorporates the bending-stretching coupling effect and also assumes an arbitrary initial imperfection. The nonlinear differential equations are solved by Newton's method using a finite-difference scheme. The growth of the delamination is also studied by applying the J-integral in order to derive a formula for the energy release rate, which includes transverse shear. Results are presented which illustrate the shear effect, especially with respect to the ratio of the in-plane extensional over shear modulus and with respect to the ratio of plate length over thickness. It is seen that transverse shear can affect largely the displacement profiles, rendering the structure more compliant, and can promote growth by increasing the energy release rate, but this latter effect is moderate and mainly noticable only at the later stages in the postbuckling regime.

Introduction

Delaminations are interface cracks and can occur frequently in laminated composites when adjacent layers become debonded. This may happen because of manufacturing imperfections or during service from low velocity impact. Under compression, the delaminated layer may buckle well before the critical point of the base plate. Although delamination buckling does not imply loss of structural integrity and the structure may still be able to carry its design load, it can lead to stiffness reductions, undesirable transverse deflections, and growth of the delamination in the postbuckling phase, or even initiation of intra-layer cracks which may be more critical from the structural integrity viewpoint (Pelegri and Kardomateas, 1998).

Postbuckling of delaminated composites has already been studied by many researchers including Kardomateas (1993) through the use of elastica theory which could not account for the effects of transverse shear and by Sheinman and Soffer (1991) through equations based on the von Karman kinematic approach, again without the shear effect. As far as just the critical load (as opposed to the postbuckling behavior), an approximate model, based on a Timoshenko-type correction for transverse shear on the Euler column buckling critical loads, had been earlier provided by Kardomateas and Schmueser (1988). In most of these cited investigations on post-buckling behavior of delaminated composites, the bending-extension coupling effect is not included (except for the Sheinman and Soffer, 1991 paper which, nevertheless, does not include transverse shear nor a study of the growth behavior). Another exception is the work of Yin (1988) which includes bending-stretching coupling and was based on using trigonometric functions to represent the displacements and strains. It should be noted that even if the plies are initially arranged in a symmetric pattern (in which case there is no coupling), this symmetry is disturbed if a delamination is present and coupling becomes possible.

Furthermore, although the classical laminated beam theory, based on the Kirchhoff hypothesis (see Reissner and Stavski, 1961) has been shown to be quite adequate for thin laminates with a high span-to-thickness ratio, due to the low transverse shear modulus relative to the inplane modulus of elasticity of polymeric composites, the effect of shear deformation should be taken into account even for moderate span-to-thickness ratios. In this regard, several improved theories have been formulated (e.g., Whitney and Pagano, 1970; Nelson and Lorch, 1974; Reissner, 1975; Lo et al., 1977, etc.). Most of these refined theories can be categorized into two basic groups: one with in-plane displacements approximated by linear variations in the thickness direction and the other by cubic polynomials. The first group requires the so-called shear correction factors as first suggested by Mindlin (1951) for homogeneous isotropic plates to account for the nonuniform distribution of transverse shear stresses and strains across the thickness. The second group, also called higher-order theories, uses higher-order approximations for shear stresses and strains and does not use shear correction factors but calls for a more involved analysis. The present paper is based on the simpler Timoshenko-Mindlin kinematic hypothesis with shear correction factors. A higher-order kinematic model has already been employed by Sheinman and Adan (1987) in the postbuckling of laminated beams (without delaminations), with the shear deformation effect taken into account. A special finite-difference scheme was used in that study to eliminate the "locking" phenomenon, which may occur in cases where the shear deformation effect is insignificant.

In this investigation, both the bending-extension coupling and the transverse shear effects are included in the study of the postbuckling and growth behavior of delaminated composites. An arbitrary initial imperfection is assumed and the resulting nonlinear problem is solved by a finite-difference scheme. Therefore, the solution procedure is not based on a preassumed displacement field. A new formula for the energy release rate, G, which includes transverse shear is derived and the modes I and II stress intensity factors including transverse shear are calculated from an adaptation of Hutchinson and Suo's (1992) formulas.

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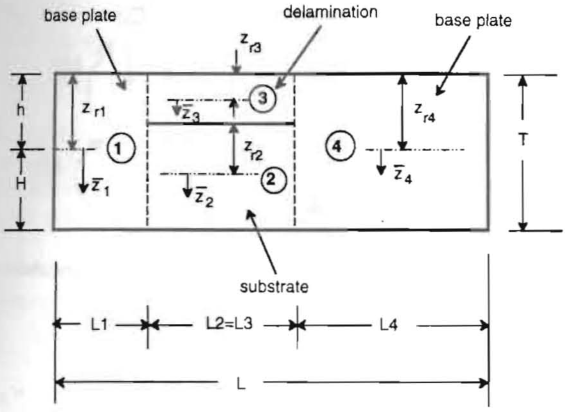

Fig. 1 Definition of the geometry for the delaminated beam/plate

Formulation

Kinematics. A laminate with a delamination can be "subdivided" into four regions, as shown in Fig. 1. In this configuration, the delamination is through the entire width.

The displacement field is defined by its components, u, in the x-direction and w, in the z-direction. In order to include transverse shear, the Kirchhoff-Love hypothesis, which requires that normals to the reference surface remain normal after deformation, is relaxed. The Mindlin (1951) model is used, which assumes that the transverse shear strain is constant over the thickness and requires the use of a shear correction factor. Thus, the displacements are in the form:

$$u(x, z) = u^0(x) + z\psi(x); \quad w(x, z) = w^0(x). \tag{1a}$$

The function $\psi(x)$ is the rotation of the normal to the reference plane, $z = 0$, and is no longer taken as $-w_{,x}$, as is the case in the classical approach. In this paper, the subscript $(,x)$ denotes derivative with respect to x.

The strains are, accordingly:

$$\epsilon_{xx}(x, z) = \epsilon^0_{xx}(x) + z\kappa_{xx}(x); \quad \gamma_{xz} = \gamma^0_{xz}(x), \tag{1b}$$

where the superscript (^{0}) denotes quantities at the reference plane, and κ_{xx} is the change in curvature. The strains at the reference plane and the change of curvature are:

$$\epsilon^0_{xx} = u^0_{,x} + \tfrac{1}{2} w^0_{,x}(w^0_{,x} + 2\bar{w}_{,x}); \quad \kappa_{xx} = \psi_{,x}, \tag{1c}$$

$$\gamma^0_{xz} = \psi + w^0_{,x}. \tag{1d}$$

The function $\bar{w}(x)$ is an assumed initial imperfection. Note that in the classical formulation $\gamma_{xz} = 0$. Therefore, this formulation, which includes transverse shear, leaves three independent variables namely, u, w and ψ.

Constitutive Relations. The stress-strain relations for each lamina j, is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{23} \\ \bar{Q}_{13} & \bar{Q}_{23} & \bar{Q}_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}, \tag{2}$$

where $\bar{Q}_{ij}$ are the stiffness constants of the lamina with fiber orientation θ and are defined in the Appendix (where $1 \equiv x$, $2 \equiv y$).

For a one-dimensional model (in the $x - z$ plane) the strains and change of curvature can be written in terms of the axial force, N_{xx}, the bending moment, M_{xx} and the shear force, Q_{xz}, as follows:

$$\epsilon^0_{xx} = \alpha_1 N_{xx} + \alpha_2 M_{xx}, \tag{3a}$$

$$\kappa_{xx} = \alpha_3 N_{xx} + \alpha_4 M_{xx}, \tag{3b}$$

$$Q_{xz} = kF\gamma^0_{xz}, \tag{3c}$$

where k is the shear correction factor. First, for a width b, fiber orientation θ and shear moduli G_{13} and G_{23} (where $1 \equiv x$, $2 \equiv y$, $3 \equiv z$) the resultant shear relation is in terms of

$$F = \bar{Q}_{44} - \bar{Q}^2_{45}/\bar{Q}_{55}, \tag{4a}$$

where

$$\bar{Q}_{44} = b\int (G_{13}\cos^2\theta + G_{23}\sin^2\theta)dz, \tag{4b}$$

$$\bar{Q}_{45} = -b\int (G_{13} - G_{23})\cos\theta\sin\theta dz, \tag{4c}$$

$$\bar{Q}_{55} = b\int (G_{13}\sin^2\theta + G_{23}\cos^2\theta)dz. \tag{4d}$$

Two different one-dimensional models are considered, (*a*) plane stress or plane strain and (*b*) cylindrical bending (see e.g., Sheinman and Kardomateas, 1997):

(*a*) First, the coefficients α_i for plane stress and plane strain are (Calcote, 1966):

$$\alpha_1 = \frac{D_{11}}{A_{11}(D_{11} - B^2_{11}/A_{11})}, \tag{5a}$$

$$\alpha_2 = \alpha_3 = -\frac{B_{11}}{A_{11}(D_{11} - B^2_{11}/A_{11})}, \tag{5b}$$

$$\alpha_4 = \frac{1}{D_{11} - B^2_{11}/A_{11}}, \tag{5c}$$

where

$$(A_{11}, B_{11}, D_{11}) = \int c_{xx}(1, z, z^2)dz, \tag{5d}$$

and

$$c_{xx} = \bar{Q}_{11} + \delta(c_1\bar{Q}_{12} + c_2\bar{Q}_{13}), \tag{5e}$$

and $\delta = 0$ corresponds to plane strain assumption ($\epsilon_{yy} = \gamma_{xy} = 0$) while $\delta = 1$ corresponds to plane stress assumption ($\sigma_{yy} = \tau_{xy} = 0$), and

$$c_1 = \frac{\bar{Q}_{23}\bar{Q}_{13} - \bar{Q}_{33}\bar{Q}_{12}}{\bar{Q}_{22}\bar{Q}_{33} - \bar{Q}^2_{23}}; \quad c_2 = \frac{\bar{Q}_{12}\bar{Q}_{23} - \bar{Q}_{22}\bar{Q}_{13}}{\bar{Q}_{22}\bar{Q}_{33} - \bar{Q}^2_{23}}, \tag{5f}$$

(*b*) For the cylindrical bending model (Sheinman, 1989), the strains and moments can be written in terms of the resultant forces N and changes in curvatures κ as:

$$\{\epsilon^0\} = [a]\{N\} - [b]\{\kappa\}, \tag{6a}$$

$$\{M\} = [b]^T\{N\} + [d]\{\kappa\}, \tag{6b}$$

where

$$[a] = [A]^{-1}; \quad [b] = [A]^{-1}[B];$$

$$[d] = [D] - [B][A]^{-1}[B], \tag{6c}$$

and this yields the following α coefficients:

$$\alpha_1 = \{[I + A^{-1}B(D - BA^{-1}B)^{-1}B]A^{-1}\}_{11}, \tag{6d}$$

$$\alpha_2 = -\{A^{-1}B(D - BA^{-1}B)^{-1}\}_{11}, \tag{6e}$$

$$\alpha_3 = -\{(D - BA^{-1}B)^{-1}BA^{-1}\}_{11}, \tag{6f}$$

$$\alpha_4 = \{(D - BA^{-1}B)^{-1}\}_{11}. \qquad (6g)$$

Here, I is the unit matrix and the subscript $(_{11})$ refers to the first term of the matrix. In addition, the laminate A, B and D matrices are 3×3 and are defined by:

$$(A_{ij}, B_{ij}, D_{ij}) = \int \bar{Q}_{ij}(1, z, z^2)dz \quad i, j = 1, 2, 3. \qquad (6h)$$

It is seen that the α's are determined not only by A_{11}, B_{11}, D_{11}, but also by A_{ij}, B_{ij}, D_{ij}, $i, j = 1, 2, 3$. These expressions for the α's would be equal to those for the classical case if the lay-up is symmetrical ($B_{ij} = 0$). For a nonsymmetrical lay-up, a significant difference between the two can be expected depending on the stacking sequence and orientation. The cylindrical bending model is the most suitable for representing a one-dimensional laminated configuration.

Equations (1*a*, *b*) can be inverted to solve for N_{xx} and M_{xx} in terms of the displacement variables and the following expressions are obtained:

$$N_{xx} = \eta\{-\alpha_4[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})] + \alpha_2\psi_{,x}\}, \qquad (7a)$$

$$M_{xx} = \eta\{\alpha_2[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})] - \alpha_1\psi_{,x}\}, \qquad (7b)$$

where

$$\eta = 1/(\alpha_2\alpha_3 - \alpha_4\alpha_1). \qquad (8)$$

Nonlinear Equilibrium Equations and Linearization Scheme. By applying now the minimum potential principle, the nonlinear equilibrium equations together with the appropriate boundary conditions can be derived. The laminate is assumed to be acted upon by external axial, transverse and moment loadings, denoted by q_x, q_z and m, respectively. A variational formulation (stationary value of total potential) gives the following expression for the variation of the strain energy and the variation in the potential of the external forces:

$$\delta\Pi = \int_x (N_{xx}\delta\epsilon^0_{xx} + M_{xx}\delta\kappa_{xx} + kQ_{xz}\delta\gamma_{xz})dx$$

$$- \int_x [q_x(x)\delta u + q_z(x)\delta w + m(x)\delta\psi]dx. \qquad (9)$$

Substituting in (1*c*), (1*d*) yields the following nonlinear equations:

$$N_{xx,x} + q_x = 0 \qquad (10a)$$

$$[N_{xx}(w_{,x} + \bar{w}_{,x})]_{,x} + Q_{xz,x} + q_z = 0 \qquad (10b)$$

$$M_{xx,x} - Q_{xz} + m = 0 \qquad (10c)$$

with the boundary conditions imposed on u or N_{xx}, w or $N_{xx}(w_{,x} + \bar{w}_{,x}) + Q_{xz}$ and ψ or M_{xx}.

Using (3*c*) and (7*a*–*c*) gives the three nonlinear equilibrium equations in terms of the displacement variables:

$$\eta\{\alpha_2\psi_{,xx} - \alpha_4[u_{,xx} + (w_{,x} + \bar{w}_{,x})w_{,xx} + w_{,x}\bar{w}_{,xx}]\} + q_x = 0, \qquad (11a)$$

$$\eta\{\alpha_2\psi_{,x} - \alpha_4[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + \bar{w}_{,x})]\}(w_{,xx} + \bar{w}_{,xx}) + kF(\psi_{,x} + w_{,xx}) - q_x(w_{,x} + \bar{w}_{,x}) + q_z = 0, \qquad (11b)$$

$$\eta\{\alpha_2[u_{,xx} + (w_{,x} + \bar{w}_{,x})w_{,xx} + w_{,x}\bar{w}_{,xx}] - \alpha_1\psi_{,xx}\} - kF(\psi + w_{,x}) + m = 0. \qquad (11c)$$

with the boundary conditions being:

$$u \quad \text{or} \quad \eta\{-\alpha_4[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})] + \alpha_2\psi_{,x}\}$$

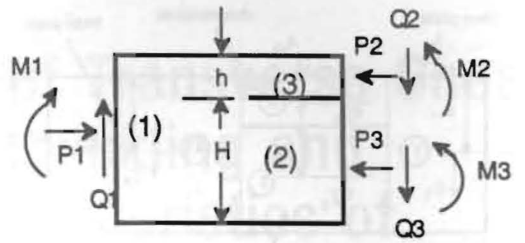

Fig. 2 Force and moment quantities at the section where the delamination starts

$$w \quad \text{or} \quad \eta\{-\alpha_4[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})] + \alpha_2\psi_{,x}\}(w_{,x} + \bar{w}_{,x}) + kF(\psi + w^0_{,x})$$

$$\psi \quad \text{or} \quad \eta\{\alpha_2[u_{,x} + \tfrac{1}{2}w_{,x}(w_{,x} + 2\bar{w}_{,x})] - \alpha_1\psi_{,x}\} \qquad (12)$$

Conditions at the Tips. The following displacement continuity conditions were applied at the crack tips, where the subscript denotes the region (Fig. 2):

First Crack Tip:

$$w_2 = w_3 = w_1, \qquad (13a)$$

$$\psi_2 = \psi_3 = \psi_1 = \psi_a, \qquad (13b)$$

$$u_2 = u_1 - \frac{h}{2}\psi_a; \quad u_3 = u_1 + \frac{h}{2}\psi_a. \qquad (13c)$$

Second Crack Tip:

$$w_2 = w_3 = w_4, \qquad (14a)$$

$$\psi_2 = \psi_3 = \psi_4 = \psi_b, \qquad (14b)$$

$$u_2 = u_4 - \frac{h}{2}\psi_b; \quad u_3 = u_4 + \frac{h}{2}\psi_b. \qquad (14c)$$

Furthermore, force and moment equilibrium were imposed at each crack tip (Fig. 2). In terms of the moments M_i, shear forces, $\hat{Q}_i$, and axial forces, N_i in each of the regions (delaminated part, substrate and base plate) at these sections,

First Crack Tip:

$$M_1 - M_2 - M_3 + N_3\frac{H}{2} - N_2\frac{h}{2} = 0, \qquad (15a)$$

$$-\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 = 0, \qquad (15b)$$

$$-N_1 + N_2 + N_3 = 0. \qquad (15c)$$

Second Crack Tip:

$$M_4 - M_2 - M_3 + N_3\frac{H}{2} - N_2\frac{h}{2} = 0, \qquad (16a)$$

$$-\hat{Q}_4 + \hat{Q}_2 + \hat{Q}_3 = 0, \qquad (16b)$$

$$-N_4 + N_2 + N_3 = 0. \qquad (16c)$$

The resultant shear force at the section, $\hat{Q}$, is defined as:

$$\hat{Q} = N_{xx}(w_{,x} + \bar{w}_{,x}) + Q_{xz} \qquad (17)$$

The foregoing equations can be written in terms of the displacements by using the relationships for M, N and Q_{xz} given in (3*c*), (7*a*–*c*).

The equilibrium equations, the boundary conditions and the continuity conditions were subsequently linearized. This was done using the linearization scheme developed by Thurston (1965) and it was implemented by defining

$$u_{n+1} = u_n + \delta u_n; \quad w_{n+1} = w_n + \delta w_n; \quad \psi_{n+1} = \psi_n + \delta\psi_n$$

The equations can be linearized directly by applying the following: Consider X and Y to be any two of the unknowns or their respective derivatives, then the product of these two unknowns with the higher order terms (δ^2) neglected, can be written as:

$$X_{n+1}Y_{n+1} = X_{n+1}Y_n + X_nY_{n+1} - X_nY_n. \quad (18a)$$

Products of three unknowns are linearized following:

$$X_{n+1}Y_{n+1}Z_{n+1} = X_{n+1}Y_nZ_n + X_nY_{n+1}Z_n + X_nY_nZ_{n+1} - 2X_nY_nZ_n. \quad (18b)$$

Once the nonlinearities are removed, the derivatives are approximated by a central finite difference scheme, in which fictitious points are used at the ends of each region to facilitate the definition of the derivatives at the beginning and ending points. Another benefit of adding these fictitious points is that they would allow applying the governing differential equations at the beginning and end points of these areas, in addition to the boundary conditions.

Upon substitution of the finite difference relations, the differential equations were converted into an algebraic sequence. Subsequently, Potter's method (Tene et al., 1974; Sheinman and Simitses, 1984) was used to solve the resulting system of linear equations.

Energy Release Rate and Stress Intensity Factors Including Transverse Shear. Several investigators have analyzed the strain energy release rate, G, with respect to predicting edge or internal delamination growth. To this extent, a Griffith-type fracture criterion is employed and it is assumed that whether further delamination occurs depends on the magnitude of the fracture energy, defined as the energy required to produce a unit of new delamination. In mixed mode Linear Elastic Fracture Mechanics configurations, this quantity is uniquely expressed in terms of the modes I and II stress intensity factors, K_I and K_{II}, respectively (e.g., Hellan, 1990).

During the initial postbuckling phase, the mode mixity changes with applied strain, and depends on the relative delamination thickness h/T. For example, the study by Kardomateas (1993) has shown that a higher mode I component is present with delaminations further away from the surface. The mode mixity also changes as the delamination grows and in the study by Kardomateas (1993) it was shown that an increased mode II component occurs as the delamination propagates under a constant applied compressive strain.

A procedure for finding the total energy release rate and the modes I and II stress intensity factors for a general laminated composite of arbitrary stacking sequence was reported in Sheinman and Kardomateas (1997). The total energy release rate was obtained by using the J-integral for a one-dimensional model of plane stress, plane strain or cylindrical bending. However, the transverse shear effect was not included in that work.

Considering Fig. 3, z_{ri} denotes the location of the reference surface of each of the four regions in which the delamination subdivides the laminate. A general geometrical procedure, based on the above constitutive relations, can be developed and be used for the stress state before $\sigma^{(b)}$ and after $\sigma^{(a)}$ the crack tip for each load level.

The stress resultants are given by:

$$[P_d, M_d] = \int_h (\sigma^{(a)} - \sigma^{(b)})[1, z]dz, \quad (19a)$$

$$[P_s, M_s] = \int_H (\sigma^{(a)} - \sigma^{(b)})[1, z]dz, \quad (19b)$$

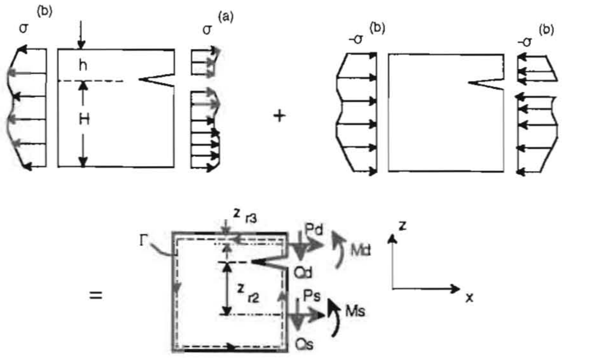

Fig. 3 Stress superposition scheme for implementing the J-integral

$$Q_d = \int_h (\tau^{(a)} - \tau^{(b)})dz; \quad Q_s = \int_H (\tau^{(a)} - \tau^{(b)})dz. \quad (19c)$$

where the subscripts d, s refer to the (upper) delaminated part and the (lower) substrate, respectively. From equilibrium, the following relationships hold:

$$P_s = -P_d; \quad Q_s = -Q_d, \quad (20a)$$

$$M_s = -M_d + P_d(h + z_{r2} - z_{r3}). \quad (20b)$$

The energy release rate can be computed from the J-integral:

$$J = \int_\Gamma \left[Wdz - T_i \frac{\partial u_i}{\partial x} ds \right], \quad (21a)$$

where W is the strain energy. Only one side has nonvanishing stresses and on this side, $dz = ds$ and

$$W = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij} = \tfrac{1}{2}(\sigma_{xx}\epsilon_{xx} + \sigma_{zz}\epsilon_{zz} + \tau_{xz}\gamma_{xz})$$

$$T_i \frac{\partial u_i}{\partial x} = \sigma_{xx}\epsilon_{xx} + \tau_{xz}w_{,x}$$

Therefore

$$J = \frac{1}{2}\int_\Gamma [-\sigma_{xx}\epsilon_{xx} + \sigma_{zz}\epsilon_{zz} + \tau_{xz}(\gamma_{xz} - 2w_{,x})]ds. \quad (21b)$$

A plane stress state with $\sigma_{zz} = 0$ is assumed. Now we make use of (1b) and (3a–c) with $N_{xx} = P$ and $M_{xx} = M$ to get the contribution of the delaminated part for the normal stress term:

$$\int \sigma_{xx}\epsilon_{xx}dz = \int \sigma_{xx}[(\alpha_{1d}P_d + \alpha_{2d}M_d) + z(\alpha_{3d}P_d + \alpha_{4d}M_d)]dz$$
$$= (\alpha_{1d}P_d + \alpha_{2d}M_d)P_d + (\alpha_{3d}P_d + \alpha_{4d}M_d)M_d,$$

and from the transverse shear term, by using (1d):

$$\int \tau_{xz}(\gamma_{xz} - 2w_{,x})dz = -\int \tau_{xz}(\gamma_{xz} - 2\psi)dz$$
$$= -\int \tau_{xz}\left(\frac{Q_{xz}}{kF} - 2\psi\right)dz = -\left(\frac{Q_{xz}}{kF} - 2\psi\right)Q_{xz}.$$

The contribution from the substrate can be found in a similar fashion and can be found in terms of P_d and M_d and Q_d by using (20a, b).

Taking into account that $\psi_d = \psi_s$ from (13b) and that $Q_d = -Q_s$ from (20a), we find the energy release rate including transverse shear:

$$G = \frac{1}{2}\left(\frac{P_d^2}{R_1} + \frac{M_d^2}{R_2} + \frac{2P_dM_d \sin\gamma}{\sqrt{R_1R_2}} + \frac{Q_d^2}{R_3}\right), \qquad (22a)$$

where

$$\frac{1}{R_1} = \alpha_{1d} + \alpha_{1s} - (\alpha_{3s} + \alpha_{2d})(h + z_{r2} - z_{r3}) + \alpha_{4d}(h + z_{r2} - z_{r3})^2, \qquad (22b)$$

$$\frac{1}{R_2} = \alpha_{4d} + \alpha_{4s}; \quad \frac{1}{R_3} = \frac{1}{k}\left(\frac{1}{F_d} + \frac{1}{F_s}\right), \qquad (22c)$$

$$\sin\gamma = \frac{\sqrt{R_1R_2}}{2}\left[\alpha_{2s} + \alpha_{2d} + \alpha_{3s} + \alpha_{3d} - 2\alpha_{4s}(h + z_{r2} - z_{r3})\right], \qquad (22d)$$

These formulas extend the formula derived in Sheinman and Kardomateas (1997), which did not include transverse shear. They are valid for an arbitrary stacking sequence with one caveat. Due to the various couplings in sublaminates with fully populated A-B-D matrices, growth will generally not be uniform across the front of a finite-width specimen and then the one-dimensional approximation would not be appropriate.

However, regarding the modes I and II stress intensity factors, at this point we shall employ the formulas of Hutchinson and Suo (1992) for a crack in an orthotropic strip, pending a solution for a fully anisotropic crack. Therefore, we use the smeared technique to obtain equivalent orthotropic properties, $\bar{s}_{ij}$, as

$$\bar{s}_{ij} = \frac{1}{t}\sum_k s_{ij}^k t_k$$

where t is the total thickness, and k denotes each lamina. The relevant stress strain in the $x - z$ plane is, therefore,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{bmatrix}_k = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix}_k \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \gamma_{xz} \end{bmatrix}_k, \qquad (23a)$$

where the stiffness constants are now:

$$s_{11} = \frac{E_{xx}}{1 - \nu_{xz}\nu_{zx}}; \quad s_{22} = \frac{E_{zz}}{1 - \nu_{xz}\nu_{zx}}; \qquad (23b)$$

$$s_{12} = \nu_{xz}s_{22} = \nu_{zx}s_{11}; \quad s_{33} = G_{xz}. \qquad (23c)$$

Let us now denote by $[p] = [\bar{s}]^{-1}$ the corresponding compliance matrix of the equivalently orthotropic laminate. Following Suo (1990) (see also Sih et al., 1965), the energy release rate for an orthotropic material can be written in terms of the mode I and mode II stress intensity factors in the form:

$$G = p_{11}n(\lambda^{-3/4}K_I^2 + \lambda^{-1/4}K_{II}^2), \qquad (24a)$$

where

$$\lambda = p_{11}/p_{22}; \quad \rho = \frac{2p_{12} + p_{33}}{2\sqrt{p_{11}p_{22}}}; \quad n = \sqrt{\frac{1+\rho}{2}}. \qquad (24b)$$

By defining

$$\frac{1}{R_1^*} = \frac{1}{R_1} + \frac{Q_d^2/P_d^2}{R_3}, \qquad (25a)$$

we can write (22a) in the equivalent form

$$G = \frac{1}{2}\left(\frac{P_d^2}{R_1^*} + \frac{M_d^2}{R_2} + \frac{2P_dM_d \sin\gamma}{\sqrt{R_1R_2}}\right), \qquad (25b)$$

Now, following the same arguments as in Suo (1990) and Sheinman and Kardomateas (1997), equating the two energy release rate equations (24a) and (26a) gives

$$\sqrt{p_{11}n}\,|\lambda^{-3/8}K_I + i\lambda^{-1/8}K_{II}| = \frac{1}{\sqrt{2}}\left|\frac{P_d}{\sqrt{R_1^*}} - ie^{i\gamma}\frac{M_d}{\sqrt{R_2}}\right|,$$

i.e., two complex quantities have the same magnitude and consequently, they can differ only by a phase angle shift, designated as ω, namely,

$$\sqrt{p_{11}n}(\lambda^{-3/8}K_I + i\lambda^{-1/8}K_{II}) = \frac{e^{i\omega}}{\sqrt{2}}\left(\frac{P_d}{\sqrt{R_1^*}} - ie^{i\gamma}\frac{M_d}{\sqrt{R_2}}\right). \qquad (26)$$

Arguments of dimensional analysis and loading lead to dependence of ω on h/H and ρ (Suo, 1990), i.e., $\omega = \omega(h/H, \rho)$.

Equation (26) leads to the following explicit relations for the stress intensity factors:

$$K_I = \frac{\lambda^{3/8}}{\sqrt{2p_{11}n}}\left[\frac{P_d}{\sqrt{R_1^*}}\cos\omega + \frac{M_d}{\sqrt{R_2}}\sin(\omega + \gamma)\right], \qquad (27a)$$

$$K_{II} = \frac{\lambda^{1/8}}{\sqrt{2p_{11}n}}\left[\frac{P_d}{\sqrt{R_1^*}}\sin\omega - \frac{M_d}{\sqrt{R_2}}\cos(\omega + \gamma)\right]. \qquad (27b)$$

where ω is restricted in the range $0 < \omega < \pi/2$. Therefore the stress intensity factors are fully determined apart from the single dimensionless real function $\omega(h/H, \rho)$. The determination of this function requires a rigorous solution of the crack problem as was performed in Suo (1990) by use of dislocation modeling. However, in that paper it was shown that ω is around 50 degrees. He showed that for $h/H = 1$, ω does not depend on ρ and is 49.1 degrees. Moreover, for $\rho = 1$, he also showed that an excellent approximation is

$$\omega = 52.1 - 3h/H, \quad \text{in degrees} \qquad (27c)$$

For arbitrary values of h/H and ρ, the integral equation solution in Suo (1990) showed a very weak dependence on ρ and confirmed that Eq. (27c) can be used as an adequate estimate for a wide range of ρ and for $0 \le h/H \le 1$.

In our present formulation, we should mention at this point that although the G derived in (22a) or (25b) is an exact expression, the K_I and K_{II} calculations are based on the smeared equivalent orthotropic properties assumption as well as the definition of (25a) for the equivalent constant, R_1^* which now depends on the ratio Q_d/P_d in addition to the geometry and material. Therefore, unlike the G formulas which are exact, the K_I and K_{II} formulas are approximate in the case of arbitrary stacking sequence.

It is useful to express the relative amounts of mode I and mode II by the mode mixity, $\bar{\psi}$, defined as:

$$\bar{\psi} = \tan^{-1}(K_{II}/K_I). \qquad (28)$$

Notice that the limit of a very thin delamination in an isotropic material (thin film model) would predict G and $\bar{\psi}$ in terms of the applied compressive strain ϵ_0 and the critical strain ϵ_{cr} as follows (e.g., Kardomateas, 1993):

$$G = \tfrac{1}{2}Eh(1 - \nu^2)(\epsilon_0 - \epsilon_{cr})(\epsilon_0 + 3\epsilon_{cr})$$

$$\tan\bar{\psi} = \frac{2.457 + 1.367\xi}{-3.156 + 1.064\xi}; \quad \xi = \left[\frac{4}{3}\left(\frac{\epsilon_0}{\epsilon_{cr}} - 1\right)\right]^{1/2}$$

Discussion of Results

A verification was done first regarding the critical load. Data were provided by Davidson and Ferrie (1994) who performed experiments and analysis on a laminate consisting of 20 plies of

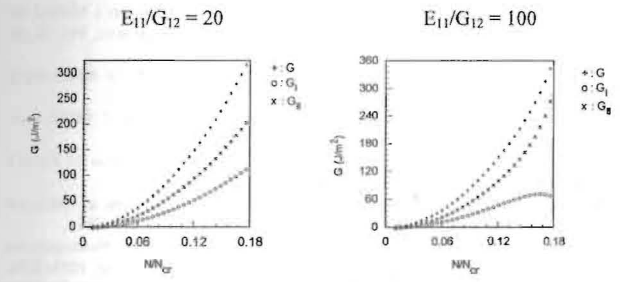

Fig. 4(a) The effect of the E_{11}/G_{12} ratio on the energy release rate for $h/T = \frac{4}{15}$

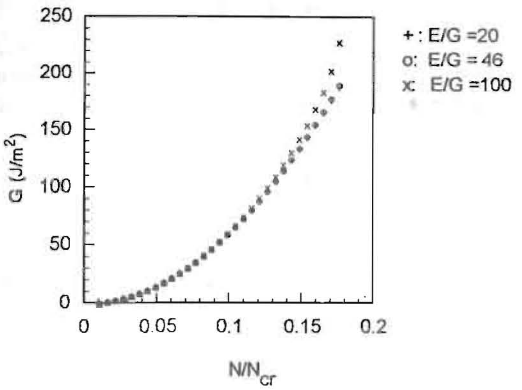

Fig. 4(b) The effect of the E_{11}/G_{12} ratio on the Mode *II* energy release rate for $h/T = \frac{7}{15}$

Ciba-Geigy C6000/R6376 graphite-epoxy in a layup $[(0_2/90/0_2)/d/(0_2/90/0_2)_{2s}]$ where d denotes the delamination. The specimen geometry was: $L_2 = 38.1$ mm, $L_1 = L_4 = 37.95$ mm, $W = 25$ mm and $h/T = \frac{1}{4}$. The unidirectional material properties in compression for graphite/epoxy are (moduli in Gpa): $E_{11} = 124.11$, $E_{22} = 10.27$, $G_{12} = 5.45$, $\nu_{12} = 0.37$ and the ply thickness was $t_{ply} = 0.127$ mm. The critical strain for delamination buckling was found from different approaches including experimental and using both the Classical Beam Theory (CBT) and the present First Order Shear Deformation Theory (FOSDT). The following table lists the results:

Method of Determination	ϵ_{cr}	% difference
Experimental results	1123 $\mu\epsilon$	—
Davidson and Ferrie (1994)	1116 $\mu\epsilon$	0.6
FOSDT	1122 $\mu\epsilon$	0.1
CBT	1143 $\mu\epsilon$	1.9

Next, a parametric study was done to examine the effect of transverse shear. The material chosen for the derivation of numerical results is boron/epoxy with material properties given as follows (in GPa): $E_{11} = 206.8$, $E_{22} = E_{33} = 18.6$, $G_{12} = 4.48$, $G_{23} = 2.55$, $\nu_{12} = 0.21$ and ply thickness $t_{ply} = 0.0889$ mm. The material was chosen because of its naturally high ratio E_{11}/G_{12}. However, a parametric study was done by keeping the same E_{11} and changing G_{12} to achieve ratios of between 20 and 100. This ensured that the critical load for the perfect structure remains nearly the same. The laminates were comprised of 15 unidirectional zero degree plies. A delamination was located symmetrically with respect to the length of the beam with a L_2/L ratio of $\frac{1}{3}$. The location of the delamination through the thickness was also changed from between first and second layer to between 7th and 8th layer. By keeping the orientation of the plies at zero degrees, the effect of the E_{11}/G_{12} ratio could be isolated from other factors such as bending-extension coupling, which could arise from angle ply configurations.

Figure 4(a) shows the energy release rate components for $h/T = \frac{4}{15}$ for the cases of $E_{11}/G_{12} = 20$ and 100. Transverse shear effects, which are more pronounced for $E_{11}/G_{12} = 100$, lead to higher energy release rate values. Another example is shown in Fig. 4(b), which gives the effect of the E_{11}/G_{12} ratio on the mode II energy release rate for $h/T = \frac{7}{15}$. Again, higher values of energy release rate occur with higher E_{11}/G_{12} ratios, this effect being more important as the postbuckling proceeds, i.e. as the load and deflections increase. The effect on the midpoint deflection curve for $h/T = \frac{7}{15}$ is plotted in Fig. 5. In the higher E_{11}/G_{12} case, large deflections are achieved at much smaller loads.

The previous results were presented by maintaining a ratio of $L/T = 10$. However, the effect of transverse shear also depends on the L/T ratio. Figure 6 shows the midpoint deflection curves for $E_{11}/G_{12} = 46$, $h/T = \frac{7}{15}$ and a range of L/T ratios.

In conclusion, the present formulation, which is geometrically nonlinear for an arbitrary stacking sequence and includes the effects of transverse shear and bending-extension coupling as well as that of an arbitrary initial imperfection, can be used to examine

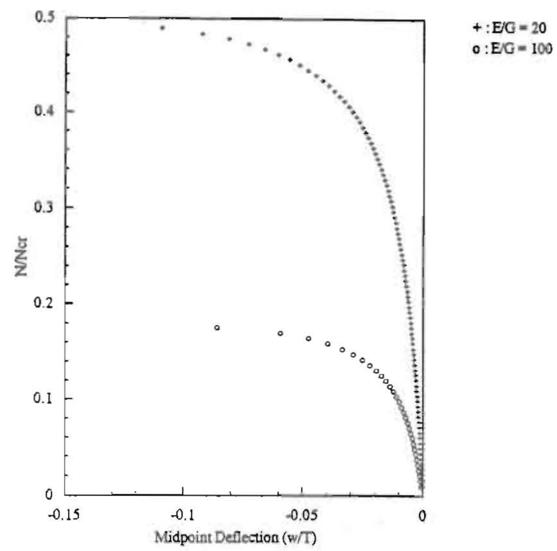

Fig. 5 The effect of the E_{11}/G_{12} ratio on the midpoint deflection for $h/T = \frac{7}{15}$ (E_{11} kept constant)

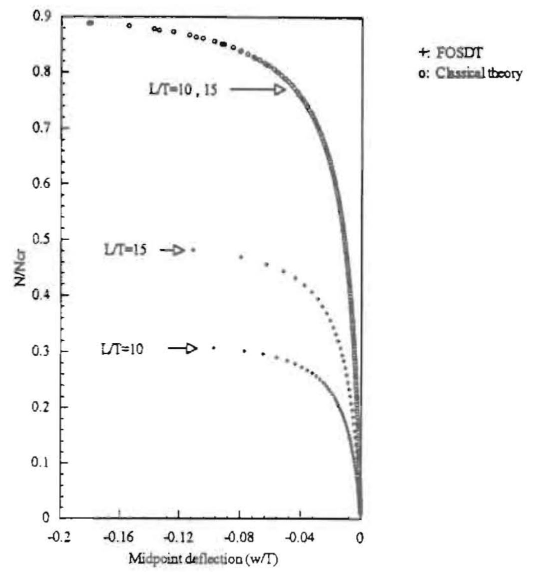

Fig. 6 Midpoint deflection curves for $L/T = 10$ and 15, using CBT and FOSDT for $h/T = \frac{7}{15}$ and $E_{11}/G_{12} = 46$

the buckling, postbuckling and growth conditions of delaminations under compression.

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APPENDIX

For a beam/plate of width b with in-plane moduli E_{11}, E_{22}, G_{12}, and transverse moduli G_{13} and G_{23} and with a fiber orientation (relative to $1 \equiv x$) θ, the stiffness constants $\bar{Q}_{ij}$ are given in terms of

$$Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}; \quad Q_{12} = \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})}, \qquad (A1)$$

$$Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}; \quad Q_{33} = G_{12}, \qquad (A2)$$

as follows:

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{33})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta, \qquad (A3a)$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{33})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta, \qquad (A3b)$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{33})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta), \qquad (A3c)$$

$$\bar{Q}_{13} = (Q_{11} - Q_{12} - 2Q_{33})\sin\theta\cos^3\theta + (Q_{11} - Q_{12} + 2Q_{33})\sin^3\theta\cos\theta, \qquad (A3d)$$

$$\bar{Q}_{23} = (Q_{11} - Q_{12} - 2Q_{33})\sin^3\theta\cos\theta + (Q_{11} - Q_{12} - 2Q_{33})\sin\theta\cos^3\theta, \qquad (A3e)$$

$$\bar{Q}_{33} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{33})\sin^2\theta\cos^2\theta + Q_{33}(\sin^4\theta + \cos^4\theta). \qquad (A3f)$$